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A BENDING AND STRETCHING ASYMPTOTIC THEORY FOR GENERAL ELASTIC SHALLOW ARCHES

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Abstract. We present a bending model for a shallow arch, namely the type of curved rod where the curvature is of the order of the diameter of the cross section. The model is deduced in a rigorous mathematical way from classical tridimensional linear elasticity theory via asymptotic techniques, by taking the limit on a suitable re-scaled formulation of that problem as the diameter of the cross section tends to zero. This model is valid for general cases of applied forces and material, and it allows us to calculate displacements, axial stresses, bending moments and shear forces. The equations present a more general form than in the classical Bernoulli-Navier bending theory for straight slender rods, so that flexures and extensions are proved to be coupled in the most general case.

Key words: elasticity, shallow arches, asymptotic methods, curved rods
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1. Introduction

Strictly speaking, we call a *rod* any three-dimensional solid occupying the volume generated by a planar connected domain when its mass center is dragged orthogonally along a space curve (the axis or centerline of the rod). The plane domain is called the cross section of the rod. Its essential characteristic is the fact that the diameter of the cross section should be much smaller than the length of the centerline (a ratio of 1 : 10 or less is admitted). We shall say the rod is straight whenever its axis is a straight line segment; a straight rod with constant cross section is usually called a prismatic rod: otherwise we shall just talk about *curved rods*.

In this work we are concerned with a particular class of curved rods. More to the point, we call *weakly curved rods* or *shallow arches* those characterized by the fact that the curvature of their centerline should have the same order of magnitude than the diameter of the cross section, both being much smaller than their length. A more precise definition will be given later on in the work.

The results contained in this work were already announced in [4, 5]. More precisely, here we undertake the aim of obtaining an asymptotic approximation to the tridimensional linear elasticity model, for *weakly curved rods*. This work is then a continuation and extension of previous results by [2, 3] on the asymptotic approximation of slender straight linear elastic rods. Below we summarize up the utilized technique:

(a) By means of a change of variable and a change of scale of displacements and stresses, the linearized three-dimensional elasticity problem posed in the curved beam is converted into an equivalent re-scaled problem posed in a straight reference beam (see[2, 3]).

(b) Then we aim to study the limit behaviour of unknowns (displacements and stress) of the re-scaled problem, as the diameter of the cross section goes to zero. Thus we characterize a limit problem which yields the approximated model in the reference domain. It is at this stage that the weak curvature assumption plays a crucial role.

(c) By undoing the change of variable in the limit model, we obtain the corresponding model for the actual curved rod.

This procedure has already been used in earlier works, most notably in [6, 7], where this method leads to the mathematical justification of the two-dimensional equations of a linear elastic shallow shell.

2. Posing the three-dimensional problem

Greek indices will take values in the set $\{1, 2\}$, whereas Latin indices will take values in $\{1, 2, 3\}$. The summation convention on repeated indices will also be used.

Let ω be an open connected bounded set on plane Ox_1x_2, having area equal to A and a Lipschitzian boundary γ. We shall suppose that Ox_1x_2 is a principal system of inertia for ω. For all ε such that $0 < \varepsilon \leq 1$ and for any given $L > 0$ ($\varepsilon \ll L$) we define:

$$\omega^\varepsilon = \varepsilon\omega, \gamma^\varepsilon = \partial\omega^\varepsilon, \Omega^\varepsilon = \omega^\varepsilon \times (0, L), \tag{1}$$

$$\Gamma_0^\varepsilon = \omega^\varepsilon \times \{0\}, \Gamma_L^\varepsilon = \omega^\varepsilon \times \{L\}, \Gamma^\varepsilon = \gamma^\varepsilon \times (0, L) \tag{2}$$

and we note $\partial_\alpha^\varepsilon = \partial/\partial x_\alpha^\varepsilon$, $\partial_3^\varepsilon = \partial_3 = \partial/\partial x_3$ and for any function ξ depending only on x_3 we shall note by ξ', ξ'', ξ''', ... the corresponding derivatives of ξ. We shall leave out superscript ε when $\varepsilon = 1$: $\Omega = \Omega^1$, $\Gamma = \Gamma^1$, $\Gamma_0 = \Gamma_0^1, \ldots$ A generic point in $\bar{\Omega}^\varepsilon$ will be noted by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3)$. For every ε we define a curve C^ε in the space $Ox_1^\varepsilon x_2^\varepsilon x_3$ of the form

