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A NEW APPROACH TO TWO OVERDETERMINED EIGENVALUE PROBLEMS OF POMPEIU TYPE.

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Abstract. We are interested in two overdetermined problems in spectral theory, known as Schiffer's conjectures and related to the Pompeiu problem. We show the connection between these problems and the critical points of the eigenvalue functional with a volume constraint. In two dimensions, we use this fact to establish an integral identity satisfied by a conformal map.

1. Introduction.

In 1928, the rumanian mathematician Pompeiu introduced the notion of Pompeiu property for a domain :
$\Omega \subset I\!\!R^N$ is said to have the Pompeiu property if the only function $f \in C(I\!\!R^N)$ satisfying $\displaystyle\int_{\sigma(\Omega)} f(x)dx = 0$ for all rigid motions σ of $I\!\!R^N$ is the zero function.

Then a natural question arises : what are the connected regular domains in $I\!\!R^N$ which have the Pompeiu Property?

Williams [14] has proved in 1976 that the Pompeiu Problem was equivalent to a conjecture of M. Schiffer in spectral theory :
let Ω be a connected regular domain in $I\!\!R^N$ such that there exists an eigenvalue μ and a Neumann-eigenfunction $u \neq 0$ satisfying

$$(SC)_\mu \quad \left\{ \begin{array}{rcll} -\Delta u & = & \lambda u & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} & = & 0 & \text{on} \quad \partial\Omega, \\ u & = & const. & \text{on} \quad \partial\Omega. \end{array} \right.$$

can we prove that Ω is a ball ?

This conjecture, for the Neumann case, can also, of course, be considered for the Dirichlet case : if $\lambda > 0$ and $u \neq 0$ are such that

$$(SC)_\lambda \quad \left\{ \begin{array}{rcll} -\Delta u & = & \lambda u & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial n} & = & const. & \text{on} \quad \partial\Omega. \end{array} \right.$$

then does it imply that Ω is a ball ?

The motivation of this work is to give another interpretation of these two spectral overdetermined boundary value problems in two dimensions. First we prove that the solutions of these two problems $(SC)_\lambda$ and $(SC)_\mu$ are critical points of the eigenvalue functional under a volume constraint. Then we use this fact to obtain an interesting integral identity satisfied by some conformal map.

In part 2, we recall the classical formulae of derivative of eigenvalues with respect to the domain. These formulae allow us to point out the link which exists between the Schiffer's conjecture $(SC)_\lambda$ and $(SC)_\mu$, and the critical points of the eigenvalue, considered as a domain functional, under a volume constraint. In this part we consider any single eigenvalue of the Laplacian operator.

In part 3, we restrict ourselves to the two-dimensional case. We consider for Ω a solution of $(SC)_\lambda$ or $(SC)_\mu$ and we introduce the conformal map ϕ which maps Ω onto the unit ball D_0. We map Ω into Ω_t thanks to a displacement field given by a perturbation ψ of ϕ and we obtain, using the result of part 2, the following integral identity satisfied by ϕ :

Theorem : $\exists\, c \in \mathbb{C}$ *such that* $\forall z \in D_0 \ : \ c\phi'(z) = \dfrac{1}{2\pi}\displaystyle\int_{D_0} \frac{\phi'(\eta)\hat{u}^2}{(1-\overline{\eta}z)^2} d\eta$
where $\hat{u}$ is the Dirichlet (or Neumann) eigenfunction for $(SC)_\lambda$ (or $(SC)_\mu$) transferred to the unit disc.

2. Critical points of the eigenvalue functional.

2.1. The Dirichlet Problem $(SC)_\lambda$

Domain derivative is now a very usual tool in shape optimization. From a mathematical point of view, it goes back to Hadamard (1905) and Garabedian-Schiffer(1953). Good references are the works of Murat-Simon(1975) which are summed up in [11], or the more recent book of Sokolowski and Zolésio [13].

The n^{th} eigenvalue of the Dirichlet-Laplacian, $\lambda_n(\Omega)$ which is characterized by

$$l\lambda_n(\Omega) = \inf_{\substack{v \in W_n^0 \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} \tag{1}$$

where $W_n^0 = (\mathrm{Span}(u_1, ..., u_{n-1}))^\perp \subset H_0^1(\Omega)$ and
$u_1, ..., u_{n-1}$ are the first $(n-1)$ eigenfunctions

can be regarded as a domain functional since it depends obviously on the domain Ω.

The differentiability of $\lambda_n(\Omega)$ with respect to the domain Ω has been extensively studied by many authors, see e.g. [3], [4], [13], [1]. Let us recall the formula for a single eigenvalue in the next theorem.

