

ESAIM: PROCEEDINGS, VOL. 4, 1998, 7-19
CONTRÔLE ET ÉQUATIONS AUX DÉRIVÉES PARTIELLES
<http://www.emath.fr/proc/Vol.4/>

**OPTIMAL CONTROL FOR THE OBSTACLE PROBLEM WITH
STATE CONSTRAINTS**

MAÏTINE BERGOUNIOUX

UMR-CNRS 6628, Université d'Orléans, U.F.R. Sciences, B.P. 6759,
F-45067 Orléans Cedex 2, France
E-mail: maitine@univ-orleans.fr

DAN TIBA

Institute for Mathematics,
Romanian Academy of Sciences,
70700 Bucuresti P.O.Box 1-764, Romania
E-mail: dtiba@imar.ro

Key Words : Constrained Control Problems, Variational Inequalities, Optimality Conditions.

AMS Subject Qualification : 49J20, 49M29.

Article published by [EDP Sciences](http://www.edpsciences.org/proc) and available at <http://www.edpsciences.org/proc> or <http://dx.doi.org/10.1051/proc:1998018>

Abstract

In this paper we investigate optimal control problems governed by elliptic variational inequalities of the obstacle type. We show how to obtain optimality conditions for a relaxed problem with or without state constraints. Then we present the optimality system related to the original problem with state constraints, using a generalized derivative.

1 Introduction

In this paper we investigate optimal control problems governed by elliptic variational inequalities of the obstacle type. This kind of problems have been studied by different methods by many authors and we quote the works of Mignot [7], Barbu [1], Mignot and Puel [8], Friedman [6], Tiba [10], He [11], Barbu and Tiba [5], Bergounioux [3, 4]. However, there are still many open questions as, for instance, the treatment of the state constraints, and a complete solution of the problem is not yet known, by our knowledge.

We fix our attention on the following problem (P) :

$$\min \left\{ J(y, v) = \frac{1}{2} \int_{\Omega} (y - y_d)^2 dx + \frac{M}{2} \int_{\Omega} u^2 dx \right\} \quad (1.1)$$

subject to $y \in [H_o^1(\Omega)]_+$, the positive cone in $H_o^1(\Omega)$, and

$$a(y, \varphi - y) \geq (u + f, \varphi - y)_{L^2(\Omega)} \quad \forall \varphi \in [H_o^1(\Omega)]_+, \quad (1.2)$$

$$u \in U_{ad} \subset L^2(\Omega) \text{ convex, closed subset.} \quad (1.3)$$

Above, Ω is an open, bounded smooth domain in \mathbb{R}^n , $f \in L^2(\Omega)$, $y_d \in L^2(\Omega)$, $M > 0$ is a constant and $a : H_o^1(\Omega) \times H_o^1(\Omega) \rightarrow \mathbb{R}$ is a bilinear form satisfying the ellipticity condition

$$a(y, y) \geq \delta |y|_{H_o^1(\Omega)}^2, \quad \delta > 0. \quad (1.4)$$

For instance, $a(\cdot, \cdot)$ may have the form

$$a(\varphi, \psi) = \sum_{i,j=1}^n \int_{\Omega} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx + \sum_{i=1}^n \int_{\Omega} b_i \frac{\partial \varphi}{\partial x_i} \psi dx + \int_{\Omega} c \varphi \psi dx, \quad (1.5)$$

where $a_{ij} \in C^{0,1}(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $c \geq 0$.

It is known that, under the above assumptions the variational inequality (1.2) has a unique solution $y \in H^2(\Omega) \cap H_o^1(\Omega)$ and we have the following estimate

$$|y|_{H^2(\Omega)} \leq C |u + f|_{L^2(\Omega)}, \quad (1.6)$$

with C a fixed positive constant. Moreover, (1.2) is equivalent with

$$Ay = u + f + \xi \text{ in } \Omega \quad (1.7)$$

$$y = 0 \text{ on } \partial\Omega \quad (1.8)$$

$$y \geq 0, \xi \geq 0, \xi \in L^2(\Omega), \quad (1.9)$$

$$(y, \xi)_{L^2(\Omega)} = 0. \quad (1.10)$$

Here $A : H_o^1(\Omega) \rightarrow H^{-1}(\Omega)$ denotes the linear bounded operator generated by $a(.,.)$, that is

$$(Au, v)_{H^{-1}(\Omega) \times H_o^1(\Omega)} = a(u, v), \quad \forall u, v \in H_o^1(\Omega) .$$

We see by (1.10) that the optimal control problem (1.1)- (1.3) is nonconvex and constraint qualification conditions (like Slater condition) will not be fulfilled. This shows the difficulty of finding the optimality system and motivates our choice to examine it. In section 2. we study a relaxed form of the problem (1.1)- (1.3) which is significant from the point of view of numerical approximation. In the last section, we investigate the problem (1.1)- (1.3) to which explicit state constraints are added and we obtain a general form of the first order necessary conditions.