$$C^\varepsilon = \{\theta^\varepsilon(x_3) = (\theta_1^\varepsilon(x_3), \theta_2^\varepsilon(x_3), x_3) \in I\!R^3 : \ x_3 \in [0, L]\} \tag{3}$$

where $\theta_\alpha^\varepsilon(x_3)$ are given functions verifying $\theta_\alpha^\varepsilon \in \mathbf{C}^3[0, L]$. Let $(t^\varepsilon, n^\varepsilon, b^\varepsilon)$ be the Frenet trihedron associated to the curve C^ε ([1]). From now on we suppose that n^ε belongs to $\mathbf{C}^1[0, L]$, so that the curve C^ε is smooth. This hypothesis is satisfied if $\theta_1^{\varepsilon\prime\prime}$ and $\theta_2^{\varepsilon\prime\prime}$ do not vanish at the same time (which is equivalent to the fact that the curvature of C^ε be strictly positive for any $x_3 \in [0, L]$). The case where C^ε has null curvature points (e. g. a straight chunk) can be treated in the same fashion, provided that we suppose that along these points we have the same degree of smoothness as before, with $t^\varepsilon, n^\varepsilon, b^\varepsilon$ appropriately chosen. We define the map $\varphi^\varepsilon : \bar{\Omega}^\varepsilon \longrightarrow \varphi^\varepsilon(\bar{\Omega}^\varepsilon) = \hat{\Omega}^\varepsilon \subset I\!R^3$ in the following manner:

$$\varphi^\varepsilon(x^\varepsilon) = (\theta_1^\varepsilon(x_3), \theta_2^\varepsilon(x_3), x_3) + x_1^\varepsilon n^\varepsilon(x_3) + x_2^\varepsilon b^\varepsilon(x_3) \tag{4}$$

and we assume that φ^ε is a $\mathbf{C}^1$-diffeomorphism (which can be proved if ε is small enough for functions $\theta_\alpha^\varepsilon$ herein considered). The body having $\{\hat{\Omega}^\varepsilon\}^-$ as its reference configuration is called a curved rod of axis C^ε. Let us bear in mind that the area of the cross section is of the order ε^2. A generic point of $\{\hat{\Omega}^\varepsilon\}^-$ will be denoted by $\hat{x}^\varepsilon = (\hat{x}_i^\varepsilon) = \varphi^\varepsilon(x^\varepsilon)$, and we set $\hat{\partial}_i^\varepsilon = \partial/\partial\hat{x}_i^\varepsilon$. Rod $\hat{\Omega}^\varepsilon$ is considered to be made of an elastic material having Young's modulus E^ε and Poisson's ratio ν^ε. The rod is acted on by volume forces $\hat{f}^\varepsilon \in [L^2(\hat{\Omega}^\varepsilon)]^3$ and surface ones $\hat{g}^\varepsilon \in [L^2(\hat{\Gamma}^\varepsilon)]^3$. Both ends $\hat{\Gamma}_0^\varepsilon = \varphi^\varepsilon(\Gamma_0^\varepsilon)$, $\hat{\Gamma}_L^\varepsilon = \varphi^\varepsilon(\Gamma_L^\varepsilon)$ are supposed to be clamped. Other limit conditions could have been tackled with a similar approach. Denoting by $\hat{u}^\varepsilon = (\hat{u}_i^\varepsilon) : \{\hat{\Omega}^\varepsilon\}^- \longrightarrow$

$I\!R^3$ the displacement field and by $\hat{\sigma}^\varepsilon = (\hat{\sigma}^\varepsilon_{ij}) : \{\hat{\Omega}^\varepsilon\}^- \longrightarrow I\!R^9_s$ the stress field, we have the following mixed variational problem in linear elasticity:

$$\hat{\sigma}^\varepsilon \in \Sigma(\hat{\Omega}^\varepsilon) := \{\hat{\tau}^\varepsilon = (\hat{\tau}^\varepsilon_{ij}) \in [L^2(\hat{\Omega}^\varepsilon)]^9 : \ \hat{\tau}^\varepsilon_{ij} = \hat{\tau}^\varepsilon_{ji}\} \tag{5}$$

$$\hat{u}^\varepsilon \in V(\hat{\Omega}^\varepsilon) := \{\hat{v}^\varepsilon = (\hat{v}^\varepsilon_i) \in [H^1(\hat{\Omega}^\varepsilon)]^3 : \ \hat{v}^\varepsilon = 0 \text{ on } \hat{\Gamma}^\varepsilon_0 \cup \hat{\Gamma}^\varepsilon_L\} \tag{6}$$

