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A BRIEF INTRODUCTION TO HOMOGENIZATION AND MISCELLANEOUS APPLICATIONS*

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Abstract. This paper is a set of lecture notes for a short introductory course on homogenization. It covers the basic tools of periodic homogenization (two-scale asymptotic expansions, the oscillating test function method and two-scale convergence) and briefly describes the main results of the more general theory of $G-$ or $H-$convergence. Several applications of the method are given: derivation of Darcy's law for flows in porous media, derivation of the porosity model and long time behavior of a diffusion equation. Numerical agorithms for homogenization are also discussed, including multiscale finite element methods.

Résumé. Cet article reprend des notes de cours, d'un niveau introductif, sur l'homogénéisation. Ces notes couvrent la théorie de l'homogénéisation périodique (développements asymptotiques à deux échelles, méthode de la fonction test oscillante, convergence à deux échelles) et décrivent brièvement les principaux résultats de la théorie plus générale de la $G-$ ou $H-$convergence. Plusieurs applications de la méthode sont données: dérivation de la loi de Darcy pour des écoulements en milieux poreux, dérivation du modèle de double porosité et comportement en temps grand d'une équation de diffusion. Des algorithmes numériques pour l'homogénéisation sont aussi présentés, dont, en particulier, les méthodes d'éléments finis multi-échelles.

Introduction

These lecture notes are a brief introduction to the mathematical theory of homogenization. These notes are intended to be pedagogical so not all technical details or complete proofs are given. Rather we prefer to focus on various applications to multiscale modelling and to numerical computations in heterogeneous media. For a more advanced presentation of homogenization, the reader is referred to the books [7], [17], [18], [29], [35], [68], [77], [87] and [89]. Roughly speaking, homogenization is a rigorous version of what is known as averaging or upscaling. In other words, homogenization extracts homogeneous effective parameters from disordered or heterogeneous media.

Homogenization has first been developed for periodic structures. Indeed, in many fields of science and technology one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting by ϵ their ratio, an asymptotic analysis, as ϵ goes to zero, is called for. Starting from a microscopic description of a problem, we seek a macroscopic, or effective, description. This process of making an asymptotic analysis and seeking an averaged formulation is called homogenization. The first section will focus on the homogenization of periodic structures. The method

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of two-scale asymptotic expansions is presented, and its mathematical justification will be briefly discussed. In particular we introduce the notion of two-scale convergence due to Nguetseng [74] and Allaire [6].

However we emphasize that homogenization is not restricted to the periodic case and can be applied to any kind of disordered media. This is the focus of the second section where the notion of G- or H-convergence is introduced. It allows to consider any possible geometrical situation without any specific assumptions like periodicity or randomness.

The goal of the following sections is to show that homogenization is a very efficient tool in the modeling of complex phenomena in heterogeneous media. In Sections 1 and 2 we consider a model problem of diffusion for which the homogenized operator is of the same type (still a diffusion equation). In this context, homogenization is really a matter of defining and computing effective diffusion tensors. On the contrary, Sections 3 and 4 will focus on models which have different homogenized limits (in the sense that the partial differential equations are of a different mathematical nature). In Section 3, we shall see that the Stokes equations for a viscous fluid in a porous medium yield the Darcy's law as an homogenized model. In Section 4 we will establish the so-called double porosity model and study the long time behavior of a diffusion equation. Therefore, in this context, homogenization is a modeling tool which can justify new models arising as homogenized limits of complex microscopic equations.

Eventually, Sections 5 and 6 are devoted to the numerical aspects of homogenization theory. Indeed, homogenization is a useful tool to design algorithms for computing approximate solutions of partial differential equations in highly heterogeneous media. In Section 5 we briefly review the classical numerical approximations for periodic media. It amounts to the separate computation of the cell and homogenized problems. We also give some indications about correctors and boundary layers. In Section 6 we briefly discuss multiscale numerical methods, inspired from the theory of periodic homogenization, but which can be applied to any kind of disordered heterogeneous media (not necessarily periodic).

1. Periodic homogenization

1.1. **Setting of the problem.**

We consider a model problem of diffusion or conductivity in a periodic medium (for example, an heterogeneous domain obtained by mixing periodically two different phases, one being the matrix and the other the inclusions; see Figure 1). To fix ideas, the periodic domain is called Ω (a bounded open set in $\mathbb{R}^N$ with $N \geq 1$ the space dimension), its period ϵ (a positive number which is assumed to be very small in comparison with the size of the domain), and the rescaled unit periodic cell $Y = (0,1)^N$. The conductivity in Ω is not constant, but varies periodically with period ϵ in each direction. It is a matrix (a second order tensor) $A(y)$, where $y = x/\epsilon \in Y$ is the fast periodic variable, while $x \in \Omega$ is the slow variable. Equivalently, x is also called the macroscopic variable, and y the microscopic variable. Since the component conductors do not need to be isotropic, the matrix A can be any second order tensor that is bounded and positive definite, i.e., there exist two positive constants $\beta \geq \alpha > 0$ such that, for any vector $\xi \in \mathbb{R}^N$ and at any point $y \in Y$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \leq \beta|\xi|^2. \tag{1}$$

At this point, the matrix A is not necessarily symmetric (such is the case when some drift is taken into account in the diffusion process). The matrix $A(y)$ is a periodic function of y, with period Y, and it may be discontinuous in y (to model the discontinuity of conductivities from one phase to the other).

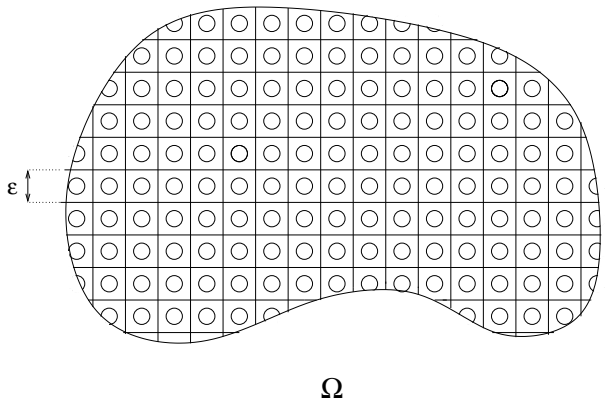

FIGURE 1. A periodic domain.

Denoting by $f(x)$ the source term (a scalar function defined in Ω), and enforcing a Dirichlet boundary condition (for simplicity), our model problem of conductivity reads

$$\begin{cases} -\text{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $u_\epsilon(x)$ is the unknown function, modeling the electrical potential or the temperature.

Remark 1.1. *From a mathematical point of view, problem (2) is well posed in the sense that, if the source term $f(x)$ belongs to the space $L^2(\Omega)$ of square integrable functions on Ω, then the Lax-Milgram lemma implies existence and uniqueness of the solution u_ϵ in the Sobolev space $H_0^1(\Omega)$ of functions which belong to $L^2(\Omega)$ along with their first derivatives. Furthermore, the following energy estimate holds*

$$\|u_\epsilon\|_{L^2(\Omega)} + \|\nabla u_\epsilon\|_{L^2(\Omega)} \leq C,$$

where the constant C does not depend on ϵ.

The domain Ω, with its conductivity $A\left(\frac{x}{\epsilon}\right)$, is highly heterogeneous with periodic heterogeneities of length-scale ϵ. Usually one does not need the full details of the variations of the potential or temperature u_ϵ, but rather some global of averaged behavior of the domain Ω considered as an homogeneous domain. In other words, an effective or equivalent macroscopic conductivity of Ω is sought. From a numerical point of view, solving equation (2) by any method will require too much effort if ϵ is small since the number of elements (or degrees of freedom) for a fixed level of accuracy grows like $1/\epsilon^N$. It is thus preferable to average or homogenize the properties of Ω and compute an approximation of u_ϵ on a coarse mesh. Averaging the solution of (2) and finding the effective properties of the domain Ω is what we call homogenization.

There is a difference of methodology between the traditional physical approach of homogenization and the mathematical theory of homogenization. In the mechanical literature, the so-called representative volume element (RVE) method is often used (see [26], or Chapter 1 in [49]). Roughly speaking, it consists in taking a sample of the heterogeneous medium of size much larger than the heterogeneities, but still much smaller than the medium, and averaging over it the gradient ∇u_ϵ and the flux $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon$. Denoting by ξ the average of the gradient and by σ that of the flux, the effective tensor of conductivity A^* of this sample is defined by the linear relationship $\sigma = A^*\xi$. It turns out that the averaged stored energy $A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla u_\epsilon$ is also equal to the effective energy $A^*\xi \cdot \xi$. Although this type of definition is very intuitive, it is not clear whether it defines correctly

an effective tensor A^*. In particular, it may depend on the choice of source term f, sample size, or boundary conditions.

The mathematical theory of homogenization works completely differently. Rather than considering a single heterogeneous medium with a fixed lengthscale, the problem is first embedded in a sequence of similar problems for which the lengthscale ϵ, becoming increasingly small, goes to zero. Then, an asymptotic analysis is performed as ϵ tends to zero, and the conductivity tensor of the limit problem is said to be the *effective* or *homogenized* conductivity. This seemingly more complex approach has the advantage of defining uniquely the homogenized properties. Further, the approximation made by using effective properties instead of the true microscopic coefficients can be rigorously justified by quantifying the resulting error.

In the case of a periodic medium Ω, this asymptotic analysis of equation (2), as the period ϵ goes to zero, is especially simple. The solution u_ϵ is written as a power series in ϵ

$$u_\epsilon = \sum_{i=0}^{+\infty} \epsilon^i u_i.$$

The first term u_0 of this series will be identified with the solution of the so-called homogenized equation whose effective conductivity A^* can be exactly computed. It turns out that A^* is a constant tensor, describing a homogeneous medium, which is independent of f and of the boundary conditions. Therefore, numerical computations on the homogenized equation do not require a fine mesh since the heterogeneities of size ϵ have been averaged out. This homogenized tensor A^* is almost never a usual average (arithmetic or harmonic) of $A(y)$. Various estimates will confirm this asymptotic analysis by telling in which sense u_ϵ is close to u_0 as ϵ tends to zero.

Remark 1.2. *From a more theoretical point of view, homogenization can be interpreted as follows. Rather than studying a single problem (2) for the physically relevant value of ϵ, we consider a sequence of such problems indexed by the period ϵ, which is now regarded as a small parameter going to zero. The question is to find the limit of this sequence of problems. The notion of limit problem is defined by considering the convergence of the sequence $(u_\epsilon)_{\epsilon>0}$ of solutions of (2): Denoting by u its limit, the limit problem is defined as the problem for which u is a solution. Of course, u will turn out to coincide with u_0, the first term in the series defined above, and it is therefore the solution of the homogenized equation. Clearly the mathematical difficulty is to define an adequate topology for this notion of convergence of problems as ϵ goes to zero.*

1.2. **Two-scale asymptotic expansions.**

Ansatz.

The method of two-scale asymptotic expansions is an heuristic method, which allows one to formally homogenize a great variety of models or equations posed in a periodic domain. We present it briefly and refer to the classical books [17], [18], and [77] for more detail. A mathematical justification of what follows is to be found in Subsection 1.3. As already stated, the starting point is to consider the following *two-scale asymptotic expansion* (also called an *ansatz*), for the solution u_ϵ of equation (2)

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \tag{3}$$

where each term $u_i(x,y)$ is a function of both variables x and y, periodic in y with period $Y = (0,1)^N$ (u_i is called a Y-periodic function with respect to y). This series is plugged into the equation, and the following derivation rule is used:

$$\nabla\left(u_i\left(x, \frac{x}{\epsilon}\right)\right) = \left(\epsilon^{-1}\nabla_y u_i + \nabla_x u_i\right)\left(x, \frac{x}{\epsilon}\right), \tag{4}$$

where ∇_x and ∇_y denote the partial derivative with respect to the first and second variable of $u_i(x,y)$. For example, one has

$$\nabla u_\epsilon(x) = \epsilon^{-1}\nabla_y u_0\left(x,\frac{x}{\epsilon}\right) + \sum_{i=0}^{+\infty}\epsilon^i\left(\nabla_y u_{i+1} + \nabla_x u_i\right)\left(x,\frac{x}{\epsilon}\right).$$

Equation (2) becomes a series in ϵ

$$\begin{aligned}
&-\epsilon^{-2}\left[\mathrm{div_y} A\nabla_y u_0\right]\left(x,\frac{x}{\epsilon}\right)\\
&-\epsilon^{-1}\left[\mathrm{div_y} A(\nabla_x u_0 + \nabla_y u_1) + \mathrm{div_x} A\nabla_y u_0\right]\left(x,\frac{x}{\epsilon}\right)\\
&-\epsilon^{0}\left[\mathrm{div_x} A(\nabla_x u_0 + \nabla_y u_1) + \mathrm{div_y} A(\nabla_x u_1 + \nabla_y u_2)\right]\left(x,\frac{x}{\epsilon}\right) \qquad (5)\\
&-\sum_{i=1}^{+\infty}\epsilon^i\left[\mathrm{div_x} A(\nabla_x u_i + \nabla_y u_{i+1}) + \mathrm{div_y} A(\nabla_x u_{i+1} + \nabla_y u_{i+2})\right]\left(x,\frac{x}{\epsilon}\right)\\
&= f(x).
\end{aligned}$$

Identifying each coefficient of (5) as an individual equation yields a cascade of equations (a series of the variable ϵ is zero for all values of ϵ if each coefficient is zero). It turns out that the three first equations are enough for our purpose. The ϵ^{-2} equation is

$$-\mathrm{div_y} A(y)\nabla_y u_0(x,y) = 0,$$

which is nothing else than an equation in the unit cell Y with periodic boundary condition. In this equation, y is the variable, and x plays the role of a parameter. It can be checked (see Lemma 1.3) that there exists a unique solution of this equation up to a constant (i.e., a function of x independent of y since x is just a parameter). This implies that u_0 is a function that does not depend on y, i.e., there exists a function $u(x)$ such that

$$u_0(x,y) \equiv u(x).$$

Since $\nabla_y u_0 = 0$, the ϵ^{-1} equation is

$$-\mathrm{div_y} A(y)\nabla_y u_1(x,y) = \mathrm{div_y} A(y)\nabla_x u(x), \qquad (6)$$

which is an equation for the unknown u_1 in the periodic unit cell Y. Again, it is a well-posed problem, which admits a unique solution up to a constant, as soon as the right hand side is known. Equation (6) allows one to compute u_1 in terms of u, and it is easily seen that $u_1(x,y)$ depends linearly on the first derivative $\nabla_x u(x)$.

Finally, the ϵ^0 equation is

$$\begin{aligned}
-\mathrm{div_y} A(y)\nabla_y u_2(x,y) \quad = \quad & \mathrm{div_y} A(y)\nabla_x u_1\\
& +\mathrm{div_x} A(y)\left(\nabla_y u_1 + \nabla_x u\right) + f(x),
\end{aligned} \qquad (7)$$

which is an equation for the unknown u_2 in the periodic unit cell Y. Equation (7) admits a solution if a compatibility condition is satisfied (the so-called *Fredholm alternative*; see Lemma 1.3). Indeed, integrating the left hand side of (7) over Y, and using the periodic boundary condition for u_2, we obtain

$$\int_Y \mathrm{div_y} A(y)\nabla_y u_2(x,y)dy = \int_{\partial Y}\left[A(y)\nabla_y u_2(x,y)\right]\cdot n\, ds = 0,$$

which implies that the right hand side of (7) must have zero average over Y, i.e.,

$$\int_Y \left[\mathrm{div_y} A(y)\nabla_x u_1 + \mathrm{div_x} A(y)\left(\nabla_y u_1 + \nabla_x u\right) + f(x)\right] dy = 0,$$

which simplifies to

$$-\mathrm{div_x}\left(\int_Y A(y)\left(\nabla_y u_1 + \nabla_x u\right) dy\right) = f(x) \ \text{ in } \Omega. \tag{8}$$

Since $u_1(x,y)$ depends linearly on $\nabla_x u(x)$, equation (8) is simply an equation for $u(x)$ involving only the second order derivatives of u.

The cell and homogenized problems.

The method of two-scale asymptotic expansions give rise to a couple of equations (6) (8) that have a mathematical, as well as physical, interpretation. In order to compute u_1 and to simplify (8), we introduce the so-called *cell problems*. We denote by $(e_i)_{1\le i\le N}$ the canonical basis of $\mathbb{R}^N$. For each unit vector e_i, consider the following conductivity problem in the periodic unit cell:

$$\begin{cases} -\mathrm{div_y} A(y)\left(e_i + \nabla_y w_i(y)\right) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic}, \end{cases} \tag{9}$$

where $w_i(y)$ is the local variation of potential or temperature created by an averaged (or macroscopic) gradient e_i. The existence of a solution w_i to equation (9) is guaranteed by the following result.

Lemma 1.3. *Let $f(y) \in L^2_\#(Y)$ be a periodic function. There exists a solution in $H^1_\#(Y)$ (unique up to an additive constant) of*

$$\begin{cases} -\mathrm{div} A(y)\nabla w(y) = f & \textit{in } Y \\ y \to w(y) & Y\textit{-periodic}, \end{cases} \tag{10}$$

if and only if $\int_Y f(y)dy = 0$ (this is called the Fredholm alternative).

By linearity, it is not difficult to compute $u_1(x,y)$, solution of (6), in terms of $u(x)$ and $w_i(y)$

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y). \tag{11}$$

In truth, $u_1(x,y)$ is merely defined up to the addition of a function $\tilde{u}_1(x)$ (depending only on x), but this does not matter since only its gradient $\nabla_y u_1(x,y)$ is used in the homogenized equation. Inserting this expression in equation (8), we obtain the homogenized equation for u that we supplement with a Dirichlet boundary condition on $\partial\Omega$,

$$\begin{cases} -\mathrm{div_x} A^* \nabla_x u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

The homogenized conductivity A^* is defined by its entries

$$A^*_{ij} = \int_Y \left[\left(A(y)\nabla_y w_i\right)\cdot e_j + A_{ij}(y)\right] dy,$$

or equivalently, after a simple integration by parts in Y,

$$A^*_{ij} = \int_Y A(y)\left(e_i + \nabla_y w_i\right)\cdot\left(e_j + \nabla w_j\right) dy. \tag{13}$$

The constant tensor A^* describes the effective or homogenized properties of the heterogeneous material $A\left(\frac{x}{\epsilon}\right)$. Note that A^* does not depend on the choice of domain Ω, source term f, or boundary condition on $\partial\Omega$.

Remark 1.4. *This method of two-scale asymptotic expansions is unfortunately not rigorous from a mathematical point of view. In other words, it yields heuristically the homogenized equation, but it does not yield a correct proof of the homogenization process. The reason is that the ansatz (3) is usually not correct after the two first terms. For example, it does not include possible boundary layers in the vicinity of $\partial\Omega$ (for details, see, e.g., [59]). Nevertheless, it is possible to rigorously justify the above homogenization process (see Subsection 1.3).*

A variational characterization of the homogenized coefficients.