For convenience, in the sequel we denote by $|| \cdot ||$ the $L^2(\Omega)$ -norm et (\cdot, \cdot) the $L^2(\Omega)$ -scalar product. $|| \cdot ||_i$ and $(\cdot, \cdot)_i$ denote respectively the usual norm and the scalar product of $H^i(\Omega)$.

2 Relaxation of the Problem

The formulation (1.7)- (1.10) of the state-equation allows us to interpret $\xi \in L^2(\Omega)$ as a supplementary control parameter satisfying the control constraints (1.9) and the mixed constraints (1.10). This point of view has been proposed by Mignot and Puel [8], Saguez and Bermudez [9] and we shall follow it here. In this section we replace (1.10) by the relaxed constraint

$$(y, \xi) \leq \alpha \tag{2.1}$$

as it has already been done in Bergounioux [3]; $\alpha > 0$ is arbitrary.

In order to simplify the exposition, we assume that $f = 0$ and

U_{ad} is a bounded subset (by a constant k) of $L^2(\Omega)$.

This is not necessary for the existence of optimal pairs $[y^*, u^*]$ for (1.1)-(1.3), since the cost functional is coercive. Then (1.6) shows that for any $\tilde{u} \in U_{ad}$, the corresponding \tilde{y} , $\tilde{\xi}$ given by (1.2) (or equivalently by (1.7)-(1.10)) satisfy :

$$|\tilde{y}|_2 \leq |\tilde{u}| \leq C k , \tag{2.2}$$

$$|\tilde{\xi}| \leq |\tilde{u}| + |\tilde{y}|_2 \leq (C + 1) k . \tag{2.3}$$

We infer that the supplementary control ξ has to be bounded in $L^2(\Omega)$ and we impose (2.3) as an explicit condition in the relaxed problem, since it is no more automatically valid. Therefore, we shall study the optimal control (\mathcal{P}_α) given by (1.1),(1.7),(1.8),(1.9),(1.3) and (2.1),(2.3).

Theorem 2.1 *For any $\alpha > 0$, there is at least one solution $[y_\alpha, u_\alpha, \xi_\alpha]$ for the problem (\mathcal{P}_α) . Moreover, for $\alpha \rightarrow 0$, on a subsequence, we have $[y_\alpha, u_\alpha, \xi_\alpha] \rightarrow [y^*, u^*, \xi^*]$ (an optimal pair for (\mathcal{P})) in the strong topology of $H^1(\Omega) \times L^2(\Omega)$ coupled with the weak topology of $L^2(\Omega)$.*

Proof.- For any $\tilde{u} \in U_{ad}$, let \tilde{y} and $\tilde{\xi}$ in $H^2(\Omega) \times L^2(\Omega)$ be associated via (1.7)-(1.10). Then $[\tilde{y}, \tilde{u}, \tilde{\xi}]$ is admissible to (\mathcal{P}_α) , for all $\alpha > 0$ since (2.3) is automatically fulfilled by (1.6).

Now, let $\alpha > 0$ be fixed and $[y_n, u_n, \xi_n]$ be a minimizing sequence to (\mathcal{P}_α) . By our assumptions $[u_n, \xi_n]$ is bounded in $L^2(\Omega) \times L^2(\Omega)$ and (1.7),(1.8) give $\{y_n\}$ bounded in $H^2(\Omega) \cap H_o^1(\Omega)$. We denote by $u_\alpha, \xi_\alpha, y_\alpha$ some weak limits (on a subsequence) in the above topology. They are obviously admissible for (\mathcal{P}_α) and the weak lower semicontinuity of the cost functional (1.1) shows that this is an optimal triple.

The above boundedness remains valid with respect to any $\alpha > 0$ and let $[\hat{y}, \hat{u}, \hat{\xi}]$ be the weak limit on a subsequence of $[y_\alpha, u_\alpha, \xi_\alpha]$ in the topology of $H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Obviously it satisfies (1.7),(1.8) and (1.9). Since $y_\alpha \rightarrow \hat{y}$ strongly in $H^1(\Omega)$, then (2.1) and (1.9) yield that (1.10) is as well satisfied. The triple $[\hat{y}, \hat{u}, \hat{\xi}]$ is consequently admissible for (\mathcal{P}) . It is optimal (see (2.4) below) and we redenote it by $[y^*, u^*, \xi^*]$.