$$\int_{\hat{\Omega}^\varepsilon} \{\frac{1+\nu^\varepsilon}{E^\varepsilon}\hat{\sigma}^\varepsilon_{ij} - \frac{\nu^\varepsilon}{E^\varepsilon}\hat{\sigma}^\varepsilon_{pp}\delta_{ij}\}\hat{\tau}^\varepsilon_{ij}\,d\hat{x}^\varepsilon = \int_{\hat{\Omega}^\varepsilon} \hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon)\hat{\tau}^\varepsilon_{ij}\,d\hat{x}^\varepsilon \text{ for all } \hat{\tau}^\varepsilon \in \Sigma(\hat{\Omega}^\varepsilon) \tag{7}$$

$$\int_{\hat{\Omega}^\varepsilon} \hat{\sigma}^\varepsilon_{ij}\hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon)\,d\hat{x}^\varepsilon = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_i\hat{v}^\varepsilon_i\,d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon} \hat{g}^\varepsilon_i\hat{v}^\varepsilon_i\,d\hat{a}^\varepsilon \text{ for all } \hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon) \tag{8}$$

where

$$\hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_i) \tag{9}$$

are the linear stress tensor components.

3. Equivalent formulation to the three-dimensional problem on the reference open set Ω

In order to study the behaviour of $(\hat{\sigma}^\varepsilon, \hat{u}^\varepsilon)$ when the area of the cross section becomes small, we follow the same trail as in the asymptotic analysis of straight rods ([2, 3]) and the analogous work on shallow shells ([6, 7]): changing the variable to a fixed open set, scaling of unknowns, assuming certain hypotheses on forces and passing to the limit. For all $x = (x_i) \in \bar{\Omega}$ we set $x^\varepsilon = \pi^\varepsilon(x) = (\varepsilon x_1, \varepsilon x_2, x_3) \in \bar{\Omega}^\varepsilon$ et $\hat{x}^\varepsilon = \varphi^\varepsilon(x^\varepsilon) = \varphi^\varepsilon(\pi^\varepsilon x)$ and we define the scaled displacement and stresses $u(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \longrightarrow I\!R^3$ et $\sigma(\varepsilon) = (\sigma_{ij}(\varepsilon)) : \bar{\Omega} \longrightarrow I\!R^9_s$ by the following equations:

$$\begin{aligned} &u_\alpha(\varepsilon)(x) = \varepsilon\hat{u}^\varepsilon_\alpha(\hat{x}^\varepsilon), u_3(\varepsilon)(x) = \hat{u}^\varepsilon_3(\hat{x}^\varepsilon) \\ &\sigma_{\alpha\beta}(\varepsilon)(x) = \varepsilon^{-2}\hat{\sigma}^\varepsilon_{\alpha\beta}(\hat{x}^\varepsilon), \sigma_{3\beta}(\varepsilon)(x) = \varepsilon^{-1}\hat{\sigma}^\varepsilon_{3\beta}(\hat{x}^\varepsilon), \sigma_{33}(\varepsilon)(x) = \hat{\sigma}^\varepsilon_{33}(\hat{x}^\varepsilon) \end{aligned} \tag{1}$$

We now suppose that there exist constants E, ν *not depending on* ε, functions $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma)$ and $\theta_\alpha \in \mathbf{C}^3[0, L]$ *independent of* ε, and an arbitrary real number t, such that:

$$E^\varepsilon = \varepsilon^t E, \ \nu^\varepsilon = \varepsilon^t \nu \tag{2}$$

$$\hat{f}^\varepsilon_\alpha(\hat{x}^\varepsilon) = \varepsilon^{t+1}f_\alpha(x), \hat{f}^\varepsilon_3(\hat{x}^\varepsilon) = \varepsilon^t f_3(x), \text{ for all } \hat{x}^\varepsilon = \varphi^\varepsilon(\pi^\varepsilon x),\ x \in \Omega \tag{3}$$

$$\hat{g}^\varepsilon_\alpha(\hat{x}^\varepsilon) = \varepsilon^{t+2}g_\alpha(x), \hat{g}^\varepsilon_3(\hat{x}^\varepsilon) = \varepsilon^{t+1}g_3(x) \text{ for all } \hat{x}^\varepsilon = \varphi^\varepsilon(\pi^\varepsilon x),\ x \in \Gamma \tag{4}$$

$$\theta^\varepsilon_\alpha(x_3) = \varepsilon\theta_\alpha(x_3) \text{ for all } x_3 \in [0, L]. \tag{5}$$

Hypothesis (5) implies that the curvature of C^ε is of the order ε, which can be considered to be a definition of a weakly curved rod.