The homogenized conductivity A^* is defined in terms of the solutions of the cell problems by equation (13). When the conductivity tensor $A(y)$ is symmetric, it is convenient to give another definition of A^* involving standard variational principles. From now on we assume that $A(y)$ is indeed symmetric. Therefore, by (13), A^* is symmetric too, and is completely determined by the knowledge of the quadratic form $A^*\xi\cdot\xi$ where ξ is any constant vector in $\mathbb{R}^N$. From definition (13) it is not difficult to check that

$$A^*\xi\cdot\xi=\int_Y A(y)\,(\xi+\nabla_y w_\xi)\cdot(\xi+\nabla_y w_\xi)\,dy, \tag{14}$$

where w_ξ is the solution of the following cell problem:

$$\left\{\begin{array}{ll} -\mathrm{div}_y A(y)\,(\xi+\nabla_y w_\xi(y))=0 & \text{in } Y,\\ y\to w_\xi(y) & Y\text{-periodic.}\end{array}\right. \tag{15}$$

It is well-known that equation (15) is the Euler-Lagrange equation of the following variational principle: Find $w(y)$ that minimizes

$$\int_Y A(y)\,(\xi+\nabla_y w)\cdot(\xi+\nabla_y w)\,dy$$

over all periodic functions w. In other words, $A^*\xi\cdot\xi$ is given by the minimization of the potential energy

$$A^*\xi\cdot\xi=\min_{w(y)\in H^1_\#(Y)}\int_Y A(y)\,(\xi+\nabla_y w)\cdot(\xi+\nabla_y w)\,dy, \tag{16}$$

where $H^1_\#(Y)$ is the Sobolev space of Y-periodic functions w with finite energy, namely,

$$\int_Y \left(w^2+|\nabla_y w|^2\right)dy<+\infty.$$

Remark that all the above equivalent definitions of A^* are not simple algebraic formulas, but rather they deliver the value of A^* at the price of a non-explicit computation of the solutions of the cell problems. However, in practice one is not always interested in the precise value of A^*, but rather in lower or upper estimates of its value. In this respect, the variational characterization (16) of A^* is useful since it provides an upper bound by choosing a specific test function $w(y)$. The simplest choice is to take $w(y)=0$, which yields the so-called *arithmetic mean upper bound*

$$A^*\xi\cdot\xi\le\left(\int_Y A(y)dy\right)\xi\cdot\xi. \tag{17}$$

A lower bound can also be obtained from (16) if the space of admissible fields in the minimization is enlarged. Indeed, remarking that the gradient $\nabla_y w(y)$ has zero-average over Y because of the periodicity of $w(y)$, this gradient can be replaced by any zero-average vector field

$$A^*\xi\cdot\xi\ge\min_{\substack{\zeta(y)\in L^2_\#(Y)^N\\ \int_Y\zeta(y)dy=0}}\int_Y A(y)\,(\xi+\zeta(y))\cdot(\xi+\zeta(y))\,dy, \tag{18}$$

where $L^2_\#(Y)$ is the space of square summable Y-periodic functions. The minimum in the right hand side of (18) is easy to compute: The optimal vector $\zeta_\xi(y)$ satisfies the following Euler-Lagrange equation

$$A(y)\left(\xi+\zeta_\xi(y)\right)=C,$$

where C is a constant (a Lagrange multiplier for the constraint $\int_Y \zeta_\xi(y)dy=0$). After some algebra, one can compute explicitly the optimal ζ_ξ, as well as the minimal value that delivers the so-called *harmonic mean lower bound*

$$A^*\xi\cdot\xi \geq \left(\int_Y A^{-1}(y)dy\right)^{-1}\xi\cdot\xi. \tag{19}$$

From a physical point of view, the harmonic mean in (19) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in series, while the arithmetic mean in (17) corresponds to an overall conductivity obtained by assuming that the values of the conductivity $A(y)$ are placed in parallel. These estimates hold true in great generality, but usually are not optimal and can be improved (see [7] in the case of two-phase composites). Actually, improving the harmonic and arithmetic mean bounds is one of the main problems of homogenization theory applied to the modeling of composite materials (see [68] for more details on this issue).

Evolution problem.

The previous analysis extends easily to evolution problems. Let us consider first a parabolic equation modeling, for example, a diffusion process. For a final time $T>0$, a source term $f(t,x)\in L^2((0,T)\times\Omega)$, and an initial data $a\in L^2(\Omega)$, the Cauchy problem is

$$\left\{\begin{array}{ll} c\left(\dfrac{x}{\epsilon}\right)\dfrac{\partial u_\epsilon}{\partial t}-\operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u_\epsilon\right)=f & \text{in } \Omega\times(0,T)\\ u_\epsilon=0 & \text{on } \partial\Omega\times(0,T)\\ u_\epsilon(0,x)=a(x) & \text{in } \Omega. \end{array}\right. \tag{20}$$

where A satisfies the coercivity assumption (1), and c is a bounded positive Y-periodic function

$$0<c^-\leq c(y)\leq c^+<+\infty \quad \forall\, y\in Y.$$

Remark 1.5. *It is a well-known result that there exists a unique solution u_ϵ of (20) in the space $L^2\left((0,T);H^1_0(\Omega)\right)\cap C([0,T];L^2(\Omega))$ which, furthermore, satisfies the energy estimate*

$$\|u_\epsilon\|_{C([0,T];L^2(\Omega))}+\|\nabla u_\epsilon\|_{L^2((0,T);L^2(\Omega))}\leq C, \tag{21}$$

where the constant C does not depend on ϵ.

One can perform the same two-scale asymptotic expansion on (20). The ansatz is

$$u_\epsilon(t,x)=\sum_{i=0}^{+\infty}\epsilon^i u_i\left(t,x,\frac{x}{\epsilon}\right),$$

where each term $u_i(t,x,y)$ is a function of time t and both space variables x and y. It is clear that the time derivative yield no contribution in the two first equations of the cascade of equations (5). However it gives a contribution for the third one. In other words the cell problem is the same as in the steady case, but the homogenized equation is changed. The reader will check easily that

$$u_0(t,x,y)\equiv u(t,x),\quad u_1(t,x,y)=\sum_{i=1}^{N}\frac{\partial u}{\partial x_i}(t,x)w_i(y),$$

and the homogenized equation is

$$\begin{cases} c^* \dfrac{\partial u}{\partial t} - \operatorname{div}\left(A^* \nabla u\right) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times]0,T[\\ u(0) = a & \text{in } \Omega, \end{cases} \tag{22}$$

where the homogenized tensor is still given by (13) and

$$c^* = \int_Y c(y)\, dy. \tag{23}$$

We now consider an hyperbolic equation modeling, for example, the propagation of waves. For a final time $T > 0$, a source term $f(t,x) \in L^2((0,T) \times \Omega)$, a pair of initial data $a \in H_0^1(\Omega)$ and $b \in L^2(\Omega)$, the Cauchy problem is

$$\begin{cases} c\left(\dfrac{x}{\epsilon}\right) \dfrac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A\left(\dfrac{x}{\epsilon}\right) \nabla u_\epsilon\right) = f & \text{in } \Omega \times (0,T) \\ u_\epsilon = 0 & \text{on } \partial\Omega \times]0,T[\\ u_\epsilon(0,x) = a(x) & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial t}(0,x) = b(x) & \text{in } \Omega. \end{cases} \tag{24}$$

where A satisfies the coercivity assumption (1), and c is a bounded positive Y-periodic function.

Remark 1.6. *It is a well-known result that there exists a unique solution u_ϵ of (24) in the space $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ which, furthermore, satisfies the energy estimate*

$$\left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{C([0,T];L^2(\Omega))} + \left\| \nabla u_\epsilon \right\|_{C([0,T];L^2(\Omega))} \leq C, \tag{25}$$

where the constant C does not depend on ϵ.

Again one can perform a two-scale asymptotic expansion on (24) with the ansatz

$$u_\epsilon(t,x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(t, x, \frac{x}{\epsilon}\right),$$

where each term $u_i(t,x,y)$ is a function of time t and both space variables x and y. As in the parabolic case, the time derivative yield no contribution in the two first equations of the cascade of equations (5). However it gives a contribution for the third one. In other words the cell problem is the same as in the steady case, but the homogenized equation is changed. The reader will check easily that

$$u_0(t,x,y) \equiv u(t,x), \quad u_1(t,x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(t,x) w_i(y),$$

and the homogenized equation is

$$\begin{cases} c^* \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(A^* \nabla u\right) = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times]0,T[\\ u(0) = a & \text{in } \Omega \\ \frac{\partial u}{\partial t}(0) = b & \text{in } \Omega, \end{cases} \tag{26}$$

where the homogenized tensor is still given by (13) and c^* is given by (23).

1.3. **Mathematical justification of homogenization**

This subsection is devoted to a brief introduction to the mathematical methods that justify the previous heuristic analysis of homogenization. We consider only two methods out of many more available.

The oscillating test function method.

The oscillating test function method is a very elegant and efficient method for rigorously homogenizing partial differential equations which was devised by Tartar [84], [73] (sometimes it is also called the *energy method*). This method is very general and does not require any geometric assumptions on the behavior of the p.d.e. coefficients: neither periodicity nor statistical properties like stationarity or ergodicity. However, for the sake of clarity we present the oscillating test function method only in the periodic setting. Let us also mention that this method works for many models, and not only diffusion equations.

Recall that our model problem of diffusion reads

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where the source term $f(x)$ belongs to $L^2(\Omega)$. By application of Lax-Milgram lemma, equation (27) admits a unique solution u_ϵ in the space $H_0^1(\Omega)$ which satisfies the a priori estimate

$$\|u_\epsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{28}$$

where C is a positive constant which does not depend on ϵ. Estimate (28) is obtained by multiplying equation (27) by u_ϵ, integrating by parts, and using Poincaré inequality. It implies that the sequence u_ϵ, indexed by a sequence of periods ϵ which goes to 0, is bounded in the Sobolev space $H_0^1(\Omega)$. Therefore, up to a subsequence, it converges weakly to a limit u in $H_0^1(\Omega)$. The goal is to find the homogenized equation satisfied by u.

Theorem 1.7. *The sequence $u_\epsilon(x)$ of solutions of (27) converges weakly in $H_0^1(\Omega)$ to a limit $u(x)$ which is the unique solution of the homogenized problem*

$$\begin{cases} -\mathrm{div}\left(A^*\nabla u(x)\right) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

where the homogenized diffusion tensor, A^*, is defined by (13).

In order to shed some light on the principles of the energy method, let us begin with a naive attempt to prove Theorem 1.7 by passing to the limit in the variational formulation. The original problem (27) admits the following variational formulation

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \tag{30}$$

for any test function $\varphi \in H_0^1(\Omega)$. By estimate (28), we can extract a subsequence, still denoted by ϵ, such that u_ϵ converges weakly in $H_0^1(\Omega)$ to a limit u. Unfortunately, the left hand side of (30) involves the product of two weakly converging sequences in $L^2(\Omega)$, $A\left(\frac{x}{\epsilon}\right)$ and $\nabla u_\epsilon(x)$, and it is not true that it converges to the product of the weak limits. Therefore, we cannot pass to the limit in (30) without any further argument.

The main idea of the energy method is to replace in (30) the fixed test function φ by a weakly converging sequence φ_ϵ (the so-called *oscillating test function*), chosen in such a way that the left hand side of (30) miraculously passes to the limit. This phenomenon is an example of the *compensated compactness* theory, developed by Murat and Tartar [72] [85], which under additional conditions permits to pass to the limit in some products of weak convergences.

Proof of Theorem 1.7. The key idea is the choice of an oscillating test function $\varphi_\epsilon(x)$. Let $\varphi(x) \in \mathcal{D}(\Omega)$ be a smooth function with compact support in Ω. Copying the two first terms of the asymptotic expansion of u_ϵ,

the oscillating test function φ_ϵ is defined by

$$\varphi_\epsilon(x) = \varphi(x) + \epsilon \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) w_i^* \left(\frac{x}{\epsilon}\right), \tag{31}$$

where $w_i^*(y)$ are not the solutions of the cell problems, defined in (9), but that of the *dual cell problems*

$$\begin{cases} -\mathrm{div_y}\left(A^t(y)\left(e_i + \nabla_y w_i^*(y)\right)\right) = 0 & \text{in } Y \\ y \to w_i^*(y) & Y\text{-periodic.} \end{cases} \tag{32}$$

The difference between (9) and (32) is that the matrix $A(y)$ has been replaced by its *transpose* $A^t(y)$. By periodicity in y of w_i^*, it is easily seen that $\epsilon w_i^*\left(\frac{x}{\epsilon}\right)$ is a bounded sequence in $H^1(\Omega)$ which converges weakly to 0 (see Lemma 1.8 below if necessary).

The next step is to insert this oscillating test function φ_ϵ in the variational formulation (30)

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi_\epsilon(x) dx = \int_\Omega f(x) \varphi_\epsilon(x) dx. \tag{33}$$

To take advantage of our knowledge of equation (32), we develop and integrate by parts in (33). Remarking that

$$\nabla \varphi_\epsilon = \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right) + \epsilon \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i}(x) w_i^*\left(\frac{x}{\epsilon}\right),$$

yields

$$\begin{aligned} \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi_\epsilon(x) dx = & \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right) dx \\ & + \epsilon \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i}(x) w_i^*\left(\frac{x}{\epsilon}\right). \end{aligned} \tag{34}$$

The last term in (34) is easily seen to be bounded by a constant time ϵ, and thus cancels out in the limit. In the first term of (34), an integration by parts gives

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right) dx = \tag{35}$$

$$- \int_\Omega u_\epsilon(x) \mathrm{div}\left(A^t\left(\frac{x}{\epsilon}\right) \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right)\right) dx.$$

Let us compute the divergence in the right hand side of (35) which is actually a function of x and $y = x/\epsilon$

$$d_\epsilon(x) = \mathrm{div}\left(A^t\left(\frac{x}{\epsilon}\right) \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*\left(\frac{x}{\epsilon}\right)\right)\right)$$

$$= \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i}(x) \cdot A^t(y)\left(e_i + \nabla_y w_i^*(y)\right) + \frac{1}{\epsilon} \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) \mathrm{div_y}\left(A^t(y)\left(e_i + \nabla_y w_i^*(y)\right)\right). \tag{36}$$

The last term of order ϵ^{-1} in the right hand side of (36) is simply zero by definition (32) of w_i^*. Therefore, $d_\epsilon(x)$ is bounded in $L^2(\Omega)$, and, since it is a periodically oscillating function, it converges weakly to its average by virtue of Lemma 1.8.

The main point of this simplification is that we are now able to pass to the limit in the right hand side of (35). Recall that u_ϵ is bounded in $H_0^1(\Omega)$: by application of Rellich theorem, there exists a subsequence (still indexed by ϵ for simplicity) and a limit $u \in H_0^1(\Omega)$ such that u_ϵ converges *strongly* to u in $L^2(\Omega)$. The right hand side of (35) is the product of a weak convergence (d_ϵ) and a strong one (u_ϵ), and thus its limit is the product of the two limits. In other words,

$$\lim_{\epsilon \to 0} \int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi_\epsilon(x) dx =$$

$$- \int_\Omega u(x) \mathrm{div}_{\mathrm{x}} \left(\int_Y A^t(y) \sum_{i=1}^N \frac{\partial \varphi}{\partial x_i}(x) \left(e_i + \nabla_y w_i^*(y)\right) dy \right) dx. \tag{37}$$

By definition (13) of A^*, it is easily seen that the right hand side of (37) is nothing else than

$$- \int_\Omega u(x) \mathrm{div}_{\mathrm{x}} \left(A^{*t} \nabla \varphi(x) \right) dx.$$

Finally, a last integration by parts yields the limit variational formulation of (33)

$$\int_\Omega A^* \nabla u(x) \cdot \nabla \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx. \tag{38}$$

By density of smooth functions in $H_0^1(\Omega)$, (38) is valid for any test function $\varphi \in H_0^1(\Omega)$. Since A^* satisfies the same coercivity condition as A, Lax-Milgram lemma shows that (38) admits a unique solution in $H_0^1(\Omega)$. This last result proves that any subsequence of u_ϵ converges to the same limit u. Therefore, the entire sequence u_ϵ, and not only a subsequence, converges to the homogenized solution u. This concludes the proof of Theorem 1.7. □

In the course of the proof of Theorem 1.7, the following lemma on periodically oscillating functions was used several times. Its proof is elementary (see if necessary [6], [29]), at least for smooth functions, by using a covering of the domain Ω in small cubes of size ϵ and the notion of Riemann integration (approximation of integrals by discrete sums).

Lemma 1.8. *Let $w(x,y)$ be a continuous function in x, square integrable and Y-periodic in y, i.e. $w(x,y) \in L^2_\#(Y; C(\Omega))$. Then, the sequence $w\left(x, \frac{x}{\epsilon}\right)$ converges weakly in $L^2(\Omega)$ to $\int_Y w(x,y)dy$.*

Two-Scale Convergence.

Unlike the oscillating test function method, the two-scale convergence method is devoted only to periodic homogenization problems. It is therefore a less general method, but it is rather more efficient and simple in this context. Two-scale convergence has been introduced by Nguetseng [74] and Allaire [6] to which we refer for most proofs.

We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ which are periodic of period Y, and by $C_\#(Y)$ the Banach space of continuous and Y-periodic functions. Eventually, $\mathcal{D}(\Omega; C^\infty_\#(Y))$ denotes the space of infinitely smooth and compactly supported functions in Ω with values in the space $C^\infty_\#(Y)$.

Definition 1.9. *A sequence of functions u_ϵ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega \times Y)$ if, for any function $\varphi(x,y)$ in $\mathcal{D}(\Omega; C^\infty_\#(Y))$, it satisfies*

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y u_0(x,y) \varphi(x,y) dx dy.$$

In the above definition we use the fact that Y is the unit cube. Otherwise, the right hand side, which is an average over the periodicity cell Y, should be divied by its measure $|Y|$.

Theorem 1.10. *From each bounded sequence u_ϵ in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.*

Here are some examples of two-scale convergence.

(1) Any sequence u_ϵ which converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.
(2) For any smooth function $u_0(x,y)$, being Y-periodic in y, the associated sequence $u_\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right)$ two-scale converges to $u_0(x,y)$.
(3) For the same smooth and Y-periodic function $u_0(x,y)$ the sequence defined by $v_\epsilon(x) = u_0(x, \frac{x}{\epsilon^2})$ has the same two-scale limit and weak-L^2 limit, namely $\int_Y u_0(x,y)dy$ (this is a consequence of the difference of orders in the speed of oscillations for v_ϵ and the test functions $\varphi\left(x, \frac{x}{\epsilon}\right)$). Clearly the two-scale limit captures only the oscillations which are in resonance with those of the test functions $\varphi\left(x, \frac{x}{\epsilon}\right)$.
(4) Any sequence u_ϵ which admits an asymptotic expansion of the type $u_\epsilon(x) = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \cdots$, where the functions $u_i(x,y)$ are smooth and Y-periodic in y, two-scale converges to the first term of the expansion, namely $u_0(x,y)$.

The next theorem shows that more information is contained in a two-scale limit than in a weak-L^2 limit ; some of the oscillations of a sequence are contained in its two-scale limit. When all of them are captured by the two-scale limit (condition (40) below), one can even obtain a strong convergence (a corrector result in the vocabulary of homogenization).