We show that the convergence of $\{u_\alpha\}$ is valid in the strong topology of $L^2(\Omega)$, on a subsequence. We notice that $[y^*, u^*, \xi^*]$ is admissible for (\mathcal{P}_α) , for all $\alpha > 0$, that is

$$\frac{1}{2} \int_{\Omega} (y_\alpha - y_d)^2 dx + \frac{M}{2} \int_{\Omega} u_\alpha^2 dx \leq \frac{1}{2} \int_{\Omega} (y^* - y_d)^2 dx + \frac{M}{2} \int_{\Omega} (u^*)^2 dx . \quad (2.4)$$

Then (2.4) and the weak lower semicontinuity of the cost functional (1.1) give that

$$\lim_{\alpha \rightarrow 0} \int_{\Omega} u_\alpha^2 dx = \int_{\Omega} (u^*)^2 dx ,$$

and a wellknown strong convergence criterion in Hilbert spaces achieves the proof.

Remark 2.1 *Numerically, it is enough to solve (\mathcal{P}_α) for α "small", instead of (\mathcal{P}) . If U_{ad} is unbounded, one can impose directly inequality (1.6) as a constraint or can make a sequential choice of large constraints in (2.3) since the above argument shows that this algorithm will stop in a finite number of steps.*

Our next goal is to obtain the optimality condition for the problem (\mathcal{P}_α) , $\alpha > 0$. We shall use an adapted penalization technique which was introduced by Barbu [1] and has the advantage to strengthen some convergence properties given by Theorem 2.1. However, this approach is not applicable for the numerical approximation of (\mathcal{P}_α) since it uses the solution itself.

Let $\alpha > 0$ be fixed and $\varepsilon > 0$ be a penalization parameter. We consider the optimization problem

$$\min J_\varepsilon(y, u, \xi) \quad (2.5)$$

for all $y \in H^2(\Omega) \cap H_o^1(\Omega)$, $u \in U_{ad}$, $\xi \in L^2(\Omega)$ satisfying (1.9) and (2.3). The

penalized functional is defined by

$$J_\varepsilon(y, u, \xi) = J(y, v) + \frac{1}{2\varepsilon}|Ay - u - \xi|^2 + \frac{1}{2}|y - y_\alpha|_2^2 + \frac{1}{2}|u - u_\alpha|^2 + \frac{1}{2\varepsilon}[(y, \xi)_{L^2(\Omega)} - \alpha]_+^2,$$

where $g_+ = \max(0, g)$.

We denote the problem (2.5) by $(\mathcal{P}_\alpha^\varepsilon)$; its feasible set

$$\mathcal{D} = \{ (y, u, \xi) \in H^2(\Omega) \cap [H_c^1(\Omega)]_+ \times U_{ad} \times L^2(\Omega)_+ ; |\xi| \leq (C + 1)k \}$$

is independent of ε and α . The same argument as before shows (we shall frequently drop the index α) :

Theorem 2.2 *The problem $(\mathcal{P}_\alpha^\varepsilon)$ has at least a solution $[y_\varepsilon, u_\varepsilon, \xi_\varepsilon]$ in \mathcal{D} .*

Concerning the asymptotic behaviour of $(\mathcal{P}_\alpha^\varepsilon)$ we have the stronger statement

Theorem 2.3 *When $\varepsilon \rightarrow 0$, $[y_\varepsilon, u_\varepsilon, \xi_\varepsilon]$ is strongly convergent to $[y_\alpha, u_\alpha, \xi_\alpha]$ in $H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.*

Proof.- (Sketch). We have

$$J_\varepsilon(y_\varepsilon, u_\varepsilon, \xi_\varepsilon) \leq J_\varepsilon(y_\alpha, u_\alpha, \xi_\alpha) = \frac{1}{2}|y_\alpha - y_d|^2 + \frac{M}{2}|u_\alpha|^2, \quad (2.6)$$