Theorem 2.1. *Assume that Ω is C^2 and that $V : \mathbb{R}^N \to \mathbb{R}^N$ is a C^2 deformation field that transforms Ω into $\Omega_t := (Id + tV)(\Omega)$. Then the derivative of a single eigenvalue $\lambda_n(\Omega)$ with respect to the domain in the direction of the deformation field V defined as*

$$\lambda_n'(\Omega; V) := d\lambda_n(\Omega; V) := \lim_{t \to 0} \frac{1}{t}\left[\lambda_n(\Omega_t) - \lambda_n(\Omega)\right] \tag{2}$$

exists and is given by

$$d\lambda_n(\Omega; V) = -\int_{\partial\Omega} \left(\frac{\partial u_n}{\partial \nu}\right)^2 V \cdot \nu \, d\sigma, \tag{3}$$

where ν is the exterior normal vector to the boundary $\partial\Omega$, and $\frac{\partial u_n}{\partial \nu}$ is the normal derivative of the n^{th} homogeneous Dirichlet-Laplace eigenfunction normalized by

$$\int_\Omega u_n^2 = 1. \tag{4}$$

For a multi-valued eigenvalue we recall that the eigenvalue is no more Frechet differentiable, but we can give it's Clarke generalized gradient (see for instance [1], [4], [10] for more details).

Since the derivative of the volume is given by

$$dVol(\Omega; V) = \int_{\partial\Omega} V.\nu \; d\sigma \tag{5}$$

we have immediately, using relation (3) :

Proposition 2.2. *Ω is a solution of problem $(SC)_\lambda$ for a single eigenvalue λ_n if and only if there exists a constant c such that*

$$d\lambda(\Omega; V) = -c^2 \; dVol(\Omega; V). \tag{6}$$

In other words, the solutions of problem $(SC)_\lambda$ can be considered as critical points of the functional $\lambda_n(\Omega)$ with a volume constraint, the constant $-c^2$ being a Lagrange multiplier.

Proposition 2.2 is not new. It appears in a lecture of M. M. Schiffer in 1957 and in the more recent paper [4]. It has also been used in [2] or [8] to prove a similar symmetry result in the case of the first eigenvalue for inhomogeneous problems.

2.2. The Neumann Problem $(SC)_\mu$

For the homogeneous Neumann Problem, the characterization of $\mu_n(\Omega)$ is now

$$l\mu_n(\Omega) = \inf_{\substack{v \in W_n \\ v \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\int_{\mathbb{R}^N} v^2} \tag{7}$$

where $W_n = (\mathrm{Span}(u_0, ..., u_{n-1}))^\perp \subset H^1(\mathbb{R}^N)$ and $u_0, ..., u_{n-1}$ are the first n eigenfunctions

and the derivative of a single eigenvalue $\mu_n(\Omega)$ with respect to the domain in the direction V is given by

$$d\mu_n(\Omega; V) = -\int_{\partial\Omega} |\nabla u_n|^2 V \cdot \nu \, d\sigma + \mu_n(\Omega) \int_{\partial\Omega} u_n^2 V \cdot \nu \, d\sigma \tag{8}$$

where ν is the exterior normal vector to the boundary $\partial\Omega$, and u_n is the n^{th} homogeneous Neumann Laplace eigenfunction normalized by (4).

With formula (5) for the derivative of the volume we obtain now

Proposition 2.3. *Ω is a solution of problem $(SC)_\mu$ for a single eigenvalue μ_n if and only if there exists a constant c such that*

$$d\mu(\Omega; V) = -c^2 \; dVol(\Omega; V). \tag{9}$$

So the solutions of problem $(SC)_\mu$ are also critical points of the functional $\mu_n(\Omega)$ with a volume constraint, the constant $-c^2$ being a Lagrange multiplier.

3. An integral identity in the two-dimensional case.

We are now going to consider some particular deformations of the domain Ω induced by holomorphic maps. The main point is to obtain another expression of the derivative of the eigenvalue with respect to such deformations and then to apply the relation (6) or (9) in order to have some information on Ω. Since the computations for the Neumann case are quite similar to the Dirichlet case, we restrict ourselves in the following to the Dirichlet case.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let ϕ be the conformal map from the unit ball D_0 onto Ω. Since classical regularity results for such free boundary problems imply that a solution Ω of $(SC)_\lambda$ or $(SC)_\mu$ in $\mathbb{R}^2$ must be analytic, we can assume without loss of generality that $\partial\Omega$ is analytic.