Theorem 1.11. *Let u_ϵ be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$.*

(1) *Then, u_ϵ converges weakly in $L^2(\Omega)$ to $u(x) = \int_Y u_0(x,y)dy$, and we have*

$$\lim_{\epsilon \to 0} \|u_\epsilon\|^2_{L^2(\Omega)} \geq \|u_0\|^2_{L^2(\Omega \times Y)} \geq \|u\|^2_{L^2(\Omega)}. \tag{39}$$

(2) *Assume further that $u_0(x,y)$ is smooth and that*

$$\lim_{\epsilon \to 0} \|u_\epsilon\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega \times Y)}. \tag{40}$$

Then, we have

$$\|u_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\|^2_{L^2(\Omega)} \to 0. \tag{41}$$

Proof. By taking test functions depending only on x in Definition 1.9, the weak convergence in $L^2(\Omega)$ of the sequence u_ϵ is established. Then, developing the inequality

$$\int_\Omega |u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx \geq 0,$$

yields easily formula (39). Furthermore, under assumption (40), it is easily obtained that

$$\lim_{\epsilon \to 0} \int_\Omega |u_\epsilon(x) - \varphi\left(x, \frac{x}{\epsilon}\right)|^2 dx = \int_\Omega \int_Y |u_0(x,y) - \varphi(x,y)|^2 dxdy.$$

If u_0 is smooth enough to be a test function φ, it yields (41). □

Theorem 1.12. *Let u_ϵ be a bounded sequence in $H^1(\Omega)$. Then, up to a subsequence, u_ϵ two-scale converges to a limit $u(x) \in H^1(\Omega)$, and ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$, where the function $u_1(x,y)$ belongs to $L^2(\Omega; H^1_\#(Y)/\mathbb{R})$.*

Proof. Since u_ϵ (resp. ∇u_ϵ) is bounded in $L^2(\Omega)$ (resp. $L^2(\Omega)^N$), up to a subsequence, it two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$ (resp. $\xi_0(x,y) \in L^2(\Omega \times Y)^N$). Thus for any $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$, we have

$$\lim_{\epsilon \to 0} \int_\Omega \nabla u_\epsilon(x) \cdot \psi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y \xi_0(x,y) \cdot \psi(x,y) dxdy. \tag{42}$$

Integrating by parts the left hand side of (42) gives

$$\epsilon \int_\Omega \nabla u_\epsilon(x) \cdot \psi\left(x, \frac{x}{\epsilon}\right) dx = -\int_\Omega u_\epsilon(x) \left(\mathrm{div_y}\psi\left(x, \frac{x}{\epsilon}\right) + \epsilon \mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right)\right) dx. \tag{43}$$

Passing to the limit yields

$$0 = -\int_\Omega \int_Y u_0(x,y) \mathrm{div_y}\psi(x,y) dxdy. \tag{44}$$

This implies that $u_0(x,y)$ does not depend on y. Thus there exists $u(x) \in L^2(\Omega)$, such that $u_0 = u$. Next, in (42) we choose a function ψ such that $\mathrm{div_y}\psi(x,y) = 0$. Integrating by parts we obtain

$$\begin{aligned} \lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right) dx &= -\int_\Omega \int_Y \xi_0(x,y) \cdot \psi(x,y) dxdy \\ &= \int_\Omega \int_Y u(x) \mathrm{div_x}\psi(x,y) dxdy. \end{aligned} \tag{45}$$

If ψ does not depend on y, (45) proves that $u(x)$ belongs to $H^1(\Omega)$. Furthermore, we deduce from (45) that

$$\int_\Omega \int_Y (\xi_0(x,y) - \nabla u(x)) \cdot \psi(x,y) dxdy = 0 \tag{46}$$

for any function $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$ with $\mathrm{div_y}\psi(x,y) = 0$. Recall that the orthogonal of divergence-free functions are exactly the gradients (this well-known result can be very easily proved in the present context by means of Fourier analysis in Y). Thus, there exists a unique function $u_1(x,y)$ in $L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that

$$\xi_0(x,y) = \nabla u(x) + \nabla_y u_1(x,y). \quad \square \tag{47}$$

Application to the model problem (27).

We now describe how the "two-scale convergence method" can justify the homogenization of (27). In a **first step**, we deduce from the a priori estimate (27) the precise form of the two-scale limit of the sequence u_ϵ. By application of Theorem 1.12, there exist two functions, $u(x) \in H^1_0(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$, such that, up to a subsequence, u_ϵ two-scale converges to $u(x)$, and ∇u_ϵ two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$. In view of these limits, u_ϵ is expected to behave as $u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right)$.

Thus, in a **second step**, we multiply equation (27) by a test function similar to the limit of u_ϵ, namely $\varphi(x) + \epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)$, where $\varphi(x) \in \mathcal{D}(\Omega)$ and $\varphi_1(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. This yields

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon \cdot \left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x, \frac{x}{\epsilon}\right) + \epsilon\nabla_x\varphi_1\left(x, \frac{x}{\epsilon}\right)\right) dx = \int_\Omega f(x) \left(\varphi(x) + \epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)\right) dx. \tag{48}$$

Regarding $A^t\left(\frac{x}{\epsilon}\right)\left(\nabla\varphi(x) + \nabla_y\varphi_1\left(x, \frac{x}{\epsilon}\right)\right)$ as a test function for the two-scale convergence (see Definition 1.9), we pass to the two-scale limit in (48) for the sequence ∇u_ϵ. Although this test function is not necessarily very smooth, as required by Definition 1.9, it belongs at least to $C\left(\bar{\Omega}; L^2_\#(Y)\right)$ which can be shown to be enough

for the two-scale convergence Theorem 1.10 to hold (see [6] for details). Thus, the two-scale limit of equation (48) is

$$\int_\Omega \int_Y A(y) \left(\nabla u(x) + \nabla_y u_1(x,y)\right) \cdot \left(\nabla \varphi(x) + \nabla_y \varphi_1(x,y)\right) dxdy = \int_\Omega f(x)\varphi(x)dx. \tag{49}$$

In a **third step**, we read off a variational formulation for (u, u_1) in (49). Remark that (49) holds true for any (φ, φ_1) in the Hilbert space $H_0^1(\Omega) \times L^2\left(\Omega; H^1_\#(Y)/\mathbb{R}\right)$ by density of smooth functions in this space. Endowing it with the norm $\sqrt{(}\|\nabla u(x)\|^2_{L^2(\Omega)} + \|\nabla_y u_1(x,y)\|^2_{L^2(\Omega\times Y)})$, the assumptions of the Lax-Milgram lemma are easily checked for the variational formulation (49). The main point is the coercivity of the bilinear form defined by the left hand side of (49): the coercivity of A yields

$$\int_\Omega \int_Y A(y) \left(\nabla \varphi(x) + \nabla_y \varphi_1(x,y)\right) \cdot \left(\nabla \varphi(x) + \nabla_y \varphi_1(x,y)\right) dxdy \geq$$

$$\alpha \int_\Omega \int_Y |\nabla \varphi(x) + \nabla_y \varphi_1(x,y)|^2 dxdy = \alpha \int_\Omega |\nabla \varphi(x)|^2 dx + \alpha \int_\Omega \int_Y |\nabla_y \varphi_1(x,y)|^2 dxdy.$$

By application of the Lax-Milgram lemma, we conclude that there exists a unique solution (u, u_1) of the variational formulation (49) in $H_0^1(\Omega) \times L^2\left(\Omega; H^1_\#(Y)/\mathbb{R}\right)$. Consequently, the entire sequences u_ϵ and ∇u_ϵ converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$. An easy integration by parts shows that (49) is a variational formulation associated to the following system of equations, the so-called "two-scale homogenized problem",

$$\left\{ \begin{array}{ll} -\mathrm{div_y}\left(A(y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right)\right) = 0 & \text{in } \Omega \times Y \\ -\mathrm{div_x}\left(\int_Y A(y)\left(\nabla u(x) + \nabla_y u_1(x,y)\right) dy\right) = f(x) & \text{in } \Omega \\ y \to u_1(x,y) & Y\text{-periodic} \\ u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{50}$$

At this point, the homogenization process could be considered as achieved since the entire sequence of solutions u_ϵ converges to the solution of a well-posed limit problem, namely the two-scale homogenized problem (50). However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one does not want to solve the small scale structure). In other words, we want to extract and decouple the usual homogenized and local (or cell) equations from the two-scale homogenized problem.

Thus, in a **fourth (and optional) step**, the y variable and the u_1 unknown are eliminated from (50). It is an easy exercise of algebra to prove that u_1 can be computed in terms of the gradient of u through the relationship

$$u_1(x,y) = \sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) w_i(y), \tag{51}$$

where $w_i(y)$ are defined as the solutions of the cell problems (9). Then, plugging formula (51) in (50) yields the usual homogenized problem (12) with the homogenized diffusion tensor defined by (13).

Due to the simple form of our model problem the two equations of (50) can be decoupled in a microscopic and a macroscopic equation, (9) and (12) respectively, but we emphasize that it is not always possible, and sometimes it leads to very complicate forms of the homogenized equation, including integro-differential operators. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, on the contrary of the two-scale homogenized system, which is in most cases of the same type as the original problem, but with a double number of variables (x and y) and unknowns (u and u_1). The supplementary microscopic variable and unknown play the role of "hidden" variables in the vocabulary of mechanics. Although their presence doubles the size of the limit problem, it greatly simplifies its structure (which could be useful for numerical purposes too), while eliminating them introduces "strange" effects (like memory or non-local effects) in the usual homogenized problem.

Remark 1.13. *It is often very useful to obtain so-called "corrector" results which permit to obtain strong (or pointwise) convergences instead of just weak ones by adding some extra information stemming from the local equations. Typically, in the above example we simply proved that the sequence u_ϵ converges weakly to the homogenized solution u in $H_0^1(\Omega)$. Introducing the local solution u_1, this weak convergence can be improved as follows*

$$\left(u_\epsilon(x) - u(x) - \epsilon u_1\left(x, \frac{x}{\epsilon}\right)\right) \to 0 \text{ in } H_0^1(\Omega) \text{ strongly.} \tag{52}$$

This type of result is easily obtained with the two-scale convergence method. This rigorously justifies the two first term in the usual asymptotic expansion of the sequence u_ϵ. Indeed we can develop

$$\int_\Omega A\left(\frac{x}{\epsilon}\right)\left(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right)\right)\cdot\left(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right)\right) dx.$$

After some algebra and passing to the two-scale limit, we deduce that $(\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\epsilon}\right))$ goes to zero in $L^2(\Omega)^N$.

Remark 1.14. *There is a variant of the method of two-scale convergence which is called the periodic unfolding method [27], [28]. The main idea of the periodic unfolding method is to introduce a so-called unfolding operator E_ϵ from $L^2(\Omega)$ into $L^2\left(\Omega; L^2_\#(Y)\right)$ defined, for any $u(x) \in L^2(\Omega)$, by*

$$E_\epsilon(u)(x,y) = \sum_{i=1}^{N(\epsilon)} u(x_i^\epsilon + \epsilon y)\chi_{Y_i^\epsilon}(x),$$

where $(Y_i^\epsilon)_{1\le i\le N(\epsilon)}$ is a tiling of Ω with cubic cells Y_i^ϵ, which are equal to ϵY translated to its origin x_i^ϵ, and $\chi_{Y_i^\epsilon}(x)$ is the i-th cell characteristic function. The unfolding operator is easily seen to be linear and bounded. The main result of [27], [28] is then to prove that a sequence $u_\epsilon(x)$ of $L^2(\Omega)$ two-scale converges to a limit $u_0(x,y)$ if and only if the sequence $E_\epsilon(u_\epsilon)(x,y)$ weakly converges to $u_0(x,y)$ in $L^2\left(\Omega; L^2_\#(Y)\right)$. Note that some of the ideas of the periodic unfolding method were anticipated in [12], [16], [57].

2. General theory of homogenization

2.1. Introduction.

The first section was devoted to a brief presentation of homogenization in a periodic setting. This second section focus on the general setting of homogenization when no geometric assumptions are available (like periodicity, or ergodicity in a probabilistic framework). It turns out that homogenization can be applied to any kind of disordered media, and is definitely not restricted to the periodic case (although the nice "explicit" formulae of the periodic setting for the homogenized conductivity tensor have no analogue). We introduce the notion of G- or H-convergence which is due to DeGiorgi and Spagnolo [39], [82], [83], and has been further generalized by Murat and Tartar [73], [84] (see also the textbooks [75], [87], [89]). It allows to consider any possible geometrical situation without any specific assumptions like periodicity or randomness. The G- or H-convergence turns out to be the adequate notion of convergence for effective properties that will be the key tool in the study of optimal shape design problems.

Finally, let us mention that there is also a stochastic theory of homogenization (see [55], [36], [76]) and a variational theory of homogenization (the Γ-convergence of De Giorgi, [37], [38], see also the books [21], [35]) that will not be described below.

2.2. Definition of G-, or H-convergence.

The G-convergence is a notion of convergence associated to sequences of symmetric operators (typically, these operators are applications giving the solution of a partial differential equation in terms of the right hand side).

The G means Green since this type of convergence corresponds roughly to the convergence of the associated Green functions. The H-convergence is a generalization of the G-convergence to the case of non-symmetric operators (it provides also an easier mathematical framework, but we shall not dwell on that). The H stands for Homogenization since it is an important tool of that theory. For the sake of simplicity, we restrict ourselves to the case of symmetric operators (i.e. diffusion equations with symmetric coefficients). In such a case, G- and H-convergence coincide. Therefore in the sequel, we use only the notation G-convergence.

The main result of the G-convergence is a compactness theorem in the homogenization theory which states that, for any bounded and uniformly coercive sequence of coefficients of a symmetric second order elliptic equation, there exist a subsequence and a *G-limit* (i.e. homogenized coefficients) such that, for any source term, the corresponding subsequence of solutions converges to the solution of the homogenized equation. In practical terms, it means that the mechanical properties of an heterogeneous medium (like its conductivity, or elastic moduli) can be well approximated by the properties of a homogeneous or homogenized medium if the size of the heterogeneities are small compared to the overall size of the medium.

The G-convergence can be seen as a mathematically rigorous version of the so-called *representative volume element* method [26], [49] for computing effective or averaged parameters of heterogeneous media.

We introduce the notion of G-convergence for the specific case of a diffusion equation with a Dirichlet boundary condition, but all the results hold for a larger class of second order elliptic operators and boundary conditions. Let Ω be a bounded open set in $\mathbb{R}^N$, and let α, β be two positive constants such that $0 < \alpha \leq \beta$. We introduce the set $\mathcal{M}(\alpha, \beta, \Omega)$ of all possible symmetric matrices defined on Ω with uniform coercivity constant α and $L^\infty(\Omega)$-bound β. In other words, $A \in \mathcal{M}(\alpha, \beta, \Omega)$ if $A(x)$ satisfies

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2.$$

We consider a sequence $A_\epsilon(x)$ of conductivity tensors in $\mathcal{M}(\alpha, \beta, \Omega)$, indexed by a sequence of positive numbers ϵ going to 0. Here, ϵ is not associated to any specific lengthscale or statistical property of the elastic medium. In other words, no special assumptions (like periodicity or stationarity) are placed on the sequence A_ϵ.

For a given source term $f(x) \in L^2(\Omega)$, there exists a unique solution u_ϵ in the Sobolev space $H_0^1(\Omega)$ of the following diffusion equation

$$\begin{cases} -\mathrm{div}\,(A_\epsilon(x)\nabla u_\epsilon) = f(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{53}$$

The G-convergence of the sequence A_ϵ is defined below as the convergence of the corresponding solutions u_ϵ.

Definition 2.1. *The sequence of tensors $A_\epsilon(x)$ is said to G-converge to a limit $A^*(x)$, as ϵ goes to 0, if, for any $f \in L^2(\Omega)$ in (53), the sequence of solutions u_ϵ converges weakly in $H_0^1(\Omega)$ to a limit u which is the unique solution of the homogenized equation associated to A^*:*

$$\begin{cases} -\mathrm{div}\,(A^*(x)\nabla u) = f(x) & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{54}$$

Remark that, by definition, the homogenized tensor A^* is independent of the source term f. We shall see that it is also independent of the boundary condition and of the domain.

This definition makes sense because of the compactness of the set $\mathcal{M}(\alpha, \beta, \Omega)$ with respect to the G-convergence, as stated in the following theorem.

Theorem 2.2. *For any sequence A_ϵ in $\mathcal{M}(\alpha, \beta, \Omega)$, there exist a subsequence (still denoted by ϵ) and a homogenized limit A^*, belonging to $\mathcal{M}(\alpha, \beta, \Omega)$, such that A_ϵ G-converges to A^*.*

The G-convergence of a general sequence A_ϵ is always stated *up to a subsequence* since A_ϵ can be the union of two sequences converging to two different limits. The G-convergence of A_ϵ is not equivalent to any other "classical" convergence. For example, if A_ϵ converges strongly in $L^\infty(\Omega)$ to a limit A (i.e. the convergence is pointwise), then its G-limit A^* coincides with A. But the converse is not true ! On the same token, the G-convergence has nothing to do with the usual weak convergence. Indeed, the G-limit A^* of a sequence A_ϵ is usually different of its weak-* $L^\infty(\Omega)$-limit. For example, a straightforward computation in one space dimension ($N = 1$) shows that the G-limit of a sequence A_ϵ is given as the inverse of the weak-* $L^\infty(\Omega)$-limit of A_ϵ^{-1} (the so-called harmonic limit). However, this last result holds true only in 1-D, and no such explicit formula is available in higher dimensions.

The G-convergence enjoys a few useful properties as enumerated in the following proposition.

Proposition 2.3. *Properties of G-convergence.*

(1) *If a sequence A_ϵ G-converges, its G-limit is unique.*
(2) *Let A_ϵ and B_ϵ be two sequences which G-converge to A^* and B^* respectively. Let $\omega \subset \Omega$ be a subset strictly included in Ω such that $A_\epsilon = B_\epsilon$ in ω. Then $A^* = B^*$ in ω (this property is called the locality of G-convergence).*
(3) *The G-limit of a sequence A_ϵ is independent of the source term f and of the boundary condition on $\partial\Omega$.*
(4) *Let A_ϵ be a sequence which G-converges to A^*. Then, the associated density of energy $A_\epsilon \nabla u_\epsilon \cdot \nabla u_\epsilon$ also converges to the homogenized density of energy $A^* \nabla u \cdot \nabla u$ in the sense of distributions in Ω.*
(5) *If a sequence A_ϵ G-converges to a limit A^*, then the sequence of fluxes $A_\epsilon \nabla u_\epsilon$ converges weakly in $L^2(\Omega)^N$ to the homogenized flux $A^* \nabla u$.*

These properties of the G-convergence implies that the homogenized medium A^* approximates the heterogeneous medium A_ϵ in many different ways. First of all, by definition of G-convergence, the fields u, u_ϵ and their gradients are closed (this is the sense of the convergence of u_ϵ to u in the Sobolev space $H_0^1(\Omega)$). Then, by application of the above proposition, the fluxes and the energy densities are also closed.

Remark also that, by locality of the G-convergence, the homogenized tensor is defined at each point of the domain Ω independently of what may happen in other regions of Ω.

Of course, a particular example of G-convergent sequences A_ϵ is given by periodic media of the type $A\left(\frac{x}{\epsilon}\right)$ as in the previous section.

3. Homogenization of Stokes equations

3.1. **Derivation of Darcy's Law**

This subsection is devoted to the derivation of Darcy's law for an incompressible viscous fluid flowing in a porous medium. Starting from the steady Stokes equations in a periodic porous medium, with a no-slip (Dirichlet) boundary condition on the solid pores, Darcy's law is rigorously obtained by periodic homogenization using the two-scale convergence method. The assumption on the periodicity of the porous medium is by no means realistic, but it allows to cast this problem in a very simple framework and to prove theorems without too much effort. We denote by ϵ the ratio of the period to the overall size of the porous medium: it is the small parameter of our asymptotic analysis since the pore size is usually much smaller than the characteristic length of the reservoir. The porous medium is contained in a domain Ω, and its fluid part is denoted by Ω_ϵ. From a mathematical point of view, Ω_ϵ is a periodically perforated domain, i.e., it has many small holes of size ϵ which represent solid obstacles that the fluid cannot penetrate.

The motion of the fluid in Ω_ϵ is governed by the steady Stokes equations, complemented with a Dirichlet boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, and f the density of forces acting on the fluid (u_ϵ and f are vector-valued functions, while p_ϵ is scalar). The fluid viscosity is a positive

constant μ that we scale by a factor ϵ^2 (where ϵ is the period). The Stokes equations are

$$\left\{ \begin{array}{ll} \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{array} \right. \tag{55}$$

The above scaling for the viscosity is such that the velocity u_ϵ has a non-trivial limit as ϵ goes to zero. Physically speaking, the very small viscosity, of order ϵ^2, balances exactly the friction of the fluid on the solid pore boundaries due to the no-slip boundary condition. Remark that this scaling is perfectly legitimate since by linearity of the equations one can always replace u_ϵ by $\epsilon^2 u_\epsilon$. To obtain an existence and uniqueness result for (55), the forcing term is assumed to have the usual regularity: $f(x) \in L^2(\Omega)^N$. Then, as is well-known (see e.g. [88]), the Stokes equations (55) admits a unique solution

$$u_\epsilon \in H_0^1(\Omega_\epsilon)^N, \ p_\epsilon \in L^2(\Omega_\epsilon)/\mathbb{R}, \tag{56}$$

the pressure being uniquely defined up to an additive constant. The homogenization problem for (55) is to find the effective equation satisfied by the limits of u_ϵ, p_ϵ. From the point of view of homogenization, the mathematical originality of system (55) is that the periodic oscillations are not in the coefficients of the operator but in the geometry of the porous medium Ω_ϵ.