and this shows that $Ay_\varepsilon - u_\varepsilon - \xi_\varepsilon \rightarrow 0$ strongly in $L^2(\Omega)$, $[(y_\varepsilon, \xi_\varepsilon) - \alpha]_+ \rightarrow 0$ in \mathbb{R} , $\{y_\varepsilon\}$ is bounded in $H^2(\Omega)$, $\{u_\varepsilon\}$, $\{\xi_\varepsilon\}$ are bounded in $L^2(\Omega)$. If $[\tilde{y}_\alpha, \tilde{u}_\alpha, \tilde{\xi}_\alpha]$ denote their weak limit in $H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, on a subsequence, then it will remain in \mathcal{D} . Moreover the strong convergence of y_ε to \tilde{y}_α in $L^2(\Omega)$ shows finally that $[\tilde{y}_\alpha, \tilde{u}_\alpha, \tilde{\xi}_\alpha]$ is an admissible triple for (\mathcal{P}_α) . Then (2.5), (2.6) yield :

$$\begin{aligned} \frac{1}{2}|\tilde{y}_\alpha - y_\alpha|_2^2 + \frac{1}{2}|\tilde{u}_\alpha - u_\alpha|^2 + \frac{1}{2}|\tilde{y}_\alpha - y_d|^2 + \frac{M}{2}|\tilde{u}_\alpha|^2 \\ \leq \frac{1}{2}|y_\alpha - y_d|^2 + \frac{M}{2}|u_\alpha|^2. \end{aligned}$$

Then, the optimality of $[y_\alpha, u_\alpha, \xi_\alpha]$ shows that $\tilde{y}_\alpha = y_\alpha$, $\tilde{u}_\alpha = u_\alpha$, $\tilde{\xi}_\alpha = \xi_\alpha$ (by (1.7)) and that the convergences are valid in the strong topology.

Now, we want to derive optimality conditions for $(\mathcal{P}_\alpha^\varepsilon)$. J_ε is \mathcal{C}^1 and the feasible domain of $(\mathcal{P}_\alpha^\varepsilon)$ is convex, so using convex variations we have

$$\forall (y, u, \xi) \in \mathcal{D} \quad \nabla J_\varepsilon(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)(y - y_\varepsilon, u - u_\varepsilon, \xi - \xi_\varepsilon) \geq 0. \quad (2.7)$$

This leads to the following penalized optimality system :

Theorem 2.4 For all $\varepsilon > 0$ (small enough), there exist $q_\varepsilon \in L^2(\Omega)$ and $\lambda_\varepsilon \in \mathbb{R}^+$ such that

$$\begin{aligned} \forall y \in H^2(\Omega) \cap H_o^1(\Omega), y \geq 0 \\ (y_\varepsilon - y_d, y - y_\varepsilon) + (q_\varepsilon, A(y - y_\varepsilon)) + \\ (y_\varepsilon - y_\alpha, y - y_\varepsilon)_2 + \lambda_\varepsilon (y - y_\varepsilon, \xi_\varepsilon) \geq 0, \end{aligned} \quad (2.8)$$

$$\forall u \in U_{ad} \quad (Mu_\varepsilon - q_\varepsilon + u_\varepsilon - u_\alpha, u - u_\varepsilon) \geq 0, \quad (2.9)$$

$$\forall \xi \geq 0, |\xi| \leq (C+1)k \quad (\lambda_\varepsilon y_\varepsilon - q_\varepsilon, \xi - \xi_\varepsilon) \geq 0. \quad (2.10)$$

Proof.- relation (2.7) may be decoupled to obtain

$$\forall y \in H^2(\Omega) \cap H_o^1(\Omega), y \geq 0 \quad \nabla_y J_\varepsilon(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)(y - y_\varepsilon) \geq 0, \quad (2.11)$$

$$\forall u \in U_{ad} \quad \nabla_u J_\varepsilon(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)(u - u_\varepsilon) \geq 0, \quad (2.12)$$

$$\forall \xi \geq 0, |\xi| \leq (C+1)k \quad \nabla_\xi J_\varepsilon(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)(\xi - \xi_\varepsilon) \geq 0. \quad (2.13)$$

Setting

$$q_\varepsilon = \frac{Ay_\varepsilon - u_\varepsilon - \xi_\varepsilon}{\varepsilon} \in L^2(\Omega) \quad \text{and} \quad \lambda_\varepsilon = \frac{[(y_\varepsilon, \xi_\varepsilon)_{L^2(\Omega)} - \alpha]_+}{\varepsilon} \in \mathbb{R}^+,$$

we get (2.8)-(2.10).