With (1)–(5), problem (5)–(8) changes into a problem posed in the fixed domain Ω of the following form:

$$\begin{aligned} &\sigma(\varepsilon) \in \Sigma(\Omega),\ u(\varepsilon) \in V(\Omega) \\ &a_0(\sigma(\varepsilon), \tau) + \varepsilon^2 a_2(\sigma(\varepsilon), \tau) + \varepsilon^4 a_4(\sigma(\varepsilon), \tau) + b(\tau, u(\varepsilon)) + \\ &\qquad a^\#(\varepsilon, \theta)(\sigma(\varepsilon), \tau) + b^\#(\varepsilon, \theta)(\tau, u(\varepsilon)) = 0 \text{ for all } \tau \in \Sigma(\Omega) \\ &b(\sigma(\varepsilon), v) + b^\#(\varepsilon, \theta)(\sigma(\varepsilon), v) = F(v) + F^\#(\varepsilon, \theta)(v) \text{ for all } v \in V(\Omega) \end{aligned} \tag{6}$$

where the bilinear forms $a_0(\cdot,\cdot)$, $a_2(\cdot,\cdot)$, $a_4(\cdot,\cdot)$ and the linear form $F(\cdot)$ are given by

$$a(\sigma,\tau) = \int_\Omega \frac{1}{E}\sigma_{33}\tau_{33}\,dx, \tag{7}$$

$$a_2(\sigma,\tau) = \int_\Omega \left[\frac{2(1+\nu)}{E}\sigma_{3\beta}\tau_{3\beta} - \frac{\nu}{E}(\sigma_{33}\tau_{\mu\mu} + \sigma_{\mu\mu}\tau_{33})\right] dx, \tag{8}$$

$$a_4(\sigma,\tau) = \int_\Omega \left[\frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\mu\mu}\delta_{\alpha\beta}\right]\tau_{\alpha\beta}\,dx, \tag{9}$$

$$b(\tau,v) = -\int_\Omega \tau_{ij}e_{ij}(\varepsilon)(v)\,dx, \tag{10}$$

$$F(v) = -\int_\Omega f_i v_i\,dx - \int_\Gamma g_i v_i\,da. \tag{11}$$

and forms $a^{\#}(\varepsilon,\theta)(\cdot,\cdot)$, $b^{\#}(\varepsilon,\theta)(\cdot,\cdot)$ and $F^{\#}(\varepsilon,\theta)(\cdot)$ represent "remainders" of order ≥ 2 with respect to ε ([4]).

4. Convergence when ε tends to zero: first order general model

The main result in this work is contained in the following theorem, the proof of which can be found in [4]. We introduce the following notations, depending on the geometric characteristics of the cross section and the centerline, supposing for simplicity that $\theta_1''\theta_2''(x_3) \neq 0$, for all $x_3 \in [0,L]$:

$$b_\alpha(x_3) = \theta_\alpha''(x_3)/[(\theta_1''(x_3))^2 + (\theta_2''(x_3))^2]^{1/2}, \tag{1}$$

$$\chi_1^b(x) = x_1 b_1(x_3) - x_2 b_2(x_3),\ \chi_2^b(x) = x_1 b_2(x_3) + x_2 b_1(x_3). \tag{2}$$

Theorem 1. *Under hypotheses (1)–(5), we have the following results:*

i) When ε tends to zero, families $(u(\varepsilon))_{\varepsilon>0}$ and $(\sigma(\varepsilon))_{\varepsilon>0}$ satisfy

$$u_i(\varepsilon) \to u_i^0 \text{ in } H^1(\Omega),\ \sigma_{33}(\varepsilon) \to \sigma_{33}^0 \text{ in } L^2(\Omega), \tag{3}$$

$$\varepsilon\sigma_{3\beta}(\varepsilon) \to 0 \text{ in } L^2(\Omega),\ \varepsilon^2\sigma_{\alpha\beta}(\varepsilon) \to 0 \text{ in } L^2(\Omega). \tag{4}$$

ii) Element $u^0 = \lim_{\varepsilon\to 0} u(\varepsilon)$ is a generalized Bernoulli-Navier field, namely u^0 is of the form

$$u_\alpha^0(x) = \xi_\alpha(x_3),\ u_3^0(x) = \xi_3(x_3) - \chi_\beta^b(x)\xi_\beta'(x_3),\ \xi_\alpha \in H_0^2(0,L), \xi_3 \in H_0^1(0,L). \tag{5}$$