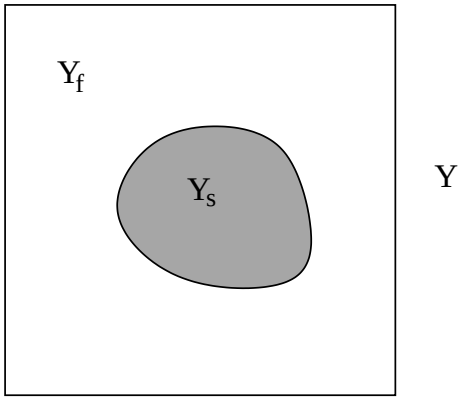

FIGURE 2. Unit cell of a porous medium.

Before stating the main result, let us describe more precisely the assumptions on the porous domain Ω_ϵ. As usual in periodic homogenization, a periodic structure is defined by a domain Ω and an associated microstructure, or periodic cell $Y = (0,1)^N$, which is made of two complementary parts : the fluid part Y_f and the solid part Y_b, satisfying $Y_f \cup Y_b = Y$ and $Y_f \cap Y_b = \emptyset$ (see Figure 2). We assume that Ω is a smooth, bounded, connected set in $\mathbb{R}^N$, and that Y_f is a smooth and connected open subset of Y, identified with the unit torus (i.e. Y_f, repeated by Y-periodicity in $\mathbb{R}^N$, is a smooth and connected open set of $\mathbb{R}^N$). The domain Ω is covered by a regular mesh of size ϵ: each cell Y_i^ϵ is of the type $(0,\epsilon)^N$, and is divided in a fluid part $Y_{f,i}^\epsilon$ and a solid part $Y_{s,i}^\epsilon$, i.e. is similar to the unit cell Y rescaled to size ϵ. The fluid part Ω_ϵ of a porous medium is defined by

$$\Omega_\epsilon = \Omega \setminus \bigcup_{i=1}^{N(\epsilon)} Y_{s,i}^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} Y_{f,i}^\epsilon \tag{57}$$

where the number of cells is $N(\epsilon) = |\Omega| \epsilon^{-N} (1 + o(1))$.

A final word of caution is in order : the sequence of solutions (u_ϵ, p_ϵ) is not defined in a fixed domain *independent of* ϵ but rather in a varying set, Ω_ϵ. To state the homogenization theorem, convergences in fixed

Sobolev spaces (defined on Ω) are used which requires first that (u_ϵ, p_ϵ) be extended to the whole domain Ω. Recall that, by definition, an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of (u_ϵ, p_ϵ) is defined on Ω and coincides with (u_ϵ, p_ϵ) on Ω_ϵ.

Theorem 3.1. *There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (55) such that the velocity $\tilde{u}_\epsilon$ converges weakly in $L^2(\Omega)^N$ to u, and the pressure $\tilde{p}_\epsilon$ converges strongly in $L^2(\Omega)/\mathbb{R}$ to p, where (u, p) is the unique solution of the homogenized problem, a Darcy's law,*

$$\begin{cases} u(x) = \frac{1}{\mu} A\left(f(x) - \nabla p(x)\right) & \text{in } \Omega \\ \mathrm{div} u(x) = 0 & \text{in } \Omega \\ u(x) \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{58}$$

where A is a symmetric, positive definite, tensor (the so-called permeability tensor) defined by its entries

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y) dy \tag{59}$$

where, $(e_i)_{1\le i\le N}$ being the canonical basis of $\mathbb{R}^N$, $w_i(y)$ denotes the unique solution in $H^1_\#(Y_f)^N$ of the local, or unit cell, Stokes problem

$$\begin{cases} \nabla q_i - \Delta w_i = e_i & \text{in } Y_f \\ \mathrm{div} w_i = 0 & \text{in } Y_f \\ w_i = 0 & \text{in } Y_b \\ y \to q_i, w_i & Y\text{-periodic.} \end{cases} \tag{60}$$

The weak convergence of the velocity can be further improved by the following *corrector* result.

Proposition 3.2. *With the same notations as in Theorem 3.1, the velocity satisfies*

$$\left(\tilde{u}_\epsilon(x) - \sum_{i=1}^N w_i(\frac{x}{\epsilon}) u_i(x)\right) \to 0 \text{ strongly in } L^2(\Omega)^N, \tag{61}$$

where $(w_i)_{1\le i\le N}$ are the local, unit cell, velocities and $(u_i)_{1\le i\le N}$ are the components of the homogenized velocity $u(x)$.

Remark 3.3. *The homogenized problem (58) is a Darcy's law, i.e., the flow rate u is proportional to the balance of forces including the pressure. The permeability tensor A depends only on the microstructure, Y_f of the porous media (and not on the exterior forces, nor on the physical properties of the fluid). Quite early, many papers have been devoted to the derivation of Darcy's law by homogenization, using formal asymptotic expansions (see for example [54], [59], [77]). The first rigorous proof (including the difficult construction of a pressure extension) appeared in [86]. Further extensions are to be found in [2], [61], and [65]. A good reference for physical aspects of this problem, as well as mathematical ones, is the book [49]. Of course, more complicated models than the incompressible Stokes equations can be homogenized to derive various variants of Darcy's law. The next subsections investigate such more general microscopic models. Of course there are other methods, apart from periodic homogenization, which permit to derive Darcy's law. It can also be established by stochastic homogenization, representative volume averaging, and so on.*

3.2. Two-scale asymptotic expansions

We apply the method of two-scale asymptotic expansions to the previous Stokes equations. We start from the following *two-scale asymptotic expansion* (or *ansatz*) of the velocity u_ϵ and pressure p_ϵ

$$u_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right), \quad p_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^i p_i\left(x, \frac{x}{\epsilon}\right), \tag{62}$$

where each term $u_i(x,y)$ or $p_i(x,y)$ is a function of both variables x and y, periodic in y with period $Y=(0,1)^N$. These series are plugged into equation (55), and the following derivation rule is used:

$$\nabla\left(u_i\left(x,\frac{x}{\epsilon}\right)\right)=\left(\epsilon^{-1}\nabla_y u_i+\nabla_x u_i\right)\left(x,\frac{x}{\epsilon}\right),$$

where ∇_x and ∇_y denote the partial derivative with respect to the first and second variable of $u_i(x,y)$. Equation (55) becomes a series in ϵ

$$\begin{cases}\epsilon^{-1}\nabla_y p_0\left(x,\frac{x}{\epsilon}\right)+\epsilon^0\left[\nabla_x p_0+\nabla_y p_1-\mu\Delta_{yy}u_0\right]\left(x,\frac{x}{\epsilon}\right)+\mathcal{O}(\epsilon)=f(x)\\ \epsilon^{-1}\mathrm{div_y}u_0\left(x,\frac{x}{\epsilon}\right)+\epsilon^0\left[\mathrm{div_x}u_0+\mathrm{div_y}u_1\right]\left(x,\frac{x}{\epsilon}\right)+\mathcal{O}(\epsilon)=0.\end{cases}\tag{63}$$

Identifying each coefficient of (63) as an individual equation yields a cascade of equations (a series of the variable ϵ is zero for all values of ϵ if each coefficient is zero). Here, only the two first equations are enough for our purpose. The ϵ^{-1} equation for the pressure is

$$\nabla_y p_0(x,y)=0,$$

which is nothing else than an equation in the unit cell Y with periodic boundary condition. This implies that p_0 does not depend on y, i.e. there exists a function $p(x)$ such that

$$p_0(x,y)\equiv p(x).$$

The ϵ^{-1} equation from the incompressibility condition and the ϵ^0 equation from the momentum equation are

$$\begin{cases}\nabla_y p_1-\mu\Delta_{yy}u_0=f(x)-\nabla_x p(x)\\ \mathrm{div_y}u_0=0\end{cases}\tag{64}$$

which is a Stokes equation for the velocity u_0 and pressure p_1 in the periodic unit cell Y. It is a well-posed problem, which admits a unique solution, as soon as the right hand side is known. Equation (64) allows one to compute u_0 in terms of f and $\nabla_x p$ which do not depend on y. By linearity we find

$$u_0(x,y)=\frac{1}{\mu}\sum_{i=1}^N w_i(y)\left(f-\frac{\partial p}{\partial x_i}\right)(x),\quad p_1(x,y)=\sum_{i=1}^N q_i(y)\left(f-\frac{\partial p}{\partial x_i}\right)(x),$$

where w_i is the cell velocity and q_i is the cell pressure, solutions of the cell Stokes problem (60).

Finally, the ϵ^0 equation from the incompressibility condition yields

$$\mathrm{div_x}u_0(x,y)+\mathrm{div_y}u_1(x,y)=0.\tag{65}$$

We average equation (65) in the unit cell Y. Taking into account the periodicity condition and the no-slip condition on the solid part Y_b leads, by application of Stokes theorem, to

$$\int_{Y_f}\mathrm{div_y}u_1(x,y)\,dy=\int_{\partial Y}u_1\cdot n\,ds+\int_{\partial Y_b}u_1\cdot n\,ds=0.$$

This implies that the first term of (65) must have zero average over Y, i.e.,

$$\int_Y\mathrm{div_x}\left[\sum_{i=1}^N w_i(y)\left(f-\frac{\partial p}{\partial x_i}\right)(x)\right]dy=0,$$

which simplifies to

$$-\mathrm{div_x} A\left(\nabla_x p(x) - f(x)\right) = 0 \quad \text{in } \Omega,$$

which is a second-order elliptic equation for the pressure p. The constant tensor A is defined by its columns

$$Ae_i = \int_Y w_i(y)\, dy,$$

which is equivalent to the previous definition (59) by a simple integration by parts (multiply the Stokes cell problem (60) by w_j).

Of course, p is the homogenized pressure, and the homogenized velocity u is defined by

$$u(x) = \int_Y u_0(x,y)\, dy = \frac{1}{\mu} A\left(f - \nabla p\right)(x).$$

3.3. **Proof of the Homogenization Theorem**

This subsection is devoted to the proof of Theorem 3.1 by the method of two-scale convergence. We assume the existence of bounded extensions of the velocity and pressure of the fluid in the porous medium (see [2], [49], [86] for a proof which is quite delicate for the pressure).

Lemma 3.4. *There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) satisfying the a priori estimates*

$$\|\tilde{u}_\epsilon\|_{L^2(\Omega)^N} + \epsilon \|\nabla \tilde{u}_\epsilon\|_{L^2(\Omega)^{N\times N}} \leq C \tag{66}$$

and

$$\|\tilde{p}_\epsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C, \tag{67}$$

where the constant C does not depend on ϵ.

We also take for granted the following generalization of Theorem 1.12, the proof of which may be found in [6].

Proposition 3.5. *Let u_ϵ be a bounded sequence in $L^2(\Omega)$ such that $\epsilon \nabla u_\epsilon$ is also bounded in $L^2(\Omega)^N$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, u_ϵ two-scale converges to $u_0(x,y)$, and $\epsilon \nabla u_\epsilon$ to $\nabla_y u_0(x,y)$.*

Let u_ϵ be a bounded sequence of vector valued functions in $L^2(\Omega)^N$ such that its divergence $\mathrm{div} u_\epsilon$ *is also bounded in $L^2(\Omega)$. Then, there exists a two-scale limit $u_0(x,y) \in L^2(\Omega \times Y)^N$ which is divergence-free with respect to y, i.e.* $\mathrm{div_y} u_0 = 0$*, has a divergence with respect to x,* $\mathrm{div_x} u_0$*, in $L^2(\Omega \times Y)$, and such that, up to a subsequence, u_ϵ two-scale converges to $u_0(x,y)$, and* $\mathrm{div} u_\epsilon$ *to* $\mathrm{div_x} u_0(x,y)$*.*

By application of the two-scale convergence method, we firstly prove

Theorem 3.6. *The extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution of (55) two-scale converges to the unique solution $(u_0(x,y), p(x))$ of the two-scale homogenized problem*

$$\begin{cases} \nabla_y p_1(x,y) + \nabla_x p(x) - \mu \Delta_{yy} u_0(x,y) = f(x) & \text{in } \Omega \times Y_f \\ \mathrm{div_y} u_0(x,y) = 0 & \text{in } \Omega \times Y_f \\ \mathrm{div_x}\left(\int_Y u_0(x,y) dy\right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_b \\ \left(\int_Y u_0(x,y) dy\right) \cdot n = 0 & \text{on } \partial\Omega \\ y \to u_0(x,y), p_1(x,y) & Y\text{-periodic.} \end{cases} \tag{68}$$

Remark 3.7. *The two-scale homogenized problem is also called a two-pressure Stokes system. It is a combination of the usual homogenized and cell problems. By elimination of the y variable, the homogenized Darcy's law will be recovered in the end.*

Proof of Theorem 3.6. Applying Proposition 3.5 there exists a two-scale limit $u_0(x,y) \in L^2\left(\Omega; H^1_\#(Y)^N\right)$ such that, up to a subsequence, the sequences $\tilde{u}_\epsilon$ and $\epsilon\nabla\tilde{u}_\epsilon$ two-scale converge to u_0 and $\nabla_y u_0$ respectively. Furthermore, u_0 satisfies

$$\begin{cases} \mathrm{div_y} u_0(x,y) = 0 & \text{in } \Omega \times Y \\ \mathrm{div_x}\left(\int_Y u_0(x,y)dy\right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_b \\ \left(\int_Y u_0(x,y)dy\right)\cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{69}$$

By the compactness theorem of two-scale convergence, there exists a two-scale limit $p_0(x,y) \in L^2(\Omega \times Y)$ such that, up to a subsequence, $\tilde{p}_\epsilon$ two-scale converges to p_0. Multiplying the momentum equation in (55) by $\epsilon\psi\left(x, \frac{x}{\epsilon}\right)$, where $\psi(x,y)$ is a smooth, vector-valued, Y-periodic function, and integrating by parts, leads to

$$\lim_{\epsilon\to 0}\int_\Omega \tilde{p}_\epsilon \mathrm{div_y}\psi\left(x, \frac{x}{\epsilon}\right)dx = \int_\Omega\int_Y p_0(x,y)\mathrm{div_y}\psi(x,y)dxdy = 0. \tag{70}$$

Another integration by parts in (70) shows that $\nabla_y p_0(x,y)$ is 0. Thus, there exists $p(x) \in L^2(\Omega)/\mathbb{R}$ such that $p_0(x,y) = p(x)$.

The next step in the two-scale convergence method is to multiply system (55) by a test function having the form of the two-scale limit u_0, and to read off a variational formulation for the limit. Therefore, we choose a test function $\psi(x,y) \in \mathcal{D}\left(\Omega; C^\infty_\#(Y)^N\right)$ with $\psi(x,y) = 0$ in $\Omega \times Y_b$ (thus, $\psi\left(x, \frac{x}{\epsilon}\right) \in H^1_0(\Omega_\epsilon)^N$). Furthermore, we assume that ψ satisfies the incompressibility conditions (69), i.e. $\mathrm{div_y}\psi(x,y) = 0$ and $\mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0$. Multiplying equation (55) by $\psi\left(x, \frac{x}{\epsilon}\right)$, and integrating by parts yields

$$-\int_{\Omega_\epsilon} p_\epsilon(x)\mathrm{div_x}\psi\left(x, \frac{x}{\epsilon}\right)dx + \mu\int_{\Omega_\epsilon}\epsilon\nabla u_\epsilon(x)\cdot\nabla_y\psi\left(x, \frac{x}{\epsilon}\right)dx = \int_{\Omega_\epsilon} f(x)\cdot\psi\left(x, \frac{x}{\epsilon}\right)dx + O(\epsilon) \tag{71}$$

where $O(\epsilon)$ stands for the the remaining terms of order ϵ. In (71) the domain of integration Ω_ϵ can be replaced by Ω since the test function is zero in $\Omega\setminus\Omega_\epsilon$. Then, passing to the two-scale limit, the first term in (71) contributes nothing because the two-scale limit of $\tilde{p}_\epsilon$ does not depend on y and ψ satisfies $\mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0$, while the other terms give

$$\mu\int_\Omega\int_{Y_f}\nabla_y u_0(x,y)\cdot\nabla_y\psi(x,y)dxdy = \int_\Omega\int_{Y_f} f(x)\cdot\psi(x,y)dxdy. \tag{72}$$

By density (72) holds for any function ψ in the Hilbert space V defined by

$$V = \left\{\psi(x,y) \in L^2\left(\Omega; H^1_\#(Y)^N\right) \text{ such that } \begin{array}{l}\mathrm{div_y}\psi(x,y) = 0 \text{ in } \Omega\times Y \\ \mathrm{div_x}\left(\int_Y \psi(x,y)dy\right) = 0 \text{ in } \Omega\end{array},\right.$$

$$\left.\text{and } \begin{array}{l}\psi(x,y) = 0 \text{ in } \Omega\times Y_b \\ \left(\int_Y \psi(x,y)dy\right)\cdot n = 0 \text{ on } \partial\Omega\end{array}\right\}. \tag{73}$$

It is not difficult to check that the hypothesis of the Lax-Milgram lemma holds for the variational formulation (72) in the Hilbert space V, which, by consequence, admits a unique solution u_0 in V. Furthermore, by Lemma 3.8 below, the orthogonal of V, a subset of $L^2\left(\Omega; H^{-1}_\#(Y)^N\right)$, is made of gradients of the form $\nabla_x q(x) + \nabla_y q_1(x,y)$ with $q(x) \in H^1(\Omega)/\mathbb{R}$ and $q_1(x,y) \in L^2\left(\Omega; L^2_\#(Y_f)/\mathbb{R}\right)$. Thus, by integration by parts, the variational formulation (72) is equivalent to the two-scale homogenized system (68). (There is a subtle point here; one must check that the pressure $p(x)$ arising as a Lagrange multiplier of the incompressibility constraint $\mathrm{div_x}\left(\int_Y u_0(x,y)dy\right) = 0$ is the same as the two-scale limit of the pressure $\tilde{p}_\epsilon$. This can easily be done by multiplying equation (55) by a test function ψ which is divergence free only in y, and identifying limits.) Since

(68) admits a unique solution, then the entire sequence $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ converges to its unique solution $(u_0(x,y), p(x))$. This completes the proof of Theorem 3.6. □

We now arrive at the last step of the two-scale convergence method which amounts to eliminate, if possible, the microscopic variable y in the homogenized system. This allows to deduce Theorem 3.1 from Theorem 3.6.