Remark 2.2 We define the simplified adjoint state corresponding to optimal control problems without constraints on the state :

$$A^* p_\varepsilon = y_\varepsilon - y_d \quad \text{in } \Omega, \quad (2.14)$$

$$p_\varepsilon = 0 \quad \text{on } \partial\Omega. \quad (2.15)$$

Then by (2.8) we obtain

$$\begin{aligned} \forall y \in H^2(\Omega) \cap H_o^1(\Omega), y \geq 0 \\ (p_\varepsilon + q_\varepsilon, A(y - y_\varepsilon)) + (y_\varepsilon - y_\alpha, y - y_\varepsilon)_2 + \lambda_\varepsilon (y - y_\varepsilon, \xi_\varepsilon) \geq 0. \end{aligned} \quad (2.16)$$

Let us note that p_ε is just an auxiliary mapping and $p_\varepsilon \rightarrow p_\alpha$ strongly in $H^2(\Omega) \cap H_o^1(\Omega)$ by (2.14),(2.15) and Theorem 2.3. Here p_α denotes the solution of (2.14),(2.15) corresponding to y_α .

In order to pass to the limit when $\varepsilon \rightarrow 0$, we impose the natural assumption

$$0 \in U_{ad} \quad (2.17)$$

which simply says that it is possible to have no control action on the system.

Theorem 2.5 Under the above assumptions, $\{\lambda_\varepsilon\}$ is bounded in \mathbb{R} and $\{q_\varepsilon\}$ is bounded in $L^2(\Omega)$.

Proof.- We add relations (2.16),(2.9), (2.10) and we group conveniently the terms, taking into account the definition of q_ε , to obtain

$$\begin{aligned} & - (q_\varepsilon, Ay - u - \xi) - \lambda_\varepsilon [(y, \xi_\varepsilon) + (y_\varepsilon, \xi)] + 2\lambda_\varepsilon \alpha \leq \\ & (p_\varepsilon, A(y - y_\varepsilon)) + (y_\varepsilon - y_\alpha, y - y_\varepsilon)_2 + \\ & (Mu_\varepsilon + u_\varepsilon - u_\alpha, u - u_\varepsilon) - \varepsilon |q_\varepsilon|^2 - 2\lambda_\varepsilon [(y_\varepsilon, \xi_\varepsilon) - \alpha] . \end{aligned} \quad (2.18)$$

If $(y_\varepsilon, \xi_\varepsilon) - \alpha \geq 0$ then $2\lambda_\varepsilon [(y_\varepsilon, \xi_\varepsilon) - \alpha] \geq 0$, and if $(y_\varepsilon, \xi_\varepsilon) - \alpha \leq 0$ then $\lambda_\varepsilon = 0$ because of its definition; in any case the term $-2\lambda_\varepsilon [(y_\varepsilon, \xi_\varepsilon) - \alpha]$ may be neglected in the above relation. We see that the right-hand side in (2.18) is bounded by a constant independent of ε , α and depending only on y , u . Here we use as well Theorem 2.3.

First, we fix $y = 0$, $u = 0$, $\xi = 0$, which is possible by (2.17), and we have

$$2\lambda_\varepsilon \alpha \leq c ,$$

that is $\{\lambda_\varepsilon\}$ is bounded in \mathbb{R} by $\frac{c}{2\alpha}$.

Now, let us fix $u = 0$ and write (2.18) in the form

$$(-q_\varepsilon, Ay - \xi) \leq C(y, \xi) \quad (2.19)$$

where $C(.,.)$ is a bounded map from $H^2(\Omega) \times L^2(\Omega)$. Consider $\rho > 0$ some "small" constant and χ arbitrary in $B(0, \rho)$ the ball of radius ρ and center 0 in $L^2(\Omega)$.

We choose $\xi_\chi = \chi_- = \chi_+ - \chi$ and $y = y_\chi$ given by

$$Ay_\chi = \chi_+ \text{ in } \Omega , , y_\chi = 0 \text{ on } \partial\Omega . \quad (2.20)$$

Obviously $y_\chi \geq 0$ by the maximum principle, $y_\chi \in H^2(\Omega) \cap H_o^1(\Omega)$ by the regularity for (2.20) and $|\xi_\chi| \leq \rho$, $|y_\chi|_{H^2(\Omega) \cap H_o^1(\Omega)} \leq \rho c$ with an absolute c . That is y_χ and ξ_χ are admissible with ρ "small" enough and are bounded test elements.

With this choice, there is a constant $\eta > 0$ such that (1.9) implies

$$(-q_\varepsilon, \chi) \leq \eta \quad (2.21)$$

for any $\chi \in B(0, \rho)$ and this ends the proof.