iii) Element $\sigma_{33}^0 \in L^2(\Omega)$ is of the form

$$\sigma_{33}^0 = E(\xi_3' - \chi_\alpha^b\xi_\alpha'' + \theta_\alpha'\xi_\alpha'). \tag{6}$$

iv) Element $(\xi_1,\xi_2,\xi_3) \in [H_0^2(0,L)]^2 \times H_0^1(0,L)$ is the only solution to the following variational problem in $(0,L)$:

$$-\int_0^L m_\beta^b\zeta_\beta''\,dx_3 + \int_0^L n_3\theta_\beta'\zeta_\beta'\,dx_3 = \int_0^L F_\beta\zeta_\beta\,dx_3 - \int_0^L M_\beta^b\zeta_\beta'\,dx_3 \quad \text{for all } (\zeta_\beta) \in [H_0^2(0,L)]^2, \tag{7}$$

$$\int_0^L n_3\zeta_3'\,dx_3 = \int_0^L F_3\zeta_3\,dx_3,\ \text{for all } \zeta_3 \in H_0^1(0,L),$$

where

$$m^b_\beta = \int_\omega \chi^b_\beta \sigma^0_{33}\, d\omega = -EI^b_{\alpha\beta}\xi''_\alpha,\ I^b_{\alpha\beta} = \int_\omega \chi^b_\alpha \chi^b_\beta\, d\omega \tag{8}$$

$$n_3 = \int_\omega \sigma^0_{33}\, d\omega = EA(\xi'_3 + \theta'_\alpha \xi'_\alpha) \tag{9}$$

$$F_i = \int_\omega f_i\, d\omega + \int_\gamma g_i\, d\gamma;\ M^b_\beta = \int_\omega \chi^b_\beta f_3\, d\omega + \int_\gamma \chi^b_\beta g_3\, d\gamma. \tag{10}$$

v) For the bending moment components $m^b_\beta(\varepsilon)$ and the shear force components $q_\beta(\varepsilon)$ when ε tends to zero we have:

$$m^b_\beta(\varepsilon) := \int_\omega \chi^b_\beta \sigma_{33}(\varepsilon)\, d\omega \longrightarrow m^b_\beta \text{ in } L^2(0,L) \tag{11}$$

$$q_\beta(\varepsilon) := \int_\omega \sigma_{3\beta}(\varepsilon)\, d\omega \rightharpoonup q^0_\beta = \left(m^b_\beta\right)' + n_3\theta'_\beta + M^b_\beta \text{ in } L^2(0,L) \text{ weakly.} \tag{12}$$

Sketch of the proof: We begin by establishing that the sequence $(u(\varepsilon))_{\varepsilon>0}$ is bounded in $[H^1(\Omega)]^3$, and that sequences $(\sigma_{33}(\varepsilon))_{\varepsilon>0}$, $\varepsilon\,(\sigma_{3\beta}(\varepsilon))_{\varepsilon>0}$ and $\varepsilon^2\,(\sigma_{\alpha\beta}(\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\Omega)$ independently of ε. Therefore we can extract weakly convergent subsequences, and we show that these limits satisfy equations (5)–(7) passing to the limit in variational equations (6). Then we establish the strong convergence. We follow a similar pattern to the straight rod case ($\theta_\alpha = 0$) [2, 3, 5] but several difficulties crop up, most remarkably to take remainders $a^\#(\varepsilon,\theta)$, $b^\#(\varepsilon,\theta)$ and $F^\#(\varepsilon,\theta)$ into account, and to obtain the generalized Bernoulli-Navier displacement form and consequently existence and uniqueness of the solution of problem (7). It is then that we resort to the techniques in [6]. Particularly, the following lemma is of the utmost importance:

LEMMA 2. *Let $\theta^\varepsilon \in \mathbf{C}^3(0,L;I\!R^3)$ be such that its Frenet trihedron $(t^\varepsilon, n^\varepsilon, b^\varepsilon)$ is a positively oriented orthonormal basis of $I\!R^3$ and satisfies Frenet equations for curvature κ^ε and torsion τ^ε ([1]). If $\theta^\varepsilon_\alpha$ satifies (3.5) then for all $\varepsilon > 0$ we have:*