Proof of Theorem 3.1. The derivation of the homogenized Darcy's law (58) from the two-scale homogenized problem (68) is just a matter of algebra. From the first equation of (68), the velocity $u_0(x,y)$ is computed in terms of the macroscopic forces and the local velocities

$$u_0(x,y) = \frac{1}{\mu}\sum_{i=1}^{N}\left(f_i(x) - \frac{\partial p}{\partial x_i}(x)\right) w_i(y). \tag{74}$$

Averaging (74) on Y, and denoting by u the average of u_0, i.e. $u(x) = \int_{Y_f} u_0(x,y)dy$, yields the Darcy relationship

$$u(x) = \frac{1}{\mu} A\Big(f(x) - \nabla p(x)\Big), \tag{75}$$

since the matrix A satisfies

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y) dy = \int_{Y_f} w_i(y) \cdot e_j dy. \tag{76}$$

Equation (76) is obtained by multiplying the i^{th} local problem (60) by w_j and integrating by parts (the boundary integrals cancel out, thanks to the periodic boundary condition). Combining (75) with the divergence-free condition on u yields the homogenized Darcy's law. Note that it is a well-posed problem since it is just a second order elliptic equation for the pressure p, complemented by a Neumann boundary condition. To complete the proof of Theorem 3.1 it remains to show that the convergence of the pressure $\tilde{p}_\epsilon$ to p is not only weak, but also strong, in $L^2(\Omega)/\mathbb{R}$: this will be done in the next subsection. □

Lemma 3.8. *Let V be the subspace of $L^2(\Omega; H^1_\#(Y)^N)$ defined by (73). Its orthogonal $V^\perp$ (with respect to the usual scalar product in $L^2(\Omega \times Y)$) has the following characterization*

$$V^\perp = \left\{\nabla_x \varphi(x) + \nabla_y \varphi_1(x,y) \text{ with } \varphi \in H^1(\Omega) \text{ and } \varphi_1 \in L^2\left(\Omega; L^2_\#(Y_f)/\mathbb{R}\right)\right\}. \tag{77}$$

Proof. Remark that $V = V_1 \cap V_2$ with

$$V_1 = \left\{v(x,y) \in L^2(\Omega; H^1_\#(Y)^N) \text{ s.t. } \mathrm{div_y} v = 0 \text{ in } \Omega \times Y, v = 0 \text{ in } \Omega \times Y_b\right\}$$

$$V_2 = \left\{v(x,y) \in L^2(\Omega; H^1_\#(Y)^N) \text{ s.t. } \mathrm{div_x}\left(\int_{Y_f} v dy\right) = 0 \text{ in } \Omega, \left(\int_{Y_f} v dy\right) \cdot n_x = 0 \text{ on } \partial\Omega\right\}$$

It is a well-known result (see, e.g., [88]) that

$$V_1^\perp = \left\{\nabla_y \varphi_1(x,y) \text{ with } \varphi_1 \in L^2\left(\Omega; L^2_\#(Y_f)/\mathbb{R}\right)\right\}$$

$$V_2^\perp = \left\{\nabla_x \varphi(x) \text{ with } \varphi \in H^1(\Omega)\right\}.$$

The lemma is proved if one can check that $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$. Since V_1 and V_2 are two closed subspaces, this equality is equivalent to $V_1 + V_2 = \overline{V_1 + V_2}$. This is indeed true because we are going to prove that $V_1 + V_2$ is equal to $L^2(\Omega; H^1_\#(Y_f)^N)$. Introducing the divergence-free solutions $(w_i(y))_{1 \le i \le N}$ of the cell Stokes problem

(60), for any given $v(x,y) \in L^2(\Omega; H^1_\#(Y_f)^N)$, we define a unique solution $q(x)$ in $H^1(\Omega)/\mathbb{R}$ of the Neumann problem

$$\left\{ \begin{array}{ll} -\mathrm{div_x}\left(A\nabla q(x) - \int_{Y_f} v(x,y)dy\right) = 0 & \text{in } \Omega \\ \left(A\nabla q(x) - \int_{Y_f} v(x,y)dy\right)\cdot n = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{78}$$

where A is the matrix A defined by (59). This allows us to decompose v as

$$v(x,y) = \sum_{i=1}^N w_i(y)\frac{\partial q}{\partial x_i}(x) + \left(v(x,y) - \sum_{i=1}^N w_i(y)\frac{\partial q}{\partial x_i}(x)\right),$$

where the first term belongs to V_1, while the second one belongs to V_2. □

3.4. **Inertia Effects**

This subsection is devoted to a generalization of the previous model when inertial effects are added to the Stokes equations which then become the Navier-Stokes equations. To simplify the exposition we shall consider successively and separately the different inertial terms arising in the equations. First, Theorem 3.11 is concerned with the linear, evolutionary Stokes equations. Second, Theorem 3.14 focuses on the non-linear, steady Navier-Stokes equations. Of course, these two cases could be combined together with no special difficulties, but at the price of unnecessary and lengthy technical details.

The geometrical situation is the same as that of Subsection 3.1. Namely, a periodic porous domain Ω and its fluid part Ω_ϵ are considered, with period ϵ and unit cell Y. For a precise description of Ω_ϵ, the reader is referred to definition (57) above.

Darcy's Law with Memory.

We consider the unsteady Stokes equations in the fluid domain Ω_ϵ with a no-slip (Dirichlet) boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, f the density of forces acting on the fluid, and u^0_ϵ an initial condition for the velocity. We assume that the density of the fluid is equal to 1, while its viscosity is very small, and indeed is exactly $\mu\epsilon^2$ (where ϵ is the pore size). The system of equations is

$$\left\{ \begin{array}{ll} \frac{\partial u_\epsilon}{\partial t} + \nabla p_\epsilon - \epsilon^2\mu\Delta u_\epsilon = f & \text{in } (0,T)\times\Omega_\epsilon \\ \mathrm{div} u_\epsilon = 0 & \text{in } (0,T)\times\Omega_\epsilon \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega_\epsilon \\ u_\epsilon(t=0,x) = u^0_\epsilon(x) & \text{in } \Omega_\epsilon \text{ at time } t=0. \end{array} \right. \tag{79}$$

The scaling ϵ^2 of the viscosity is the same as that in Subsection 3.1. However, here it is not a simple change of variable since the density in front of the inertial term has been scaled to 1. The scalings in system (79) are precisely those which give a non-zero limit for the velocity u_ϵ and a limit problem depending on time. In particular, (79) is not equivalent to the following system (which gives rise to a different homogenized system with no inertial term in the limit)

$$\left\{ \begin{array}{ll} \frac{\partial u_\epsilon}{\partial t} + \nabla p_\epsilon - \mu\Delta u_\epsilon = f & \text{in } (0,T)\times\Omega_\epsilon \\ \mathrm{div} u_\epsilon = 0 & \text{in } (0,T)\times\Omega_\epsilon \\ u_\epsilon = 0 & \text{on } (0,T)\times\partial\Omega_\epsilon \\ u_\epsilon(t=0,x) = u^0_\epsilon(x) & \text{in } \Omega_\epsilon \text{ at time } t=0. \end{array} \right. \tag{80}$$

In some sense, the scaling of the viscosity in (79) can also be interpreted as a choice of the time scale of the order of the pore size squared. System (79) has been first studied by J.-L. Lions [59], using formal asymptotic expansions. Rigorous homogenization results have been proved later in [5]. A study of the different system (80) may be found in [65].

To obtain an existence result and convenient a priori estimates for the solution of (79), the force $f(t,x)$ is assumed to belong to $L^2\left((0,T)\times\Omega\right)^N$, and the initial condition $u_\epsilon^0(x)$ to $H_0^1(\Omega_\epsilon)^N$. Furthermore, denoting by $\tilde{u}_\epsilon^0$ the extension by zero in the solid part $\Omega\setminus\Omega_\epsilon$ of the initial condition, we assume that it satisfies

$$\left\{\begin{array}{l} \|\tilde{u}_\epsilon^0\|_{L^2(\Omega)} + \epsilon\|\nabla\tilde{u}_\epsilon^0\|_{L^2(\Omega)} \leq C \\ \mathrm{div}\tilde{u}_\epsilon^0 = 0 \text{ in } \Omega \\ \tilde{u}_\epsilon^0(x) \text{ two-scale converges to a unique limit } v^0(x,y). \end{array}\right. \tag{81}$$

Then, standard theory (see e.g. [88]) yields the following

Proposition 3.9. *Under assumption (81) on the initial condition, the Stokes equations (79) admits a unique solution $u_\epsilon \in L^2\left((0,T);H_0^1(\Omega_\epsilon)\right)^N$, and $p_\epsilon \in L^2\left((0,T);L^2(\Omega_\epsilon)/\mathbb{R}\right)$. Furthermore, the extension by zero in the solid part $\Omega\setminus\Omega_\epsilon$ of the velocity $\tilde{u}_\epsilon$ satisfies the a priori estimates*

$$\|\tilde{u}_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \epsilon\|\nabla\tilde{u}_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C \text{ and } \|\frac{\partial\tilde{u}_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leq C, \tag{82}$$

where the constant C does not depend on ϵ.

The following homogenization theorem states that the limit problem is a Darcy's law with memory (due to the convolution in time) which generalizes the usual Darcy's law.

Theorem 3.10. *There exists an extension $(\tilde{u}_\epsilon,\tilde{p}_\epsilon)$ of the solution (u_ϵ,p_ϵ) of (79) which converges weakly in $L^2\left((0,T);L^2(\Omega)^N\right)\times L^2\left((0,T);L^2(\Omega)/\mathbb{R}\right)$ to the unique solution (u,p) of the homogenized problem*

$$\left\{\begin{array}{ll} u(t,x) = v(t,x) + \frac{1}{\mu}\int_0^t A(t-s)\left(f-\nabla p\right)(s,x)ds & \text{in } (0,T)\times\Omega \\ \mathrm{div}u(t,x) = 0 & \text{in } (0,T)\times\Omega \\ u(t,x)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega, \end{array}\right. \tag{83}$$

where $v(t,x)$ is an initial condition which depends only on the sequence u_ϵ^0 and on the microstructure Y_f, and $A(t)$ is a symmetric, positive definite, time-dependent, permeability tensor which depends only on the microstructure Y_f (their precise form is to be found in the proof of the present theorem).

The complicated form of the homogenized problem (83), which is not a parabolic p.d.e. but an integro-differential equation, is due to the elimination of a *hidden* microscopic variable. Actually, to prove Theorem 3.10 we first prove a result on the corresponding two-scale homogenized system which includes this hidden variable and has a much nicer form.

Theorem 3.11. *Under assumption (81) on the initial condition, there exists an extension $(\tilde{u}_\epsilon,\tilde{p}_\epsilon)$ of the solution of (79) which two-scale converges to the unique solution $(u_0(x,y),p(x))$ of the two-scale homogenized problem*

$$\left\{\begin{array}{ll} \frac{\partial u_0}{\partial t}(x,y) + \nabla_y p_1(x,y) + \nabla_x p(x) - \mu\Delta_{yy}u_0(x,y) = f(x) & \text{in } (0,T)\times\Omega\times Y_f \\ \mathrm{div}_y u_0(x,y) = 0 & \text{in } (0,T)\times\Omega\times Y_f \\ \mathrm{div}_x\left(\int_Y u_0(x,y)dy\right) = 0 & \text{in } (0,T)\times\Omega \\ u_0(x,y) = 0 & \text{in } (0,T)\times\Omega\times Y_b \\ \left(\int_Y u_0(x,y)dy\right)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ y \rightarrow u_0, p_1 & Y\text{-periodic} \\ u_0(0,x,y) = v^0(x,y) & \text{at time } t=0. \end{array}\right. \tag{84}$$

Remark 3.12. *The two-scale homogenized problem (84) is also called a two-pressures Stokes system (see [59]). Eliminating y in (84) yields the Darcy's law with memory (83). It is not difficult to check that both $v(t,x)$ and $A(t,x)$ decay exponentially in time. Thus, the permeability keeps track mainly of the recent history. If the force f is steady (i.e. does not depend on t), asymptotically, for large time t, the usual steady Darcy's law for u and*

p is recovered. As we shall see, the two-scale homogenized problem (84) is equivalent to (83) complemented with the cell problems (85)-(86).

Proof of theorem 3.11. The proof is completely parallel to that of Theorem 3.6 in Subsection 3.1 (see [5]). The form of the two-scale homogenized problem (84) can also be obtained by using two-scale asymptotic expansions as in Subsection 3.2. The only difference is that the time derivative $\frac{\partial u_0}{\partial t}$ has to be added in the ϵ^0 equation (64) which yields (84). □

Proof of theorem 3.10. The only thing to prove is that eliminating the microscopic variable y in (84) leads to the Darcy's law with memory (83). The solution u_0 is decomposed in two parts $u_1 + u_2$ where u_1 is just the evolution (without any forcing term) of the initial condition v^0. Thus, at each point $x \in \Omega$, u_1 is the unique solution of an equation posed solely in Y_f

$$\begin{cases} \frac{\partial u_1}{\partial t}(x,y) + \nabla_y q(x,y) - \mu \Delta_{yy} u_1(x,y) = 0 & \text{in } (0,T) \times Y_f \\ \mathrm{div}_y u_1(x,y) = 0 & \text{in } (0,T) \times Y_f \\ u_1(x,y) = 0 & \text{in } (0,T) \times Y_b \\ y \to u_1, q & Y\text{-periodic} \\ u_1(0,x,y) = v^0(x,y) & \text{at time } t = 0. \end{cases} \tag{85}$$

The average of u_1 in y is just $v(t,x)$ (the initial condition in the homogenized system (83)). On the other hand, u_2 is given by

$$u_2(t,x,y) = \int_0^t \sum_{i=1}^N \left(f_i - \frac{\partial p}{\partial x_i} \right)(s,x) \frac{\partial w_i}{\partial t}(t-s,y) ds$$

where, for $1 \leq i \leq N$, w_i is the unique solution of the cell problem, which does not depend on the macroscopic variable x. The cell problem is defined by

$$\begin{cases} \frac{\partial w_i}{\partial t}(y) + \nabla_y q_i(y) - \mu \Delta_{yy} w_i(y) = e_i & \text{in } (0,T) \times Y_f \\ \mathrm{div}_y w_i(y) = 0 & \text{in } (0,T) \times Y_f \\ w_i(y) = 0 & \text{in } (0,T) \times Y_b \\ y \to w_i, q_i & Y\text{-periodic} \\ w_i(0,y) = 0 & \text{at time } t = 0. \end{cases} \tag{86}$$

Introducing the matrix A defined by

$$A_{ij}(t) = \mu \int_{Y_f} \frac{\partial w_i}{\partial t}(t,y) e_j dy, \tag{87}$$

the Darcy's law with memory is easily deduced from the two-scale homogenized problem by averaging u_1 and u_2 with respect to y. Eventually, using semi-group theory and integrating by parts in the cell problem (87), one can prove that A is symmetric, positive definite, and decays exponentially in time. □

Non-linear Darcy's Law.

We consider the steady Navier-Stokes equations

$$\begin{cases} \epsilon^\gamma u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \epsilon^2 \mu \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \mathrm{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{88}$$

As before, the fluid viscosity μ has been scaled by a factor ϵ^2, which implies precisely that the velocity u_ϵ has a non-zero limit. The non-linear convective term has also been scaled by a factor ϵ^γ, where γ is a positive constant such that $\gamma \geq 1$. The limit case $\gamma = 1$ corresponds exactly to the scaling which yields a non-linear homogenized problem. The case $\gamma = 4$ allows to replace $\epsilon^2 u_\epsilon$ by a new velocity v_ϵ which satisfies the usual *unscaled* Navier-Stokes equations. Intermediate values of γ are analyzed below. For larger values, the convective

terms are much smaller than the viscous ones, and the Navier-Stokes equations are just a small perturbation of the Stokes ones. For values smaller than 1, the opposite situation arises: convective terms dominate viscous ones. Unfortunately, in this last case, the homogenized limit is unclear.

We begin with a result of Mikelić [65] which states that, for $\gamma > 1$, the convective term of the Navier-Stokes equations disappears in the limit and the homogenized system is the usual Darcy's law (as in Subsection 3.1). The only price to pay is a weaker convergence of the pressure: the closer γ to 1, the weaker the estimate on the pressure.

Theorem 3.13. *Let $\gamma > 1$. Define a constant $\beta > 1$ by*

$$\beta = \min\left(2, \frac{N}{N-2}, \frac{N}{N+2-2\gamma}\right). \tag{89}$$

There exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (88) such that the velocity $\tilde{u}_\epsilon$ converges weakly in $L^2(\Omega)^N$ to u, and the pressure $\tilde{p}_\epsilon$ converges strongly in $L^{q'}(\Omega)/\mathbb{R}$ to p, for any $1 < q' < \beta$, where (u, p) is the unique solution of the homogenized problem, a linear Darcy's law,

$$\begin{cases} u(x) = \frac{1}{\mu} A\left(f(x) - \nabla p(x)\right) & \text{in } \Omega \\ \mathrm{div} u(x) = 0 & \text{in } \Omega \\ u(x) \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{90}$$

In (90), the permeability tensor A is the usual homogenized matrix for Darcy's law, defined by (59) in Theorem 3.1.

We consider the limit case $\gamma = 1$ which yields a non-linear Darcy-type law (sometime called a Dupuit-Forchheimer-Ergun law). The non-linear convective term does not disappear in the homogenized problem which indicates a non-linear behavior of Darcy's law.

Theorem 3.14. *Let $\gamma = 1$. Let f be a smooth function such that its norm in $C^{1,\alpha}(\overline{\Omega})$, with $0 < \alpha < 1$, is sufficiently small. Then, there exists an extension $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ of the solution (u_ϵ, p_ϵ) of (88) and a unique solution (u_0, p_1, p) of the homogenized system*

$$\begin{cases} \nabla_y p_1 + u_0 \cdot \nabla_y u_0 - \mu \Delta_{yy} u_0 = f(x) - \nabla p(x) & \text{in } Y_f \times \Omega \\ \mathrm{div}_y u_0 = 0 & \text{in } Y_f \times \Omega \\ \mathrm{div}_x \left(\int_Y u_0 dy\right) = 0 & \text{in } \Omega \\ u_0 = 0 & \text{in } Y_b \times \Omega, \\ \left(\int_Y u_0 dy\right) \cdot n = 0 & \text{on } \partial\Omega, \\ y \to (u_0, p_1) & Y\text{-periodic}, \end{cases} \tag{91}$$

such that $\tilde{p}_\epsilon$ converges strongly in $L^q(\Omega)/\mathbb{R}$ to p, for any $1 < q < 2$, and

$$\|\tilde{u}_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\|_{L^2(\Omega)^2} \leq C\epsilon^l \quad \text{with} \quad 0 < l < 1/6. \tag{92}$$

The homogenized system (91), called a two-pressure Navier-Stokes system, is very similar to the two-scale homogenized system (68). It is not possible to eliminate the y variable to obtain an explicit macroscopic effective law. Therefore, the non-linear Darcy's law is not a local, explicit, partial differential equation. Such a homogenized problem has formally been derived in [77] and [59]. A rigorous proof of convergence has recently been given in [63] (see also [66] in the two-dimensional case). The proof of Theorem 3.14 is very technical and is not reproduced here (the key argument is to prove an existence and uniqueness result for the homogenized system (91) by using a monotonicity argument).

The homogenization of the inviscid Euler equations for incompressible flows in a periodic porous medium has been studied in [60], [67].

3.5. **Derivation of Brinkman's Law**

This subsection is devoted to the derivation of Brinkman's law for an incompressible viscous fluid flowing in a porous medium. As in the previous subsections, we start from the steady Stokes equations in a periodic porous medium, with a no-slip (Dirichlet) boundary condition on the solid pores. We assume that the solid part of the porous medium is a collection of periodically distributed obstacles. We denote by ϵ the period, or the inter-obstacles distance. The major difference with the previous subsections is that the solid obstacles are assumed to be much smaller than the period ϵ. Their size is denoted by $a_\epsilon \ll \epsilon$. There are now two small parameters in our asymptotic analysis which means that two-scale asymptotic expansions cannot be used in the sequel. The assumption on the periodicity of the porous medium allows to simplify greatly the results, although it is not strictly necessary (and not very realistic). As before, the porous medium is denoted by Ω, and its fluid part by Ω_ϵ.

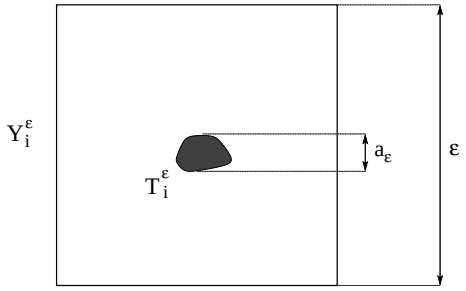

FIGURE 3. Scaling of the periodicity cell of a porous medium.

The motion of the fluid in Ω_ϵ is governed by the steady Stokes equations, complemented with a Dirichlet boundary condition. We denote by u_ϵ and p_ϵ the velocity and pressure of the fluid, μ its viscosity (a positive constant), and f the density of forces acting on the fluid (u_ϵ and f are vector-valued functions, while p_ϵ is scalar). The microscopic model is

$$\left\{\begin{array}{ll} \nabla p_\epsilon - \mu\Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \text{div} u_\epsilon = 0 & \text{in } \Omega_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon, \end{array}\right. \tag{93}$$

which admits a unique solution (u_ϵ, p_ϵ) in $H^1_0(\Omega_\epsilon)^N \times L^2(\Omega_\epsilon)/\mathbb{R}$ if $f(x) \in L^2(\Omega)^N$ (see e.g. [88]).