Remark 2.3 *The above argument shows that the qualification condition (Bergounioux and Tiba [5]) :*

$$\begin{aligned} & \exists \rho > 0, \forall \chi \in B(0, 1), \exists [y_\chi, u_\chi, \xi_\chi] \text{ bounded in } \mathcal{D} \\ & \text{by a constant independent of } \chi \\ & \text{such that } Ay_\chi = u_\chi + \xi_\chi + \rho\chi \text{ in } \Omega , \end{aligned}$$

is automatically fulfilled in the case of problem $(\mathcal{P}_\alpha^\varepsilon)$.

We may pass to the limit on a subsequence as $\varepsilon \rightarrow 0$ and we obtain

Theorem 2.6 Under the above hypotheses, if $[y_\alpha, u_\alpha, \xi_\alpha]$ is a solution of (\mathcal{P}_α) , there exist Lagrange multipliers λ_α, q_α in $\mathbb{R}^+ \times L^2(\Omega)$ such that

$$\begin{aligned} \forall y \in H^2(\Omega) \cap H_o^1(\Omega), y \geq 0 \\ (p_\alpha + q_\alpha, A(y - y_\alpha)) + \lambda_\alpha (y - y_\alpha, \xi_\alpha) \geq 0, \end{aligned} \quad (2.22)$$

$$\forall u \in U_{ad} \quad (Mu_\alpha - q_\alpha, u - u_\alpha) \geq 0, \quad (2.23)$$

$$\forall \xi \geq 0, |\xi| \leq (C + 1) k \quad (\lambda_\alpha y_\alpha - q_\alpha, \xi - \xi_\alpha) \geq 0, \quad (2.24)$$

$$\lambda_\alpha [(y_\alpha, \xi_\alpha) - \alpha] = 0. \quad (2.25)$$

Proof.- It is obvious to get relations (2.22)-(2.24). We just have to comment relation (2.25) which is a complementarity condition related to (2.1).

If $(y_\alpha, \xi_\alpha) - \alpha < 0$ then the convergence results imply that $(y_\varepsilon, \xi_\varepsilon) - \alpha < 0$ for any ε small enough. So $\lambda_\varepsilon = 0$ and the limit value is $\lambda_\alpha = 0$ as well.

Remark 2.4 We claim that we may similarly obtain first order optimality conditions if we add a state constraint of the type “ $y \in K$ ”, where K is a closed convex subset of $H_o^1(\Omega)$. All the previous convergence results remain valid. The boundedness for q_ε is obtained by setting the qualification assumption mentioned in Remark 2.3.

The optimality conditions (2.22)-(2.25) give the solution of (\mathcal{P}_α) , which suffices for the numerical approximation of (\mathcal{P}) , with α “small”. However the estimate on $\{\lambda_\varepsilon\}$ is of order α^{-1} and it seems impossible to take $\alpha \rightarrow 0$ in (2.22)-(2.25). The first order necessary conditions for the problem (\mathcal{P}) will be discussed in the next section by a related method and under the presence of state-constraints.

3 The Obstacle Problem with State Constraints

In this section, we study the problem (\mathcal{P}) with the additional state constraint

$$y \in K \subset H_o^1(\Omega) \text{ closed convex subset.} \quad (3.1)$$

By using a technique similar to the previous section we shall prove a result of Kuhn-Tucker type but we don't consider a relaxed form of the original problem. We formulate the penalized problem ($\varepsilon > 0$):

$$\min \left\{ \frac{1}{2} |y - y_d|^2 + \frac{M}{2} |u|^2 + \frac{1}{2\varepsilon} |Ay - u - \xi|^2 + \frac{1}{\varepsilon} (y, \xi) + \frac{1}{2} |u - u^*|^2 + \frac{1}{2} |\xi - \xi^*|^2 \right\} \quad (3.2)$$

subject to $y \in K \cap [H_o^1(\Omega)]_+, u \in U_{ad}, \xi \geq 0$, where $[y^*, u^*, \xi^*]$ is optimal for the problem (\mathcal{P}) with the additional constraint (3.1). This can be established, under the standard admissibility and coercivity assumptions, in the usual way. Moreover, we may suppose for simplicity that $[0, 0, 0]$ is an admissible triple. The main difference from the previous sections is

that we penalize the condition $(y, \xi) = 0$ and not $(y, \xi) \leq \alpha$; the approximating optimization problem is given by (3.1), (3.2),

$$u \in U_{ad} , \quad (3.3)$$

and

$$y \geq 0 , \quad \xi \geq 0 , \quad y \in H^2(\Omega) \cap H_o^1(\Omega), \quad \xi \in L^2(\Omega) . \quad (3.4)$$

As before the cost criterion in (3.2) is not a convex mapping, while the constraint set (3.1),(3.3),(3.4) is convex in this case. Moreover K and U_{ad} are not necessarily bounded subsets. We denote this minimization problem by $(\mathcal{P}_\varepsilon)$ (since there is no possible confusion with the previous sections).