Let us fix, in all the following, a single Dirichlet eigenvalue, say $\lambda(\Omega)$, and let us denote by u its eigenfunction normalized by (4). We define $\hat{u}$ on D_0 by :

$$\hat{u}(\hat{z}) = u(\phi(\hat{z})). \tag{10}$$

Now, for all holomorphic function ψ, let us define a deformation of Ω by :

$$\Omega_t = (\phi + t\psi)(D_0) = (Id + t\psi o \phi^{-1})(\Omega). \tag{11}$$

We also define u^t on Ω_t by transferring $\hat{u}$ to Ω_t :

$$\hat{u}(\hat{z}) = u^t((\phi + t\psi)(\hat{z})). \tag{12}$$

It is well-known that

$$\int_{\Omega_t} |\nabla u^t(z)|^2 dz = \int_\Omega |\nabla u(z)|^2 dz \tag{13}$$

and a simple change of variables yields

$$\begin{aligned} \textstyle\int_{\Omega_t} u^t(z)^2 dz &= \textstyle\int_{D_0} \hat{u}(\hat{z})^2 |\phi' + t\psi'(\hat{z})|^2 d\hat{z} \\ &= \textstyle\int_{D_0} \hat{u}(\hat{z})^2 |\phi'|^2 \ + 2t \ \int_{D_0} \hat{u}(\hat{z})^2 Re(\phi'\overline{\psi'}) \ + O(t^2) \\ &= \textstyle\int_\Omega u(z)^2 dz \ + 2t \ \int_{D_0} \hat{u}(\hat{z})^2 Re(\phi'\overline{\psi'}) \ + O(t^2). \end{aligned} \tag{14}$$

Now, (13) and (14) give :

$$\frac{\int_{\Omega_t} |\nabla u^t(z)|^2 dz}{\int_{\Omega_t} u^t(z)^2 dz} = \lambda(\Omega) - 2t\lambda(\Omega) \int_{D_0} \hat{u}(\hat{z})^2 Re(\phi'\overline{\psi'}) d\hat{z} \ + O(t^2) \tag{15}$$

and we have the following lemma :

Lemma 3.1. *Let us denote by u_t the (normalized) eigenfunction on Ω_t associated with $\lambda(\Omega_t)$. Then*

$$\frac{\int_{\Omega_t} |\nabla u^t(z)|^2 dz}{\int_{\Omega_t} u^t(z)^2 dz} = \frac{\int_{\Omega_t} |\nabla u_t(z)|^2 dz}{\int_{\Omega_t} u_t(z)^2 dz} \ + \ O(t^2).$$

Proof : Let us assume that $\lambda(\Omega_t)$ is the n^{th} eigenvalue and let us denote by u_{it} , $1 \leq i \leq n-1$, the previous eigenfunctions on Ω_t. We recall that u^t is defined in (12) by transferring $\hat{u}$ to Ω_t. We can write :

$$\begin{aligned} (u^t, u_{it}) &= (u, u_i) + (u, u_{it} - u_i) + (u^t - u, u_{it}) \\ (u^t, u_t) &= (u, u) + (u, u_t - u) + (u^t - u, u_t) \end{aligned}$$

where $(.,.)$ denotes the L^2 scalar product.

Since the variations of the domain Ω are regular, it follows from classical regularity results that ∇u_{it}, ∇u_i, u_{it} and u_i are uniformly bounded on $\overline{\Omega \cup \Omega_t}$.
Now, we use the differentiability of $t \mapsto u_t$ and $t \mapsto u^t$ together with the variational definition of the eigenfunctions to obtain :

$$|(u^t, u_t) - (u, u)| \leq tM \quad \text{and} \quad |(u^t, u_{it})| \leq tM.$$

So we can write

$$u^t = \sum_{i=1}^{k-1} t\alpha_i u_{it} \ + \ (1 + \alpha t)u_t \ + \ o(t).$$

Then the lemma follows by a straightforward calculation using the orthogonality of the eigenfunctions.

□

Now, (15) and lemma 3.1 give :

$$\lambda(\Omega_t) = \lambda(\Omega) - 2t\lambda(\Omega) \int_{D_0} \hat{u}(\hat{z})^2 Re(\phi'\overline{\psi'}) d\hat{z} \ + O(t^2) \tag{16}$$

and passing to the limit in (16) as $t \to 0$, we obtain another expression of the derivative of λ :

Proposition 3.2. *The derivative with respect to the domain of $\lambda(\Omega)$ in the direction defined by the perturbation of the domain induced by ψ (see (11)) is given by*

$$d\lambda(\Omega; \psi) = -2\lambda(\Omega) \int_{D_0} \hat{u}(\hat{z})^2 Re(\phi'\overline{\psi'}) d\hat{z} \tag{17}$$

where $\hat{u}$ is given by (10).

In the same way, we have :

Lemma 3.3. *The derivative with respect to the domain of $Vol(\Omega)$ in the direction defined by the perturbation ψ of ϕ is*

$$dVol(\Omega, \psi) = \frac{1}{2} Re \int_0^{2\pi} e^{i\theta} (\phi'\overline{\psi} + \psi'\overline{\phi})(e^{i\theta}) d\theta.$$

Proof : Stokes' formula gives

$$Vol(\Omega_t) = \frac{1}{2} Re \int_0^{2\pi} \mathrm{e}^{i\theta} (\phi' + t\psi')(\mathrm{e}^{i\theta}) + (\overline{\phi + t\psi})(\mathrm{e}^{i\theta}) d\theta,$$

and using the definition of the derivative with respect to the domain we obtain the desired result.