Let us describe more precisely the assumptions on the porous domain. It is contained in a bounded domain $\Omega \subset \mathbb{R}^N$, and its fluid part is denoted by Ω_ϵ. The set Ω is covered by a regular periodic mesh of period ϵ. At the center of each cell Y_i^ϵ (equal to $(0,\epsilon)^N$, up to a translation), a solid obstacle T_i^ϵ is placed which is obtained by rescaling a unit obstacle T to the size a_ϵ (i.e. $T_i^\epsilon = a_\epsilon T$, up to a translation, see Figure 3). The unit obstacle T is a non-empty, smooth, closed set included in the unit cell and such that $Y \setminus T$ is a smooth connected open set. The fluid part Ω_ϵ of the porous medium is defined by

$$\Omega_\epsilon = \Omega \setminus \bigcup_{i=1}^{N(\epsilon)} T_i^\epsilon, \tag{94}$$

where the number of obstacles is $N(\epsilon) = |\Omega|\epsilon^{-N}\,(1+o(1))$. A fundamental assumption is that the obstacles are *much smaller* than the period,

$$\lim_{\epsilon\to 0} \frac{a_\epsilon}{\epsilon} = 0. \tag{95}$$

According to the various scaling of the obstacle size a_ϵ in terms of the inter-obstacle distance ϵ, different limit problems arise. To sort these different regimes, we introduce a ratio σ_ϵ defined by

$$\sigma_\epsilon = \begin{cases} \left(\frac{\epsilon^N}{a_\epsilon^{N-2}}\right)^{1/2} & \text{for } N \geq 3, \\ \epsilon \left|\log\left(\frac{a_\epsilon}{\epsilon}\right)\right|^{1/2} & \text{for } N = 2. \end{cases} \tag{96}$$

As usual in perforated domains like Ω_ϵ, the sequence of solutions (u_ϵ, p_ϵ), being not defined in a fixed Sobolev space, *independent of* ϵ, needs to be extended to the whole domain Ω. We denote by $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ such an extension, which coincides, by definition, with (u_ϵ, p_ϵ) on Ω_ϵ.

Theorem 3.15. *According to the scaling of the obstacle size, there are three different flow regimes.*

(1) *If the obstacles are too small, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = +\infty$, *then the extended solution* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of (93) converges strongly in* $H_0^1(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of the homogenized Stokes equations*

$$\begin{cases} \nabla p - \mu \Delta u = f & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{97}$$

(2) *If the obstacles have a critical size, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = \sigma > 0$, *then the extended solution* $(\tilde{u}_\epsilon, \tilde{p}_\epsilon)$ *of (93) converges weakly in* $H_0^1(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of the Brinkman law*

$$\begin{cases} \nabla p - \mu \Delta u + \frac{\mu}{\sigma^2} M u = f & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{98}$$

(3) *If the obstacles are too big, i.e.* $\lim_{\epsilon\to 0} \sigma_\epsilon = 0$, *then the rescaled solution* $(\frac{\tilde{u}_\epsilon}{\sigma_\epsilon^2}, \tilde{p}_\epsilon)$ *of (93) converges strongly in* $L^2(\Omega)^N \times L^2(\Omega)/\mathbb{R}$ *to* (u, p), *the unique solution of Darcy's law*

$$\begin{cases} u = \frac{1}{\mu} M^{-1} (f - \nabla p) & \text{in } \Omega \\ \mathrm{div} u = 0 & \text{in } \Omega \\ u \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{99}$$

In all regimes, M *is the same* $N \times N$ *symmetric matrix, which depends only on the model obstacle* T *(its inverse,* M^{-1}*, plays the role of a permeability tensor).*

The following proposition gives the precise definition of M in terms of *local* problems around the unit obstacle T. Mathematically speaking, M can be interpreted in terms of the *hydrodynamic capacity* of the set T. From a physical point of view, the i^{th} column of M is the drag force of the local Stokes flow around T in the i^{th} direction. Thus, M may be interpreted as the slowing effect of the obstacles in the homogenized limit.

Proposition 3.16. *According to the space dimension* N, *the local Stokes problem and the matrix* M *are defined as follows.*

(1) *If* $N \geq 3$, *for* $1 \leq i \leq N$, *the cell problems are*

$$\begin{cases} \nabla q_i - \Delta w_i = 0 & \text{in } \mathbb{R}^N \setminus T \\ \mathrm{div} w_i = 0 & \text{in } \mathbb{R}^N \setminus T \\ w_i = 0 & \text{in } T \\ w_i \to e_i & \text{at } \infty. \end{cases} \tag{100}$$

The matrix M *is defined by its entries*

$$M_{ij} = \int_{\mathbb{R}^N \setminus T} \nabla w_i \cdot \nabla w_j dx, \tag{101}$$

or equivalently by its columns, equal to the drag forces applied on T by the local Stokes flows

$$Me_i = \int_{\partial T} \left(\frac{\partial w_i}{\partial n} - q_i n \right). \tag{102}$$

(2) *If $N = 2$, for $1 \leq i \leq 2$, the cell problems are*

$$\begin{cases} \nabla q_i - \Delta w_i = 0 & \text{in } \mathbb{R}^2 \setminus T \\ \text{div} w_i = 0 & \text{in } \mathbb{R}^2 \setminus T \\ w_i = 0 & \text{in } T \\ w_i(x) \sim e_i \log(|x|) & \text{as } |x| \to \infty. \end{cases} \tag{103}$$

The matrix M is defined by its columns, equal to the drag forces applied on T by the local Stokes flows

$$Me_i = \int_{\partial T} \left(\frac{\partial w_i}{\partial n} - q_i n \right). \tag{104}$$

Furthermore, whatever the shape or the size of the obstacle T, in two space dimension the matrix M is always the same

$$M = 4\pi Id. \tag{105}$$

In the super-critical case, the homogenized problem is a Darcy's law with a permeability tensor M^{-1} (see (99)). This tensor has nothing to do with that, denoted by A, obtained in a two-scale periodic setting (see Theorem 3.1 in Subsection 3.1). Recall that here the obstacles are much smaller than the period (see assumption (95)). The two matrices are not computed with the same local problems which are posed in a single periodicity cell in Subsection 3.1 and in the whole space $\mathbb{R}^N$ here. However, it has been proved in [4] that M^{-1} is the rescaled limit of A, when the obstacle size goes to zero in the unit periodic cell Y.

Remark 3.17. *The surprising result in 2-D, that M is always equal to $4\pi Id$, is actually a consequence of the well-known* Stokes paradox. *This paradox asserts that there exist no solution of the local problem (100) in 2-D (it explains why the growth condition at infinity in (103) is different from the higher dimensional cases). Notice also that the critical size yielding Brinkman's law changes drastically from 2-D, $a_\epsilon = e^{-\sigma^2/\epsilon^2}$, to 3-D, $a_\epsilon = \sigma^2 \epsilon^3$.*

The proof of Theorem 3.15 relies on the oscillating test function method of Tartar as adapted to the present framework by Cioranescu and Murat [30]. Let us sketch the main idea of this method. It consists in multiplying the Stokes equation (93) by a test function, integrating by parts, and passing to the limit, as $\epsilon \to 0$, in order to obtain the variational formulation of the homogenized problem. The key difficulty here is that the test function must belong to $H_0^1(\Omega_\epsilon)^N$, namely it has to vanish on the obstacles for any value of ϵ. Of course it is not the case for a non-zero fixed test function φ. Therefore, *boundary layers* $(w_i^\epsilon)_{1 \leq i \leq N}$ have to be constructed such that, φ being a smooth vector-field, $\sum_{i=1}^N \varphi_i w_i^\epsilon$ belongs to $H_0^1(\Omega_\epsilon)^N$ and converges to φ as ϵ goes to zero. These boundary layers $(w_i^\epsilon)_{1 \leq i \leq N}$ are built with the help of the solutions $(w_i)_{1 \leq i \leq N}$ of the local Stokes problems from Proposition 3.16 by rescaling them to the size a_ϵ around each obstacle and pasting each contribution at the cell boundary ∂Y_i^ϵ. Loosely speaking, w_i^ϵ is a divergence-free vector field which vanishes on the obstacles and is almost equal to the unit vector e_i far from the obstacles. Using this oscillating test function, $\sum_{i=1}^N \varphi_i w_i^\epsilon$ yields the desired result (see [3] for details). Another interest of the boundary layers $(w_i^\epsilon)_{1 \leq i \leq N}$ is that they permit to obtain *corrector results*. For example, in the critical case and under a mild smoothness assumption on the Brinkman velocity u, the weak convergence in $H_0^1(\Omega)^N$ of $\tilde{u}_\epsilon$ can be improved in

$$\left(\tilde{u}_\epsilon - \sum_{i=1}^N v_i w_i^\epsilon \right) \to 0 \text{ strongly in } H_0^1(\Omega)^N.$$

Finally, as in Subsection 3.1 on the derivation of Darcy's law, a technical difficulty is the construction of a bounded extension of the pressure p_ϵ. As usual, extending the velocity is easier: it suffices to take it equal to 0 inside the obstacles,

$$\left\{ \begin{array}{ll} \tilde{u}_\epsilon = u_\epsilon & \text{in } \Omega_\epsilon, \\ \tilde{u}_\epsilon = 0 & \text{in } \Omega \setminus \Omega_\epsilon. \end{array} \right.$$

Remark 3.18. *In space dimension $N = 2, 3$, Theorem 3.15 can be generalized easily to the non-linear Navier-Stokes equations (see Remark 1.1.10 in [3]). The microscopic equations in the porous medium are*

$$\left\{ \begin{array}{ll} \nabla p_\epsilon + u_\epsilon \cdot \nabla u_\epsilon - \mu \Delta u_\epsilon = f & \textit{in } \Omega_\epsilon \\ \mathrm{div} u_\epsilon = 0 & \textit{in } \Omega_\epsilon \\ u_\epsilon = 0 & \textit{on } \partial\Omega_\epsilon. \end{array} \right. \tag{106}$$

Then, there are still three limit flow regimes, corresponding to the same obstacle sizes, and the definitions of the local problems and of the matrix M are still given by Proposition 3.16. In the critical case, the homogenized problem is a non-linear Brinkman's law

$$\left\{ \begin{array}{ll} \nabla p + u \cdot \nabla u - \mu \Delta u + \frac{\mu}{\sigma^2} M u = f & \textit{in } \Omega \\ \mathrm{div} u = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{array} \right. \tag{107}$$

while in the super-critical case it is still the same linear Darcy's law (99).

The rigorous derivation of Brinkman's law by homogenization of Stokes equations in a periodic porous medium has first been established by Marchenko and Khruslov [62]. The description of all limit regimes and the two-dimensional paradoxical result of Proposition 3.16 are due to Allaire [3]. Brinkman's law has also been obtained formally by *three-scale* asymptotic expansion by Lévy [58] and Sanchez-Palencia [78]. It has also been derived in a variational framework by Brillard [22].

4. Homogenization of diffusion equations

4.1. **Double Permeability**

This subsection is devoted to the derivation of the so-called *double porosity* model for describing single-phase flows in fractured porous media. This model is well known in the engineering literature [49]. It has rigorously been derived by means of homogenization techniques [16]. A fractured porous medium possesses two porous structures, one associated with the system of cracks or fractures, and the other with the matrix of porous rocks. In each of these structures, the fluid flow is assumed to be governed by Darcy's law. On the contrary of the previous section where the starting model was a *microscopic* model (Stokes equations at the pore level), here the original model is already an averaged model (Darcy's law in both the matrix and the fractures). Therefore, we shall obtain a *macroscopic* model starting from a *mesoscopic* one. More precisely, we shall prove, under suitable assumptions, that the homogenization of Darcy's law in a periodic fractured porous medium yields a double porosity model.

As before, we denote by Ω the periodic porous medium with its period ϵ. The rescaled unit cell is $Y = (0,1)^N$, which is made of two complementary parts: the matrix block Y_b, and the fracture set Y_f ($Y_b \cup Y_f = Y$ and $Y_f \cap Y_b = \emptyset$). The matrix block Y_b is assumed to be completely surrounded by the fracture set Y_f, i.e. Y_b is strictly included in Y. We define the matrix and fracture parts of Ω by

$$Y_b^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} {Y_b}^\epsilon{}_i, \quad Y_f^\epsilon = \Omega \cap \bigcup_{i=1}^{N(\epsilon)} {Y_f}^\epsilon{}_i, \tag{108}$$

where ${Y_b}^\epsilon{}_i$ and ${Y_f}^\epsilon{}_i$ are the ϵ-size copies of Y_b and Y_f covering Ω.

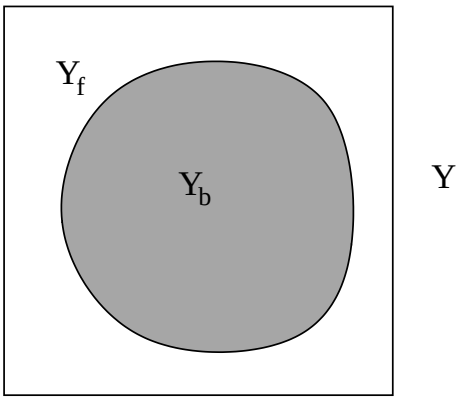

FIGURE 4. Unit cell of a fractured porous medium.

The reservoir Ω is periodic since its porosity ϕ_ϵ and permeability k_ϵ are periodic functions defined by

$$\begin{cases} k_\epsilon(x) = \epsilon^2 k_{Y_b}, \quad \phi_\epsilon(x) = \phi_{Y_b} & \text{in } Y_b^\epsilon \\ k_\epsilon(x) = k_{Y_f}, \quad \phi_\epsilon(x) = \phi_{Y_f} & \text{in } Y_f^\epsilon \end{cases} \tag{109}$$

where ϕ_{Y_b}, ϕ_{Y_f} are positive constants, and k_{Y_b}, k_{Y_f} are positive definite tensors (they could also depend on x and y). The fluid is assumed to be compressible, and the state law giving the relationship between its density ρ_ϵ and its pressure p_ϵ is linearized. Combining Darcy's law and the conservation of mass, and neglecting gravity effects, yields the following equation

$$\begin{cases} \phi_\epsilon \frac{\partial \rho_\epsilon}{\partial t} - \operatorname{div}(k_\epsilon \nabla \rho_\epsilon) = f & \text{in } (0,T) \times \Omega \\ (k_\epsilon \nabla \rho_\epsilon) \cdot n = 0 & \text{on } (0,T) \times \partial\Omega \\ \rho_\epsilon(0,x) = \rho^{init}(x) & \text{at time } t = 0. \end{cases} \tag{110}$$

Remark 4.1. *We emphasize the particular scaling of the permeability defined in (109): the matrix is much less permeable than the fractures. Equivalently, the time scale of filtration inside the matrix is much smaller than that inside the fractures. On the other hand, the porosities have the same order of magnitude in both regions. Such scalings have been found to yield a homogenized double porosity model by Arbogast, Douglas, and Hornung in [16]. If the permeabilities of both phases (matrix and fractures) were of the same order, the homogenized system would easily be seen to be a single Darcy's law with effective coefficients computed with the usual rules of homogenization.*

To obtain an existence result and convenient a priori estimates for the solution of (93), the source term $f(t,x)$ is assumed to belong to $L^2((0,T) \times \Omega)$, and the initial condition $\rho^{init}(x)$ to $H^1(\Omega)$ (the initial condition could vary with ϵ as soon as it converges sufficiently smoothly). Then, standard theory yields the following

Proposition 4.2. *There exists a unique density $\rho_\epsilon \in L^2\left((0,T); H^1(\Omega)\right)$ solution of system (79). Furthermore, it satisfies the a priori estimates*

$$\begin{aligned} &\|\rho_\epsilon\|_{L^\infty((0,T);L^2(\Omega))} + \|\nabla \rho_\epsilon\|_{L^\infty\left((0,T);L^2(Y_f^\epsilon)\right)} + \epsilon \|\nabla \rho_\epsilon\|_{L^\infty\left((0,T);L^2(Y_b^\epsilon)\right)} \leq C, \\ &\|\tfrac{\partial \rho_\epsilon}{\partial t}\|_{L^2((0,T)\times\Omega)} \leq C, \end{aligned} \tag{111}$$

where the constant C does not depend on ϵ.

The following homogenization theorem states that the homogenized problem is a double permeability model.

Theorem 4.3. *Under assumptions (109), the density ρ_ϵ, solution of (110), two-scale converges to $\rho_0(x,y) \in L^2((0,T)\times\Omega\times Y)$ such that*

$$\begin{cases} \rho_0(x,y) = \rho_{Y_f}(x) & \text{if } (x,y)\in\Omega\times Y_f \\ \rho_0(x,y) = \rho_{Y_b}(x,y) & \text{if } (x,y)\in\Omega\times Y_b \end{cases}$$

where $\left(\rho_{Y_f}(x), \rho_{Y_b}(x,y)\right) \in L^2\left((0,T); H^1(\Omega)\right) \times L^2\left((0,T)\times\Omega; H^1(Y_b)\right))$ is the unique solution of the coupled homogenized problem

$$\begin{cases} \theta\phi_{Y_f}\frac{\partial \rho_{Y_f}}{\partial t} - \mathrm{div_x}\left(k^*\nabla_x\rho_{Y_f}\right) = f - \phi_{Y_b}\int_{Y_b}\frac{\partial\rho_{Y_b}}{\partial t}(x,y)dy & \text{in } (0,T)\times\Omega \\ \left(k^*\nabla\rho_{Y_f}\right)\cdot n = 0 & \text{on } (0,T)\times\partial\Omega \\ \rho_{Y_f}(0,x) = \rho^{init}(x) & \text{at time } t=0 \\ \phi_{Y_b}\frac{\partial\rho_{Y_b}}{\partial t} - \mathrm{div_y}\left(k_{Y_b}\nabla_y\rho_{Y_b}\right) = f(t,x) & \text{in } (0,T)\times Y_b \\ \rho_{Y_b}(x,y) = \rho_{Y_f}(x) & \text{on } (0,T)\times\partial Y_b \\ \rho_{Y_b}(0,x,y) = \rho^{init}(x) & \text{at time } t=0, \end{cases} \tag{112}$$

where $\theta = \frac{|Y_f|}{|Y|}$ is the volume fraction of fractures, and k^* is the homogenized permeability tensor defined by its entries

$$k^*_{ij} = \int_{Y_f} k_{Y_f}\left(e_i + \nabla_y\chi_i\right)\cdot\left(e_j+\nabla_y\chi_j\right)dy,$$

where $\chi_i(y)$ are the unique solutions in $H^1_\#(Y_f)/\mathbb{R}$ of the cell problems

$$\begin{cases} -\mathrm{div_y} k_{Y_f}\left(e_i+\nabla_y\chi_i(y)\right) = 0 & \text{in } Y_f \\ k_{Y_f}\left(e_i+\nabla_y\chi_i(y)\right)\cdot n = 0 & \text{on } \partial Y_b \\ y\to\chi_i(y) & Y\text{-periodic}, \end{cases}$$

with $(e_i)_{1\le i\le N}$ the canonical basis of $\mathbb{R}^N$.

The homogenized problem (112) is called a *double porosity model* since it couples a macroscopic equation for ρ_{Y_f} (the first line of (112)) and a microscopic equation for ρ_{Y_b} (the fourth line of (112)). Remark that the (weak) limit of ρ_ϵ is not ρ_{Y_f}, but rather a combination of ρ_{Y_f} and ρ_{Y_b}

$$\rho_\epsilon \rightharpoonup \theta\rho_{Y_f} + (1-\theta)\rho_{Y_b} \quad \text{as } \epsilon\to 0.$$

Therefore, one can not eliminate the microscopic equation in the homogenized problem.

We apply the method of two-scale asymptotic expansions to (110). We start from the following *two-scale asymptotic expansion* (or *ansatz*)

$$\rho_\epsilon(t,x) = \sum_{i=0}^{+\infty}\epsilon^i\rho_i\left(t,x,\frac{x}{\epsilon}\right), \tag{113}$$

where each term $\rho_i(t,x,y)$ is a Y-periodic function. Plugging (113) into (110) yields a cascade of equations. One must be careful because there is a difference of order in ϵ in the matrix block Y_b and in the fracture set Y_f.