Theorem 3.1 *There exists a unique optimal triple $[y_\varepsilon, u_\varepsilon, \xi_\varepsilon]$ for $(\mathcal{P}_\varepsilon)$ and taking $\varepsilon \rightarrow 0$ we have*

$$u_\varepsilon \rightarrow u^* \text{ strongly in } L^2(\Omega) \quad (3.5)$$

$$\xi_\varepsilon \rightarrow \xi^* \text{ strongly in } L^2(\Omega) \quad (3.6)$$

$$y_\varepsilon \rightarrow y^* \text{ strongly in } H^2(\Omega) \cap H_o^1(\Omega) . \quad (3.7)$$

Proof - (Sketch). If $[y_n, u_n, \xi_n]$ is a minimizing sequence for $(\mathcal{P}_\varepsilon)$, then it is bounded in $H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ by the coercivity of the functional (3.2). On a subsequence, we may assume that $y_n \rightarrow \tilde{y}_\varepsilon$, $u_n \rightarrow \tilde{u}_\varepsilon$, $\xi_n \rightarrow \tilde{\xi}_\varepsilon$ and we may pass to the inf-limit in (3.2) since $H^2(\Omega) \subset L^2(\Omega)$ compactly. The argument follows as in Theorem 2.3.

Since the variables y , u , ξ in $(\mathcal{P}_\varepsilon)$ are independent each other, then we may take partial variations of convex type (i.e. $u_\varepsilon + \lambda(u - u_\varepsilon)$, $u \in U_{ad}$ and so on ...) and we get :

Theorem 3.2 *If $q_\varepsilon = \frac{1}{\varepsilon}(Ay_\varepsilon - u_\varepsilon - \xi_\varepsilon) \in L^2(\Omega)$ then :*

$$\begin{aligned} \forall y \in H^2(\Omega) \cap K , \quad y \geq 0 , \\ (y_\varepsilon - y_d, y_\varepsilon - y) + \frac{1}{\varepsilon} (\xi_\varepsilon, y_\varepsilon - y) + (q_\varepsilon, A(y_\varepsilon - y)) \leq 0 ; \end{aligned} \quad (3.8)$$

$$\begin{aligned} \forall u \in U_{ad} , \\ M(u_\varepsilon, u_\varepsilon - u) + (u_\varepsilon - u^*, u_\varepsilon - u) - (q_\varepsilon, u_\varepsilon - u) \leq 0 ; \end{aligned} \quad (3.9)$$

$$\begin{aligned} \forall \xi \in L^2(\Omega), \quad \xi \geq 0 , \\ \frac{1}{\varepsilon} (y_\varepsilon, \xi_\varepsilon - \xi) + (\xi_\varepsilon - \xi^*, \xi_\varepsilon - \xi) - (q_\varepsilon, \xi_\varepsilon - \xi) \leq 0 ; \end{aligned} \quad (3.10)$$

Moreover, if $0 \in \text{Int}(U_{ad})$, then $\{q_\varepsilon\}$ is bounded in $L^2(\Omega)$.

Proof - We add relations (3.8)-(3.10) and we transform them as follows :

$$\begin{aligned} (y_\varepsilon - y_d, y_\varepsilon - y) + M(u_\varepsilon, u_\varepsilon - u) + \frac{1}{\varepsilon} (y_\varepsilon, \xi_\varepsilon - \xi) + \frac{1}{\varepsilon} (\xi_\varepsilon, y_\varepsilon - y) \\ + (u_\varepsilon - u^*, u_\varepsilon - u) + (\xi_\varepsilon - \xi^*, \xi_\varepsilon - \xi) + (q_\varepsilon, \varepsilon q_\varepsilon - Ay + u + \xi) \leq 0 . \end{aligned}$$

We fix $y = 0$, $\xi = 0$ and $u = \rho v \in U_{ad}$, with $\rho > 0$ small and $v \in L^2(\Omega)$, $|v| = 1$. By (3.5)-(3.7) all the terms, except the last, are bounded, that is

$$\forall v \in L^2(\Omega), |v| = 1, \quad \rho(q_\varepsilon, v) \leq \text{Cst} . \quad (3.11)$$

Here, we also use that $(y_\varepsilon, \xi_\varepsilon) \geq 0$ and may be neglected. Relation (3.11) shows that $\{q_\varepsilon\}$ is bounded in $L^2(\Omega)$ and the proof is finished.