$$\begin{aligned}
\{\mathbf{A}^\varepsilon\}^a &= 1 + \varepsilon^2 s_0(\varepsilon,\theta)\ (a \in I\!R \text{ arbitrary}), && (13)\\
\kappa^\varepsilon &= \varepsilon\left\{c + \varepsilon^2 s_1(\varepsilon,\theta)\right\}, && (14)\\
\tau^\varepsilon &= d + \varepsilon^2 s_2(\varepsilon,\theta), && (15)\\
n^\varepsilon_1 &= b_1 + \varepsilon^2 s_3(\varepsilon,\theta), && (16)\\
n^\varepsilon_2 &= b_2 + \varepsilon^2 s_4(\varepsilon,\theta), && (17)\\
n^\varepsilon_3 &= \varepsilon\left\{h_1 + \varepsilon^2 s_5(\varepsilon,\theta)\right\} && (18)\\
b^\varepsilon_1 &= -b_2 + \varepsilon^2 s_6(\varepsilon,\theta), && (19)\\
b^\varepsilon_2 &= b_1 + \varepsilon^2 s_7(\varepsilon,\theta), && (20)\\
b^\varepsilon_3 &= \varepsilon\left\{h_2 + \varepsilon^2 s_8(\varepsilon,\theta)\right\}, && (21)\\
t^\varepsilon_1 &= \varepsilon\{t_1 + \varepsilon^2 s_9(\varepsilon,\theta)\}, && (22)\\
t^\varepsilon_2 &= \varepsilon\{t_2 + \varepsilon^2 s_{10}(\varepsilon,\theta)\}, && (23)\\
t^\varepsilon_3 &= t_3 + \varepsilon^2 s_{11}(\varepsilon,\theta) && (24)
\end{aligned}$$

where s_i, $i = 0,..,11$ are uniformly bounded constants on $\varepsilon > 0$:

$$\sup_{0<\varepsilon} \max_{x_3\in[0,L]} |s_i(\varepsilon,\theta)(x_3)| < +\infty, \tag{25}$$

and also

$$c=\sqrt{(\theta_1'')^2+(\theta_2'')^2}, \tag{26}$$

$$h_1=\frac{-\theta_1'\theta_1''+\theta_2'\theta_2''}{c},\ h_2=\frac{\theta_1'\theta_2''-\theta_2'\theta_1''}{c}, \tag{27}$$

$$t_\alpha=\theta_\alpha',\ t_3=1, \tag{28}$$

$$d=\frac{\theta_1''\theta_2'''-\theta_2''\theta_1'''}{c^2}. \tag{29}$$

5. The obtained model

Problem (7) is equivalent to the following differential problem:

$$E(I^b_{\alpha\beta}\xi_\alpha'')''-AE[\theta_\beta'(\xi_3'+\theta_\alpha'\xi_\alpha')]'=F_\beta+M_\beta^{b\prime}\text{ in }(0,L), \tag{30}$$
$$-AE[\xi_3'+\theta_\alpha'\xi_\alpha']'=F_3\text{ in }(0,L),$$
$$\xi_i(0)=\xi_i(L)=\xi_\alpha'(0)=\xi_\alpha'(L)=0.$$

The preceding equations amount to being the base of the general first order model for the curved rod, which is valid regardless of the applied forces, and has been obtained with no *a priori* hypotheses. The equations are written in terms of the limit quantities on the reference rod Ω; in order to obtain the approximate model for $\hat{\Omega}^\varepsilon$ $(\varepsilon>0)$ we have to come back to quantities $\hat{u}^{0\varepsilon}$ and $\hat{\sigma}^{0\varepsilon}_{33}$ defined in $\hat{\Omega}^\varepsilon$ by (cf. (1)):

$$\hat{u}^{0\varepsilon}_\alpha(\hat{x}^\varepsilon)=\varepsilon^{-1}u^0_\alpha(x),\ \hat{u}^{0\varepsilon}_3(\hat{x}^\varepsilon)=u^0_3(x),\ \hat{\sigma}^{0\varepsilon}_{33}(\hat{x}^\varepsilon)=\sigma^0_{33}(x). \tag{31}$$

From (5) and (6) we obtain that $\hat{u}^{0\varepsilon}$ and $\hat{\sigma}^{0\varepsilon}_{33}$ are of the following form:

$$\hat{u}^{0\varepsilon}_\alpha(\hat{x}^\varepsilon)=\hat{\xi}^\varepsilon_\alpha(x_3),\ \hat{u}^{0\varepsilon}_3(\hat{x}^\varepsilon)=\hat{\xi}^\varepsilon_3(x_3)-\chi^{b,\varepsilon}_\beta(\hat{x}^\varepsilon)\hat{\xi}^{\varepsilon\prime}_\beta(x_3) \tag{32}$$