The ϵ^{-2} equation holds true only in Y_f

$$-\mathrm{div_y}\left(k_{Y_b}\nabla_y\rho_0(t,x,y)\right) = 0 \quad \text{in } Y_f,$$

which implies that ρ_0 is a function that does not depend on y in Y_f, i.e., there exists a function $\rho_{Y_f}(t,x)$ such that

$$\rho_0(t,x,y) \equiv \rho_{Y_f}(t,x) \quad \text{for any } y\in Y_f.$$

The ϵ^{-1} equation in Y_f is

$$-\mathrm{div_y}\left(k_{Y_b}\nabla_y\rho_1(t,x,y)\right) = \mathrm{div_y}\left(k_{Y_b}\nabla_x\rho_{Y_f}(t,x)\right) \quad \text{in } Y_f,$$

with a Neumann boundary condition on ∂Y_b, which allows one to compute ρ_1 in terms of $\nabla_x \rho_{Y_f}$.

Finally, the ϵ^0 equation is

$$\begin{array}{ll} \phi_{Y_b} \frac{\partial \rho_0}{\partial t} - \mathrm{div_y}\left(k_{Y_b} \nabla_y \rho_0\right) = f(t,x) & \text{in } Y_b \\ \phi_{Y_f} \frac{\partial \rho_0}{\partial t} - \mathrm{div_y}\left(k_{Y_f} \nabla_y \rho_2(x,y)\right) = g(t,x,y) & \text{in } Y_f, \end{array} \tag{114}$$

with

$$g(t,x,y) = \mathrm{div_y}\left(k_{Y_f} \nabla_x \rho_1\right) + \mathrm{div_x}\left(k_{Y_f} \nabla_y \rho_1\right) + \mathrm{div_x}\left(k_{Y_f} \nabla_x \rho_0\right) + f(x).$$

The first line of (114) is a parabolic equation for ρ_0 in the block Y_b with Dirichlet boundary condition on ∂Y_b. Writing $\rho_0 = \rho_{Y_b}$ in Y_b, it is precisely the fourth line of the homogenized problem (112). The second line of (114) is an elliptic equation for the unknown ρ_2 in the fracture Y_f with Neumann boundary conditions on ∂Y_b. The compatibility condition of this equation (in order that it admits a solution) is

$$\int_Y \left[g(t,x,y) - \phi_{Y_f} \frac{\partial \rho_0}{\partial t} \right] dy = 0,$$

which is exactly the first line of the homogenized problem (112).

4.2. **Long time behavior of a diffusion equation**

In this subsection we discuss the homogenized limit of a time-evolution diffusion equation. This is a parabolic equation for which we are interested in the long time behavior. As we shall see, the homogenized limit is quite different in this setting from the usual one as described in Subsection 1.2. For an initial data $a \in L^2(\Omega)$, the equation is

$$\left\{ \begin{array}{ll} c\left(\frac{x}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial t} - \epsilon^2 \mathrm{div}\left(A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon\right) = \sigma\left(\frac{x}{\epsilon}\right) u_\epsilon & \text{in } \Omega \times \mathbb{R}^+ \\ u_\epsilon = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u_\epsilon(0) = a & \text{in } \Omega. \end{array} \right. \tag{115}$$

where A is a symmetric matrix satisfying the coercivity assumption

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^N A_{ij}(y) \xi_i \xi_j \leq \beta |\xi|^2, \quad \text{with} \quad 0 < \alpha \leq \beta,$$

and c is a bounded positive Y-periodic function

$$0 < c^- \leq c(y) \leq c^+ < +\infty \quad \forall\, y \in Y.$$

Equation (115) models a reaction-diffusion problem (there is no sign restriction on the reaction coefficient σ). It is frequently used also in nuclear physics or neutronics [14], [11] (see also the lecture notes [13] where additional references and physical motivation are given).

Remark the ϵ^2 scaling in front of the diffusion term. One possible justification of this scaling is that, upon the change of variables $y = x/\epsilon$, the ϵ^2 factor disappears in front of the diffusion, the periodicity cell $(0,1)^N$ is independent of ϵ while the domain size is of order $1/\epsilon$. Another interpretation of the ϵ^2 scaling is concerned with the *long time behavior* of this reaction-diffusion equation. Indeed, if we change the time scale by the change of variable $\tau = \epsilon^2 t$, we can divide all terms of the equation by ϵ^2 and suppress this scaling (except in front of the reaction term). Clearly, if the new time variable τ is of order 1, then the original time variable t is of order ϵ^{-2}, i.e. we investigate the asymptotic of (115) for very long times.

In order to state the main convergence result for (115) we need to introduce an auxiliary problem which is a *cell spectral problem*. Let $\lambda \in \mathbb{R}$ and $w(y) \in L^2_\#(Y)$ be the first eigenvalue and the first eigenfunction of the

cell spectral problem

$$\left\{ \begin{array}{ll} -\lambda c(y)w - \mathrm{div}_{\mathrm{y}}\left(A(y)\nabla_y w\right) = \sigma(y)w & \text{in } Y, \\ y \to w(y) & Y-\text{periodic.} \end{array} \right. \tag{116}$$

The existence of (λ, w) is standard, and the Krein-Rutman theorem implies furthermore that the first eigenvalue is simple and the first eigenvector is the only one that can be chosen positive, $w(y) > 0$ in Y. Physically, the first eigencouple models the local equilibrium between the diffusion and reaction terms.

Theorem 4.4. *Let u_ϵ be the unique solution of (115). Define a new unknown*

$$v_\epsilon(\epsilon^2 t, x) = \frac{u_\epsilon(t,x)}{w\left(\frac{x}{\epsilon}\right)} e^{\lambda t}. \tag{117}$$

Then, for any time $T > 0$, $v_\epsilon(\tau, x)$ converges weakly in $L^2\left((0,T); H^1(\Omega)\right)$ to $u(\tau, x)$ which is the unique solution of the homogenized problem

$$\left\{ \begin{array}{ll} \overline{c}\dfrac{\partial u}{\partial \tau} - \mathrm{div}\left(\overline{A}\nabla u\right) = 0 & \textit{in } \Omega \times (0,T) \\ u = 0 & \textit{on } \partial\Omega \times (0,T) \\ u(0) = \overline{u}_0 & \textit{in } \Omega, \end{array} \right. \tag{118}$$

with

$$\overline{c} = \int_Y c(y) w(y)^2 dy,$$

and $\overline{A}$ the homogenized diffusion tensor defined by its entries

$$\overline{A}_{ij} = \int_Y w^2 A\left(e_i + \nabla_y \chi_i\right) \cdot \left(e_j + \nabla_y \chi_j\right) dy,$$

where $\chi_i(y)$ are the unique solutions in $H^1_{\#}(Y)/\mathbb{R}$ of the cell problems

$$\left\{ \begin{array}{ll} -\mathrm{div}_{\mathrm{y}}\left(w^2 A\left(e_i + \nabla_y \chi_i(y)\right)\right) = 0 & \textit{in } Y \\ y \to \chi_i(y) & Y\textit{-periodic,} \end{array} \right.$$

with $(e_i)_{1 \le i \le N}$ the canonical basis of $\mathbb{R}^N$.

In other words, Theorem 4.4 gives the following asymptotic expansion for the solution of (115)

$$u_\epsilon(t,x) \approx e^{-\lambda t} w\left(\frac{x}{\epsilon}\right) u\left(\epsilon^2 t, x\right).$$

Proof. Applying the change of unknown (117) and using the cell spectral equation (116), we find a simplified equation for v_ϵ

$$\left\{ \begin{array}{ll} (cw^2)\left(\dfrac{x}{\epsilon}\right)\dfrac{\partial v_\epsilon}{\partial \tau} - \mathrm{div}\left((w^2 A)\left(\dfrac{x}{\epsilon}\right)\nabla v_\epsilon\right) = 0 & \text{in } \Omega \times (0,T) \\ v_\epsilon = 0 & \text{on } \partial\Omega \times (0,T) \\ v_\epsilon(0) = v_0^\epsilon & \text{in } \Omega \end{array} \right. \tag{119}$$

with $v_0^\epsilon = a(x)/w(x/\epsilon)$. Remark that all scalings have disappear from (119) and we can therefore apply the standard homogenization theory to (119) (see Subsection 1.2). This easily yields (118). □

Remark 4.5. *It is possible to consider a source term in the reaction-diffusion equation (115) of the form*

$$\epsilon^2 f(\epsilon^2 t, x) e^{-\lambda t}.$$

It yields a source term in the homogenized equation (118) of the type

$$\left(\int_Y w(y)\,dy\right) f(\tau,x).$$

In order to understand how we guessed the clever change of unknowns (117) in the previous paragraph, we apply now the method of two-scale asymptotic expansions to the reaction-diffusion equation (115) (without any a priori knowledge of the result). Since there are two time scales, the fast one t and the slow one $\tau=\epsilon^2 t$, we start from the following *two-scale asymptotic expansion* (or *ansatz*)

$$u_\epsilon(t,x)=\sum_{i=0}^{+\infty}\epsilon^i u_i\left(t,\epsilon^2 t,x,\frac{x}{\epsilon}\right), \tag{120}$$

where each term $u_i(t,\tau,x,y)$ is a Y-periodic function. Remark however that there is no periodicity with respect to the time variables. This series is plugged into equation (115) and we get the usual cascade of equations. The ϵ^0 equation is

$$\left\{\begin{array}{ll} c(y)\frac{\partial u_0}{\partial t}-\mathrm{div_y}\left(A(y)\nabla_y u_0\right)=\sigma(y)u_0 & \text{in } Y,\\ y\to u_0(t,\tau,x,y) & Y-\text{periodic.}\end{array}\right. \tag{121}$$

Since we are interested in the long time asymptotic of the problem, and because t is the fast time variable (compared to τ), we ignore the initial condition in (121) and we decide to retain only the behavior of u_0 for very large time t. (The initial condition will be applied to the Cauchy problem with respect to the slow time variable τ.) It is well-known that any solution of (121) has the same asymptotic profile, as time t goes to infinity, whatever the initial condition. This limit profile is precisely

$$Ce^{-\lambda t}w(y),$$

where C is a constant depending on the initial condition, and (λ,w) is the first eigencouple of the underlying operator (116). Therefore, we deduce that, for large t,

$$u_0(t,\tau,x,y)=u(\tau,x)e^{-\lambda t}w(y).$$

The ϵ equation is

$$\left\{\begin{array}{ll} c(y)\frac{\partial u_1}{\partial t}-\mathrm{div_y}\left(A(y)\nabla_y u_1\right)=\sigma(y)u_1+g_1 & \text{in } Y,\\ y\to u_1(t,\tau,x,y) & Y-\text{periodic,}\end{array}\right. \tag{122}$$

with the source term

$$g_1=\mathrm{div_x}\left(A(y)\nabla_y u_0\right)+\mathrm{div_y}\left(A(y)\nabla_x u_0\right).$$

If we want the series (120) to converge for large time t, its second term u_1 should not grow faster in t than its first term u_0. To avoid any resonance effect in (122), the right hand side g_1 must be orthogonal to the first eigenfunction $w(y)$, namely

$$\int_Y g_1(y)w(y)\,dy=0. \tag{123}$$

This condition is always satisfied since

$$\int_Y g_1(y)w(y)\,dy=e^{-\lambda t}\int_Y\left(A(y)\nabla_y w(y)\cdot\nabla_x u(x)w(y)-A(y)w(y)\nabla_x u(x)\cdot\nabla_y w(y)\right)dy=0$$

because A is a symmetric matrix. Thus, for large time t, the solution u_1 is

$$u_1(t,\tau,x,y)=e^{-\lambda t}\chi(\tau,x,y),$$

where χ solves

$$\begin{cases} -\lambda c(y)\chi - \mathrm{div_y}\left(A(y)\nabla_y\chi\right) = \sigma(y)\chi + g_1 & \text{in } Y, \\ y \to \chi(t,\tau,x,y) & Y-\text{periodic.} \end{cases}$$

The ϵ^2 equation is

$$\begin{cases} c(y)\frac{\partial u_2}{\partial t} - \mathrm{div_y}\left(A(y)\nabla_y u_2\right) = \sigma(y)u_2 + g_2 & \text{in } Y, \\ y \to u_2(t,\tau,x,y) & Y-\text{periodic,} \end{cases} \tag{124}$$

with the source term

$$g_2 = \mathrm{div_x}\left(A\nabla_y u_1\right) + \mathrm{div_y}\left(A\nabla_x u_1\right) - c\frac{\partial u_0}{\partial \tau} + \mathrm{div_x}\left(A\nabla_x u_0\right).$$

Once again, if we want the series (120) to converge, its third term u_2 should not grow faster in t than u_0. To avoid any resonance effect in (124), the right hand side g_2 must be orthogonal to the first eigenfunction $w(y)$, namely

$$\int_Y g_2(y)w(y)\,dy = 0. \tag{125}$$

After some algebra, the compatibility condition (125) yield a macroscopic equation

$$\overline{c}\frac{\partial u}{\partial \tau} - \mathrm{div_x}\left(\overline{A}\nabla_x u\right) = 0$$

to which we add the initial condition in order to recover the homogenized problem (118).

5. Numerical methods for periodic homogenization

5.1. **Homogenized and cell problems**

We consider only periodic media as described in Section 1. This is of course a serious restriction in many applications (the next section is devoted to the non-periodic case). For simplicity we restrict ourselves to a model problem of diffusion. The periodic domain is denoted by Ω (see Figure 1), its period by ϵ (assumed to be small in comparison with the size of the domain), and the rescaled unit periodic cell by $Y = (0,1)^N$. The conductivity tensor in Ω is $A\left(\frac{x}{\epsilon}\right)$ where $A(y)$ is Y-periodic and satisfies the coercivity assumption

$$\alpha|\xi|^2 \le \sum_{i,j=1}^{N} A_{ij}(y)\xi_i\xi_j \le \beta|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N,\ \forall\, y \in Y,$$

with $\beta \ge \alpha > 0$. Denoting by $f(x)$ the source term, and enforcing a Dirichlet boundary condition (for simplicity), our model problem reads

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{126}$$

which admits a unique solution $u_\epsilon(x)$.

If one wants to compute numerically the solution u_ϵ, any method (finite differences, finite elements, finite volumes) will require a mesh spacing h that must be smaller than ϵ which the characteristic length of the medium. If ϵ is too small, it yields a very fine mesh and thus a very large number of degrees of freedom. Such discrete problems may be too costly or even impossible to solve because the CPU time, as well as the memory storage, being proportional to some power of the total number of degrees of freedom, are too large.

The classical approach to numerically solve (126) is rather to compute the solution of the homogenized problem corresponding to (126), namely

$$\begin{cases} -\mathrm{div_x}\left(A^*\nabla_x u(x)\right) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{127}$$

Since the homogenized tensor A^* is constant, the discretization of (127) does not require a small mesh size h. However, we first need to compute the N cell solutions of the cell problems, for $1 \leq i \leq N$,

$$\begin{cases} -\mathrm{div}_y A(y)\left(e_i + \nabla_y w_i(y)\right) = 0 & \text{in } Y \\ y \to w_i(y) & Y\text{-periodic}, \end{cases} \tag{128}$$

which yield the value of the tensor A^*

$$A^*_{ij} = \int_Y A(y)(e_i + \nabla_y w_i) \cdot e_j \, dy = \int_Y A(y)\left(e_i + \nabla_y w_i\right) \cdot \left(e_j + \nabla w_j\right) dy.$$

An additional advantage of the cell problems (128) is that it allows to improve the approximation of u_ϵ by u by adding the so-called *correctors* to the homogenized solution. Indeed, recall the beginning of the two-scale asymptotic expansion for u_ϵ (and its mathematical justification, see Remark 1.13)

$$u_\epsilon(x) \approx u(x) + \epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right). \tag{129}$$

In (129) the term $\epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right)$ is called the corrector term. When ϵ is small, the corrector term is not very important if one is interested in the values of u_ϵ. However, if the physical quantity of interest is the gradient ∇u_ϵ (some type of flux or of strain), then the corrector is of the same order than the homogenized gradient ∇u, even if ϵ is small, because the approximation in (129) is in the sense of the $H^1(\Omega)$-norm and it implies

$$\nabla u_\epsilon(x) \approx \nabla u(x) + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x)(\nabla_y w_i)\left(\frac{x}{\epsilon}\right).$$

This classical approach of numerical homogenization has been pursued by many authors. Let us mention just a few references: [19], [20], [31], etc.

Remark 5.1. *The results of periodic homogenization are also valid for macroscopically modulated oscillating coefficients of the type $A\left(x, \frac{x}{\epsilon}\right)$, where $A(x, y)$ is a smooth function of x, periodic with respect to y. Numerically this type of problems is solved exactly as above, except that we compute an effective tensor $A^*(x)$ which is piecewise constant in each cell of the coarse mesh of size h (used for the computation of the homogenized problem (127)).*

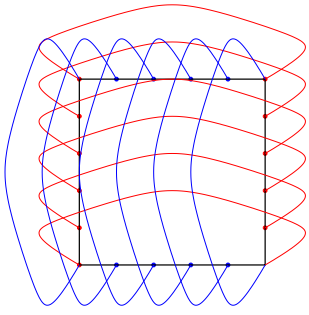

FIGURE 5. Periodic boundary conditions in the unit cell.

The only numerical difficulty in this classical approach is the periodic boundary condition in the cell problems (128). Such periodic boundary conditions are frequently not available in usual numerical codes (although they

are easy to implement: just merge the degrees of freedom on the boundary with their counterparts on the opposite face of the unit cell Y, see Figure 5). Fortunately, under appropriate symmetry conditions, one can replace the periodic boundary condition by a simpler combination of Dirichlet and Neumann conditions.

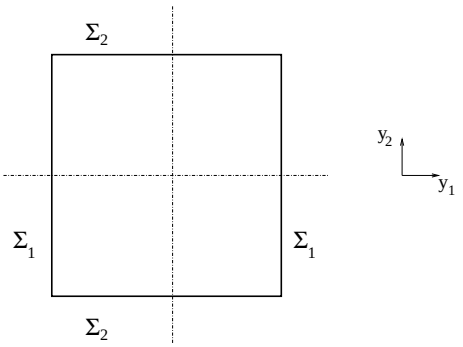

FIGURE 6. Cubic symmetry in the unit cell Y.

Let us assume that the coefficient matrix $A(y)$ has cubic symmetry in Y, i.e. is a diagonal matrix, symmetric with respect to the hyperplanes, parallel to the faces of Y, running through its center (see Figure 6). Then, it is easily seen that $-w_i(-y)$ and $w_i(y', y_j)$ for any $j \neq i$ (with $y = (y', -y_j)$) are also solutions of (128). This implies that w_i is even with respect to y_j for $j \neq i$, and odd with respect to y_i. Therefore, denoting by Σ_i the faces of the unit cube Y which are normal to the vector e_i, the cell problem (128) is equivalent to

$$\begin{cases} -\mathrm{div}_y A(y)\left(e_i + \nabla_y w_i(y)\right) = 0 & \text{in } Y \\ w_i(y) = 0 & \text{on } \Sigma_i \\ \frac{\partial w_i}{\partial n} = 0 & \text{on } \Sigma_j, j \neq i. \end{cases} \tag{130}$$

Of course, one can further reduce the computational domain Y in (130) to one of its quadrant because the same type of boundary conditions hold true on the hyperplanes, parallel to the faces of Y, running through its center.

Remark 5.2. *Another approach, suggested in [48], is to numerically solve the two-scale homogenized problem (50), or its variational formulation (49). In other words, the cell and homogenized problems are solved in a coupled way. This is especially interesting if the coefficients $A(x, y)$ depend on the macroscopic variable x too. The computational cost is reduced by using sparse tensor product finite element methods.*

There is an alternative method to the computation of the cell and homogenized problems and to the approximation formula (129): the so-called Bloch wave method [32], [33], [34]. A variant of this method has also been investigated in [64], [69], [70].