□

Now, we can give our integral identity which is satisfied for Ω solution of problem $(SC)_\lambda$:

THEOREM 3.4. *Let Ω be a solution of $(SC)_\lambda$ or $(SC)_\mu$ and let ϕ be the conformal map from the unit ball D_0 onto Ω. Then :*

$$\exists c \in \mathbb{C} \text{ such that } \forall z \in D_0 \ : \ c\phi'(z) = \frac{1}{2\pi}\int_{D_0} \frac{\phi'(\eta)\hat{u}^2}{(1-\overline{\eta}z)^2} d\eta$$

where $\hat{u}$ is the eigenfunction transferred to the unit ball.

Proof : Using polar coordinates, we introduce :

$$\hat{z} = re^{i\theta} \in D_0 \tag{18}$$

$$\hat{u}^2(re^{i\theta}) = \sum_{n=-\infty}^{n=+\infty} u_n(r)e^{in\theta} \tag{19}$$

$$\phi(\hat{z}) = \sum_{n=0}^{n=+\infty} a_n \hat{z}^n \tag{20}$$

$$\psi(\hat{z}) = \sum_{n=0}^{n=+\infty} b_n \hat{z}^n \tag{21}$$

and we have easily

$$dVol(\Omega;\psi) = 2\pi\sum_{n=1}^{+\infty} nRe(a_n\overline{b_n}). \tag{22}$$

Now, using (18) – (21) and denoting $A_j = ja_j$, $B_j = jb_j$, we can write

$$\begin{aligned}
\int_{D_0} \hat{u}^2 Re(\phi'\overline{\psi'}) &= 2\pi Re\int_0^1 \sum_{n=-\infty}^{+\infty} u_n(r) \sum_{k\geq Max(0,-n)} A_{k+1}\overline{B_{k+n+1}}r^{2k+n}dr \\
&= 2\pi Re\int_0^1 \sum_{N=0}^{+\infty}\Big(\sum_{n=-\infty}^{N} u_n(r)A_{N-n+1}\overline{B_{N+1}}r^{2N-n}\Big)rdr \\
&= 2\pi Re\int_0^1 \sum_{N=1}^{+\infty}\overline{B_N} \sum_{n=-(N-1)}^{+\infty} u_{-n}(r)A_{N+n}r^{2N+n-2}rdr \\
&= 2\pi\sum_{N=1}^{+\infty} Re\left(\left[\sum_{k=1}^{+\infty} A_k\int_0^1 u_{N-k}(r)r^{N+k-1}dr\right]\overline{B_N}\right)
\end{aligned}$$

So, replacing in (17) we obtain

$$d\lambda(\Omega;\psi) = -4\pi\lambda(\Omega)\sum_{n=1}^{+\infty} nRe(\Gamma_n\overline{b_n}) \tag{23}$$

with $\Gamma_n = \sum_{k=1}^{+\infty} ka_k\int_0^1 u_{n-k}(r)r^{n+k-1}dr$.

Now, proposition 2.2 (or 2.3) implies

$$\exists c \in \mathbb{C} \text{ such that } (\Gamma_n)_{n\geq 1} = c(a_n)_{n\geq 1} \tag{24}$$

(otherwise we can find some ψ such that $dVol(\Omega;\psi)=0$ and $d\lambda(\Omega;\psi)\neq 0$).

By a further calculation we get

$$\Gamma_n = \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\phi'(re^{i\theta})\hat{u}^2(re^{i\theta})r^{n-1}e^{-i(n-1)\theta}rdrd\theta.$$

Writing, for $\hat{z}\in D_0$

$$c\phi'(\hat{z}) = \sum_{n=0}^{+\infty}(n+1)(ca_{n+1})\hat{z}^n$$

and using (24) we obtain our integral identity.

4. Concluding remarks

Remark 1 : The two problems $(SC)_\lambda$ and $(SC)_\mu$ give the same integral identity (with different eigenfunctions $\hat{u}$ of course) of theorem 3.4. So if we were able to prove that this kind of integral identity be true only for $\phi = az+b$ (i.e. only for a ball), we would have proved that the two problems have the same solution.

Remark 2 : According to Theorem 3.4, in order to solve the Schiffer's conjectures (and the Pompeiu problem) in two dimensions, a good strategy could consist in showing that the integral identity of theorem 3.4 is true only for ϕ a polynomial function. So we would be able to conclude thanks to the result of Garofalo-Segala [6].

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