$$\hat{\sigma}^{0\varepsilon}_{33}(\hat{x}^\varepsilon)=E^\varepsilon[\hat{\xi}^{\varepsilon\prime}_3(x_3)-\chi^{b,\varepsilon}_\alpha(\hat{x}^\varepsilon)\hat{\xi}^{\varepsilon\prime\prime}_\alpha(x_3)+\theta^{\varepsilon\prime}_\alpha(x_3)\hat{\xi}^{\varepsilon\prime}_\alpha(x_3)] \tag{33}$$

$$\chi^{b,\varepsilon}_1(\hat{x}^\varepsilon)=x^\varepsilon_1b_1(x_3)-x^\varepsilon_2b_2(x_3),\ \chi^{b,\varepsilon}_2(\hat{x}^\varepsilon)=x^\varepsilon_1b_2(x_3)+x^\varepsilon_2b_1(x_3), \tag{34}$$

where $\hat{\xi}^\varepsilon_\alpha=\varepsilon^{-1}\xi_\alpha$, $\hat{\xi}^\varepsilon_3=\xi_3$. Therefore $\hat{\xi}^\varepsilon_\alpha$ and $\hat{\xi}^\varepsilon_3$ satisfy differential equations of the same type as (30):

$$E^\varepsilon(I^{b,\varepsilon}_{\alpha\beta}\hat{\xi}^{\varepsilon\prime\prime}_\alpha)''-E^\varepsilon A^\varepsilon[\theta^{\varepsilon\prime}_\beta(\hat{\xi}^{\varepsilon\prime}_3+\theta^{\varepsilon\prime}_\alpha\hat{\xi}^{\varepsilon\prime}_\alpha)]'=F^\varepsilon_\beta+(M^{b,\varepsilon}_\beta)'\text{ in }(0,L), \tag{35}$$

$$-E^\varepsilon A^\varepsilon[\hat{\xi}^{\varepsilon\prime}_3+\theta^{\varepsilon\prime}_\alpha\hat{\xi}^{\varepsilon\prime}_\alpha]'=F^\varepsilon_3\text{ in }(0,L), \tag{36}$$

$$\hat{\xi}^\varepsilon_i(0)=\hat{\xi}^\varepsilon_i(L)=0\text{ and }\hat{\xi}^{\varepsilon\prime}_\alpha(0)=\hat{\xi}^{\varepsilon\prime}_\alpha(L)=0. \tag{37}$$

where

$$F^\varepsilon_i=\int_{\omega^\varepsilon}f^\varepsilon_i\,d\omega^\varepsilon+\int_{\gamma^\varepsilon}g^\varepsilon_i\,d\gamma^\varepsilon,\ M^{b,\varepsilon}_\beta=\int_{\omega^\varepsilon}\chi^{b,\varepsilon}_\beta f^\varepsilon_3\,d\omega^\varepsilon+\int_{\gamma^\varepsilon}\chi^{b,\varepsilon}_\beta g^\varepsilon_3\,d\gamma^\varepsilon, \tag{38}$$

$$I^{b,\varepsilon}_{\alpha\beta}=\int_{\omega^\varepsilon}\chi^{b,\varepsilon}_\alpha\chi^{b,\varepsilon}_\beta \tag{39}$$

We remark that the model here found is indeed a generalization of simpler models found in literature on shallow arch theory. In this way we have simultaneously justified *a priori* assumptions of a geometrical nature by showing the "limit displacement" field $\hat{u}^{0\varepsilon}$ is a Bernoulli-Navier field in the sense (32). As in the shallow shell case studied in [6], no confusion arises between variables $x^\varepsilon\in\Omega^\varepsilon$ and $\hat{x}^\varepsilon\in\hat{\Omega}^\varepsilon$, carefully distinguished throughout the de-scaling process.

6. Some examples

Further discussion of shallow arch theory would involve the study of how well the obtained model ties in with classical theories. However we have not met with the same success as in classical straight rod theories (see [2, 3, 4] among others). The long and short of it is that whenever we have tried to look into classical shallow arch theory, we feel the authors make use of simplifying *a priori* hypotheses wherever they meet lengthy or unwieldy calculations, which makes the whole theory appear rather mystifying from the mathematician's viewpoint, though perhaps not so from the engineer's.