5.2. **Boundary layers**

An important issue in the previous classical approach is the possible improvement of the computation by adding *boundary layers*. The starting point is to recognize that the right hand side of (129) does not satisfy the Dirichlet boundary condition which is actually imposed to the true solution u_ϵ. Therefore, the approximation (129) can be improved, *at least near the boundary* $\partial\Omega$, by adding to its right hand side a so-called *boundary layer*. Introduce a function $u^\epsilon_{1,bl}(x)$, solution of

$$\begin{cases} -\mathrm{div}\left(A\left(\dfrac{x}{\epsilon}\right)\nabla u^\epsilon_{1,bl}\right) = 0 & \text{in } \Omega \\ u^\epsilon_{1,bl} = -\displaystyle\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{131}$$

Then, we replace (129) by

$$u_\epsilon(x) \approx u(x) + \epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right) + \epsilon u^\epsilon_{1,bl}(x). \tag{132}$$

By construction, the right hand side of (132) satisfies the Dirichlet boundary condition as the true solution u_ϵ. One can prove that (132) is a better approximation than (129), mainly near the boundary. More precisely, without boundary layer the optimal error estimate is

$$\left\| u_\epsilon(x) - u(x) - \epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right) \right\|_{H^1(\Omega)} \leq C\sqrt{\epsilon},$$

while taking into account the boundary layer improves the estimate

$$\left\| u_\epsilon(x) - u(x) - \epsilon \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i\left(\frac{x}{\epsilon}\right) - \epsilon u^\epsilon_{1,bl}(x) \right\|_{H^1(\Omega)} \leq C\epsilon. \tag{133}$$

However, the function $u^\epsilon_{1,bl}(x)$ is not explicit. It is only for rectangular domains with faces aligned with those of the unit cell that one can find an explicit approximation $u^\epsilon_{1,bl}(x)$.

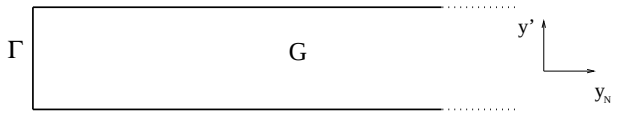

FIGURE 7. Semi infinite band.

For simplicity, assume that $\Omega = (0, L)^N$ where L is a positive length. Denote by Γ that face of Ω where $x_N = 0$ (we use the same name for the corresponding face of the unit cell Y). We define a *semi-infinite band* G by

$$G = (0, 1)^{N-1} \times (0, +\infty),$$

such that $\Gamma = \overline{G} \cap \{y_N = 0\}$ (see Figure 7). We use the notation $y = (y', y_N)$ with $y' = (y_1, ..., y_{N-1})$. We define a *semi-infinite band problem*

$$\begin{cases} -\mathrm{div}_y(A(y)\nabla_y w_{i,bl}) = 0 & \text{in } G \\ w_{i,bl} = -w_i & \text{on } \Gamma \\ y' \to w_i(y', y_N) & (0,1)^{N-1}\text{-periodic.} \end{cases} \tag{134}$$

One can prove that (134) admits a unique solution. Furthermore, this solution converges exponentially fast to a constant when y_N tends to $+\infty$, and its gradient converges exponentially fast to zero. More precisely, there exists a limit c_i and a positive exponent $\gamma > 0$ such that

$$\lim_{y_N \to +\infty} e^{\gamma y_N} \left(|w_{i,bl} - c_i| + |\nabla w_{i,bl}| \right) = 0.$$

The solution $w_{i,bl}$ of (134) is called a boundary layer. One can prove that, near Γ, we have

$$u^\epsilon_{1,bl}(x) \approx \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_{i,bl}\left(\frac{x}{\epsilon}\right). \tag{135}$$

Of course a similar result holds true for any face of Ω. Finally, plugging the boundary layer approximation (135) in (132) gives an approximation of u_ϵ which satisfies the same improved error estimate (133). Numerically, it is easy to compute approximate solutions of the semi-infinite band problem (134): beacuse of the exponential decay of its solution $w_{i,bl}$, one can truncate the semi-infinite band G to just a few cells (typically of the order of 5).

Boundary layers are discussed at length in many papers including [8], [59], [71]. Boundary layers are also very important for the homogenization of oscillating boundaries and for the determination of *effective boundary conditions* (see, e.g. [1], [23], [46], [52]).

6. Numerical methods for heterogeneous non-periodic media

6.1. **Generalities**

The goal is to compute the solution of a partial differential equation in a heterogeneous medium with one, several, or a continuum of lengthscales, characterized by a small parameter ϵ. If the lengthscale ϵ is very small, a direct computation is impossible or too costly. Therefore, we want to use a mesh with a mesh-spacing of size $h >> \epsilon$.

The main idea is to use an homogenization paradigm (i.e. a specific model) to devise an adapted numerical algorithm. For example, in this subsection and in the next one, we consider again the model problem

$$\begin{cases} -\mathrm{div}\left(A^\epsilon \nabla u_\epsilon\right) = f & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{136}$$

where $A^\epsilon(x)$ is not necessarily a periodic function of x. Subsection 6.3 will focus on another model problem.

The point is that we do not satisfy ourselves with the mere computation of the solution of the homogenized problem of (136). We want, not only the homogenized behavior of the true solution u_ϵ, but also its microscopic fluctuations (the correctors in the terminology of periodic homogenization).

Several multiscale finite element methods for the numerical solution of (136) have been proposed. We are going to discuss at length that of Hou et al. [50], [51], but let us also mention a method due to Arbogast [15] for mixed finite element algorithms, a method of Matache, Babuska and Schwab [64] using Bloch waves, the HMM approach of E and Engquist [43], a wavelet-based method [24], [41] and a finite-volume method [53]. There are many other methods devoted to numerical homogenization (multigrid, residual free bubble, etc.). For a more complete overview we refer to the books [42] and [44] (and references therein).

6.2. **Multiscale finite element methods**

In this subsection we describe the multiscale finite element method of Hou et al. [50], [51], as well as its extension in [10]. The model problem under consideration is (136). We get some inspiration from the periodic case $A^\epsilon = A\left(\frac{x}{\epsilon}\right)$, but the method will be of use for more general cases. However, the convergence proofs are available only in the periodic case, or at least in the *non-resonant case*, i.e. when all heterogeneities length scales are smaller than the mesh size, $h >> \epsilon$.

The main idea is very close to the method of the oscillating test function due to Tartar [73]. Indeed, instead of using the usual P_1 (or affine) finite element basis, we first build an oscillating finite element basis and then compute the solution of (136) with this specially adapted basis.

We start with a coarse mesh of the domain Ω, denoted by $\mathcal{T}_h = (K_l)_{l\in I_h}$. It can be a rectangular or a triangular mesh (see Figure 8). We denote by $h > 0$ the mesh spacing in this coarse mesh. We denote by $(x_i)_{1\le i\le N_h}$ the vertices of this coarse mesh. We build a special finite element basis adapted to the problem. Each mesh cell K has its own fine mesh, independent from the other ones (see Figure 8). For each mesh cell K

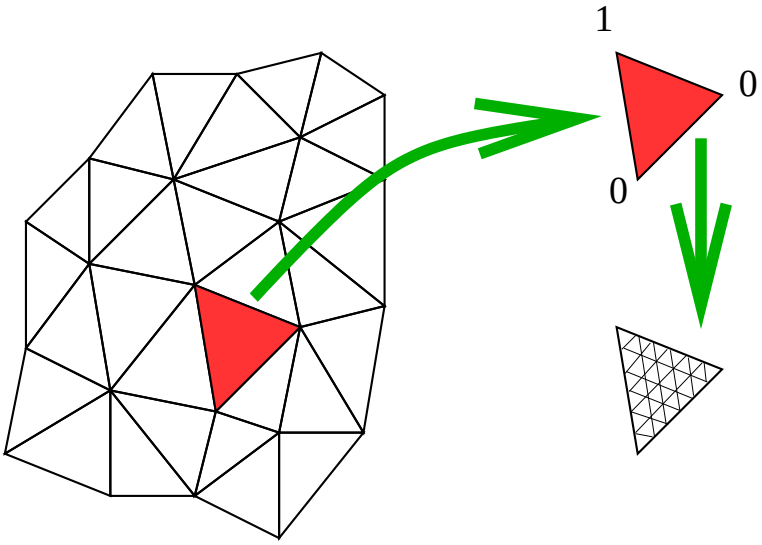

FIGURE 8. Multiscale finite element method.

and for each of its vertex x_i we compute a base function $\phi^\epsilon_{i,K}$ as the solution of

$$\begin{cases} -\mathrm{div}\left(A^\epsilon \nabla \phi^\epsilon_{i,K}\right) = 0 & \text{in } K \\ \phi^\epsilon_{i,K}(x_j) = \delta_{ij} & \text{at the vertex } x_j \\ \phi^\epsilon_{i,K} & \text{affine on } \partial K \end{cases} \tag{137}$$

The boundary value problem (137) is similar to the cell problem (128). Indeed, introducing the affine function $e \cdot x + c$ which coincides with the boundary condition of (137) (namely $e \cdot x_j + c = \delta_{ij}$), and defining $w^\epsilon_{i,K} = \phi^\epsilon_{i,K} - e \cdot x - c$, (137) is equivalent to

$$\begin{cases} -\mathrm{div}\left(A^\epsilon (e + \nabla w^\epsilon_{i,K})\right) = 0 & \text{in } K \\ w^\epsilon_{i,K} = 0 & \text{on } \partial K \end{cases} \tag{138}$$

The main difference between (138) and the cell problem (128) is the boundary condition: Dirichlet for the former, periodic for the latter.

Collecting these functions $\phi^\epsilon_{i,K}$ for all cells K around a single vertex x_i we get a base function ϕ^ϵ_i, with compact support (see Figure 9), such that

$$(\phi^\epsilon_i)_{|K} \equiv \phi^\epsilon_{i,K}, \quad \phi^\epsilon_i(x_j) = \delta_{ij} \ \text{ at any vertex } x_j. \tag{139}$$

Remark that the computations of the base functions $\phi^\epsilon_{i,K}$ can be made in parallel since they are completely independent.

Finally it remains to compute an approximation u^h_ϵ of the true solution u_ϵ of (136) by using the finite element basis $(\phi^\epsilon_i)_{1 \le i \le N_h}$ on the coarse mesh $\mathcal{T}_h$. This last problem is of moderate size, and thus of low cost. However, it incorporates the oscillations of the heterogeneous tensor A^ϵ. This multiscale finite element method is thus a simple conforming method. When implementing this method, there are two delicate issues. First, the rigidity matrix must be computed with a quadrature rule applied on the fine mesh. Second, the numerical solution must be plotted on the fine mesh. This is very important if one want to see the fine oscillations which are incorporated in the finite element basis functions.

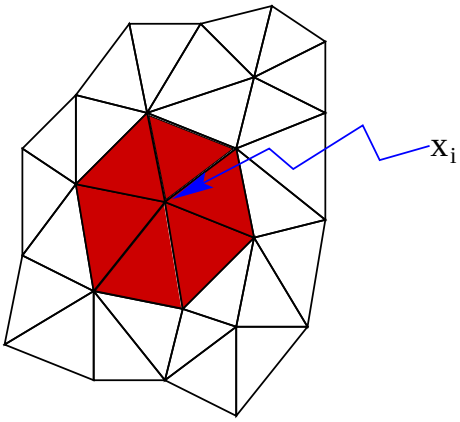

FIGURE 9. Support of the finite element basis function ϕ_i^ϵ.

In the periodic case, the following convergence result has been proved [50], [51]: there exists a constant $C > 0$ such that

$$\|u_\epsilon - u_\epsilon^h\|_{H_0^1(\Omega)} \leq C\left(h + \sqrt{\frac{\epsilon}{h}}\right). \tag{140}$$

It is clear that estimate (140) is interesting only when $h >> \epsilon$, and that there is a resonance effect when h is of the same order as ϵ. The main idea in the proof of estimate (140) is to use asymptotic expansions of the type of (129) for u_ϵ and for each basis function ϕ_i^ϵ.

There is a variant of this method [51], called the multi scale finite element method *with oversampling*, which improves the estimate (140) by getting rid of the square root. This generalization amounts to compute the base functions $\phi_{i,K}^\epsilon$ on a fine mesh K' which is slightly larger than K, namely $K \subset\subset K'$. The method becomes non-conforming since the different functions $\phi_{i,K}^\epsilon$ do not match at the interface between neighbouring cells, but this variant suppresses all boundary layers effects near the cell boundaries ∂K.

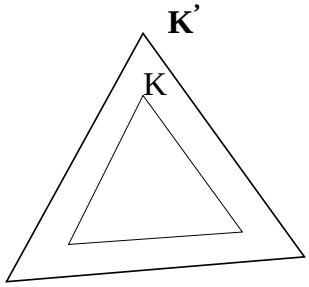

FIGURE 10. Fine mesh for the oversampling method.

Another variant [10] starts by recognizing that the approximation (129) is also equivalent (by a first order Taylor expansion) to

$$u_\epsilon(x) \approx u\left(x + \epsilon w\left(\frac{x}{\epsilon}\right)\right) \quad \text{with} \quad w = (w_i)_{1\leq i\leq N}. \tag{141}$$

Formula (141) suggests a change of variables. Let $(\psi_i)_{1\leq i\leq N_h}$ be a standard finite element basis on the coarse mesh (for example, P_k Lagrange finite elements). This basis is suitable for approximating the homogenized solution u. Then, by composing it with oscillating functions of the type $x + \epsilon w\left(\frac{x}{\epsilon}\right)$, we should obtain a good basis to approximate u_ϵ.

For example, taking a combination of the oscillating functions $\phi^\epsilon_{i,K}$, defined by (137), we build a vector ϕ^ϵ_K solution of

$$\left\{ \begin{array}{ll} -\mathrm{div}\left(A^\epsilon \nabla \phi^\epsilon_K\right) = 0 & \text{in } K \\ \phi^\epsilon_K = x & \text{on } \partial K. \end{array} \right. \tag{142}$$

As already remarked, ϕ^ϵ_K is similar to $x + \epsilon w\left(\frac{x}{\epsilon}\right)$ (in the periodic case). Thus, by composition, we define an oscillating finite element basis

$$\psi^\epsilon_i(x) = \psi_i\left(\phi^\epsilon_K(x)\right) \quad \text{in each cell } K. \tag{143}$$

Finally, this basis $(\psi^\epsilon_i)_{1\leq i\leq N_h}$ is used to compute an approximate solution of u_ϵ.

If the coarse basis $(\psi_i)_{1\leq i\leq N_h}$ were P_1 (piecewise affine) finite elements, then this method is exactly that proposed by T. Hou in [50]. However, the advantages of this new method are the following.

(1) One can take higher order elements for $(\psi_i)_{1\leq i\leq N_h}$, thus improving the convergence rate. More precisely, in estimate (140) the term h can be replaced by h^k if P_k Lagrange finite elements are used.
(2) The idea can be generalized to the non-periodic case.
(3) One can change the definition of ϕ^ϵ_K, and do, for example, an oversampling method: this will still yield a conforming method.

Remark 6.1. *Roughly speaking, the method of Arbogast [15] amounts to replace the Dirichlet boundary condition in (137) by a Neumann boundary condition (but it works for mixed finite elements), while the method of Babuska and Schwab [64] uses periodic boundary conditions.*

6.3. **Factorization method in neutronics**

In this subsection we describe another method of numerical homogenization which does not rely on the usual model problem (136) but rather on the following model of reaction-diffusion equation

$$\left\{ \begin{array}{ll} c\left(\frac{x}{\epsilon}\right)\frac{\partial u_\epsilon}{\partial t} - \epsilon^2 \mathrm{div}\left(D\left(\frac{x}{\epsilon}\right)u_\epsilon\right) = \sigma\left(\frac{x}{\epsilon}\right)u_\epsilon & \text{in } \Omega\times\mathbb{R}^+_* \\ u_\epsilon = 0 & \text{on } \partial\Omega\times\mathbb{R}^+_* \\ u_\epsilon(0) = u_0 & \text{in } \Omega. \end{array} \right. \tag{144}$$

This model, as well as the following numerical method, is frequently used in nuclear reactor physics (or neutronics, see e.g. [9], [11], [13]). Let us note that it works for a transport model (linear Boltzmann equation) too [56], [79].

Recall from Subsection 4.2 that the asymptotic behavior of (144) is given by

$$u_\epsilon(t,x) \approx e^{-\lambda t} w\left(\frac{x}{\epsilon}\right) u\left(\epsilon^2 t, x\right), \tag{145}$$

where (λ, w) is the first eigencouple of the following cell spectral problem

$$\left\{ \begin{array}{ll} -\lambda c(y)w - \mathrm{div}\left(D(y)w\right) = \sigma(y)w & \text{in } Y \\ y \to w(y) & Y-\text{periodic}, \end{array} \right. \tag{146}$$

and u is the solution of the homogenized problem

$$\left\{ \begin{array}{ll} \overline{c}\frac{\partial u}{\partial \tau} - \mathrm{div}\left(\overline{D}u\right) = 0 & \text{in } \Omega\times\mathbb{R}^+_* \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+_* \\ u(0) = \overline{u}_0 & \text{in } \Omega. \end{array} \right. \tag{147}$$

We refer to Theorem 4.4 for the precise values of the homogenized coefficients in (147). The asympotic result (145) allows to compute a correct approximation of the true solution u_ϵ without using a fine mesh. As an

example, we reproduce a result of [11] in Figure 11 (to which we refer for a precise description of the test case). It is a one-dimensional result (with 20 cells) comparing the true solution u_ϵ and the right hand side of (145). The solution is plotted for a very large time (thus, its spatial profile is given by the first eigenfunction).

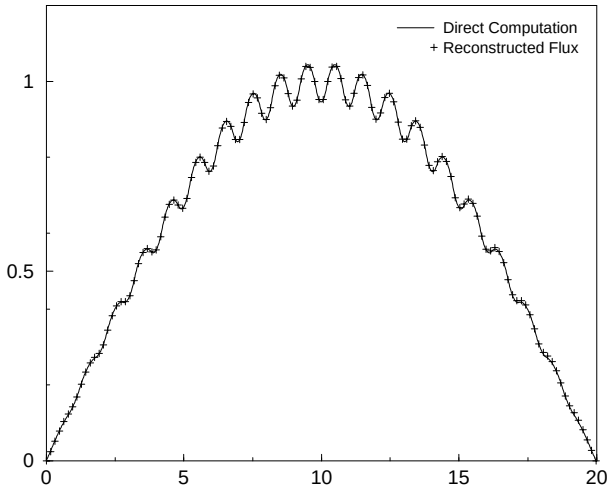

FIGURE 11. Comparison of the left and right hand sides of (145).

As another example of the application of (145) we display a two-dimensional result of V. Siess [80] in Figure 12.

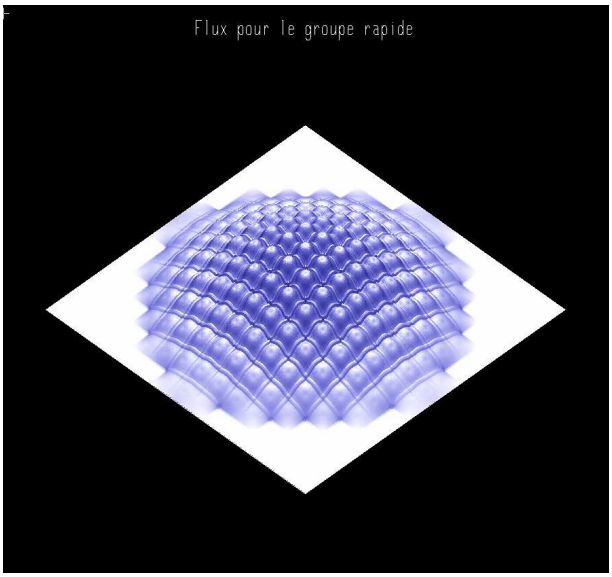

FIGURE 12. Reconstructed solution with (145).

Although the asympotic result (145) is restricted to the periodic setting, it can be used to derive a multiscale numerical method in the non-periodic case (see e.g. [25], [40], [81]). When the coefficients in (144) are not

periodic, one can still compute a cell spectral problem (146) for each cell of the domain Ω. Then, in the spirit of (145), a change of unknown is performed

$$v_\epsilon\left(\epsilon^2 t, x\right) = \frac{u_\epsilon(t,x)e^{\lambda t}}{w\left(\frac{x}{\epsilon}\right)},$$

and a standard discretization scheme is applied to the new function v_ϵ (the coefficients in the equation for v_ϵ are simply averaged in each cell). Remark that the function v_ϵ is not continuous through the interfaces between cells. Therefore, if a finite element method is used for v_ϵ, it should be a non conforming one.

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