Nevertheless we shall verify that our asymptotic curved rod theory is consistent with asymptotic straight rod theory. Thus we shall examine the particular case of a planar centerline. Let us so suppose that $\theta_2^\varepsilon \equiv 0$ for example, so that $\mathbf{C}^\varepsilon$ is a plane curve. Then the simplified model for the curved rod would be written as:

$$\begin{aligned}&E^\varepsilon I_1^\varepsilon(\hat{\xi}_1^\varepsilon)^{(4)} - E^\varepsilon A^\varepsilon[\theta_1^{\varepsilon\prime}(\hat{\xi}_3^{\varepsilon\prime} + \theta_1^{\varepsilon\prime}\hat{\xi}_1^{\varepsilon\prime})]' = F_1^\varepsilon + (M_1^\varepsilon)' \text{ in } (0,L),\\ &\hat{\xi}_1^\varepsilon(0) = \hat{\xi}_1^\varepsilon(L) = \hat{\xi}_1^{\varepsilon\prime}(0) = \hat{\xi}_1^{\varepsilon\prime}(L) = 0,\end{aligned} \tag{1}$$

$$\begin{aligned}&-E^\varepsilon A^\varepsilon[\hat{\xi}_3^{\varepsilon\prime} + \theta_1^{\varepsilon\prime}\hat{\xi}_1^{\varepsilon\prime}]' = F_3^\varepsilon \text{ in } (0,L),\\ &\hat{\xi}_3^\varepsilon(0) = \hat{\xi}_3^\varepsilon(L) = 0,\end{aligned} \tag{2}$$

$$\begin{aligned}&E^\varepsilon I_2^\varepsilon(\hat{\xi}_2^\varepsilon)^{(4)} = F_2^\varepsilon + (M_2^\varepsilon)' \text{ in } (0,L),\\ &\hat{\xi}_2^\varepsilon(0) = \hat{\xi}_2^\varepsilon(L) = \hat{\xi}_2^{\varepsilon\prime}(0) = \hat{\xi}_2^{\varepsilon\prime}(L) = 0,\end{aligned} \tag{3}$$

with

$$I_\alpha^\varepsilon = \int_{\omega^\varepsilon} (x_\alpha^\varepsilon)^2\, d\omega, \quad M_\alpha^\varepsilon = \int_{\omega^\varepsilon} x_\alpha^\varepsilon f_3^\varepsilon\, d\omega + \int_{\gamma^\varepsilon} x_\alpha^\varepsilon g_3^\varepsilon\, d\gamma. \tag{4}$$

Thus flexure $\hat{\xi}_2^\varepsilon$ is now decoupled from $\hat{\xi}_1^\varepsilon$ and stretch $\hat{\xi}_3^\varepsilon$, and the last two are coupled themselves.

In the case of a straight rod we would have $\theta_1^\varepsilon(x_3^\varepsilon) = a^\varepsilon x_1^\varepsilon$, where a^ε is a nonzero constant in general, and equals zero if system $O\hat{x}_1^\varepsilon\hat{x}_2^\varepsilon\hat{x}_3^\varepsilon$ is principal of inertia. In this last case we would have:

$$\begin{aligned}&E^\varepsilon I_1^\varepsilon(\hat{\xi}_1^\varepsilon)^{(4)} = F_1^\varepsilon + (M_1^\varepsilon)' \text{ in } (0,L),\\ &\hat{\xi}_1^\varepsilon(0) = \hat{\xi}_1^\varepsilon(L) = \hat{\xi}_1^{\varepsilon\prime}(0) = \hat{\xi}_1^{\varepsilon\prime}(L) = 0\end{aligned} \tag{5}$$

$$\begin{aligned}&E^\varepsilon I_2^\varepsilon(\hat{\xi}_2^\varepsilon)^{(4)} = F_2^\varepsilon + (M_2^\varepsilon)' \text{ in } (0,L),\\ &\hat{\xi}_2^\varepsilon(0) = \hat{\xi}_2^\varepsilon(L) = \hat{\xi}_2^{\varepsilon\prime}(0) = \hat{\xi}_2^{\varepsilon\prime}(L) = 0\end{aligned} \tag{6}$$

$$\begin{aligned}&-E^\varepsilon A^\varepsilon \hat{\xi}_3^{\varepsilon\prime\prime} = F_3^\varepsilon \text{ in } (0,L),\\ &\hat{\xi}_3^\varepsilon(0) = \hat{\xi}_3^\varepsilon(L) = 0.\end{aligned} \tag{7}$$

Consequently we get the classical Bernoulli-Navier flexion-extension theory for straight rods (see [2, 3]).

7. Conclusions

By taking the limit in the linear three-dimensional elasticity problem, we have obtained a one-dimensional model (a new one, to our knowledge) generalizing classical Bernoulli-Navier theory for straight rods, to the case of weakly curved rods, also furnishing a justified definition for the latter: the curvature is of the order of the diameter of the cross section.

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