Remark 3.1 *The necessary conditions (3.8)-(3.10) are not sufficient in general, since the problem $(\mathcal{P}_\varepsilon)$ is nonconvex.*

It is known that the original variational inequality may be equivalently expressed by using the maximal monotone operator $\partial I_{[H^1_0(\Omega)]_+}$, i.e. the subdifferential of the indicator function of the convex $[H^1_0(\Omega)]_+$. Then, the optimality conditions involve a generalized derivative of this operator. In finite dimensional spaces, Mignot [7], Theorem 1.3, has shown that maximal monotone operators are differentiable a.e. in their domain. In our setting, we introduce the following definition, which extends the one used by Barbu and Tiba [2] :

Definition 3.1

$$\begin{aligned} & \tilde{D}\partial I_{K \cap L^2(\Omega)_+}(y^*, w)q = \\ & \left\{ \delta \in (H^2(\Omega))' \mid r_\varepsilon \rightharpoonup q \text{ weakly in } L^2(\Omega), \tilde{w}_\varepsilon \in \partial I_{K_\varepsilon \cap L^2(\Omega)_+}(y_\varepsilon + \varepsilon r_\varepsilon), \right. \\ & w_\varepsilon \in \partial I_{K \cap H^2(\Omega)_+}(y_\varepsilon), \tilde{w}_\varepsilon \rightarrow w \text{ strongly in } L^2(\Omega), w_\varepsilon \rightarrow w \text{ strongly in } \\ & (H^2(\Omega))' \\ & \left. \frac{1}{\varepsilon}(\tilde{w}_\varepsilon - w_\varepsilon) \rightarrow \delta \text{ weakly in } (H^2(\Omega))', y_\varepsilon \rightarrow y^* \text{ strongly in } H^2(\Omega) \right\} . \end{aligned}$$

Here m is a constant bounding $\{q_\varepsilon - \xi_\varepsilon + \xi^*\}$ and $K_\varepsilon = \{s \in L^2(\Omega) \mid \text{dist}(s, K) \leq \varepsilon m\}$.

Theorem 3.3 *Under the above hypotheses, if $[y^*, u^*]$ is an optimal pair for the problem (\mathcal{P}) with the additional state-constraint (3.1), there exists $q^* \in L^2(\Omega)$ such that :*

$$-A^* q^* \in y^* - y_d - \tilde{D}\partial I_{K \cap L^2(\Omega)_+}(y^*, -\xi^*)(-q^*) , \quad (3.12)$$

$$q^* \in Mu^* + \partial I_{U_{ad}}(u^*) . \quad (3.13)$$

Proof - We notice that $y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*) \in K_\varepsilon \cap L^2(\Omega)_+$ by the above estimates. Relation (3.10) may be rewritten as

$$y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*) \in \partial I_-(-\xi_\varepsilon) ,$$

or equivalently

$$-\xi_\varepsilon \in \partial I_+(y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*)) ,$$

(here I_- , I_+ are the indicator functions of the negative, positive cones in $L^2(\Omega)$.) Since we know that $y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*) \in K_\varepsilon$ as well, we may write that

$$-\xi_\varepsilon \in \partial I_{K_\varepsilon \cap L^2(\Omega)_+}(y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*)) . \quad (3.14)$$

Similarly, relation (3.8) gives that

$$-A^* q_\varepsilon - \frac{1}{\varepsilon} \xi_\varepsilon \in y_\varepsilon - y_d + \frac{1}{\varepsilon} \partial I_{K \cap H^2(\Omega)_+}(y_\varepsilon), \quad (3.15)$$

where A^* is the adjoint of the operator $A : H^2(\Omega) \rightarrow L^2(\Omega)$. Then (3.14) and (3.15) yield that

$$-A^* q_\varepsilon \in y_\varepsilon - y_d + \frac{1}{\varepsilon} \partial I_{K \cap H^2(\Omega)_+}(y_\varepsilon) - \frac{1}{\varepsilon} \partial I_{K_\varepsilon \cap L^2(\Omega)_+}(y_\varepsilon - \varepsilon q_\varepsilon + \varepsilon(\xi_\varepsilon - \xi^*)).$$

The convergence properties of y_ε , q_ε , ξ_ε and Definition 3.1 give the point (3.12). The maximum principle (3.13) is a consequence of (3.5) and (3.9).

Remark 3.2 *It is possible to discuss the above approximation scheme in the spaces $H_o^1(\Omega) \times H^{-1}(\Omega)$ instead of $(H^2(\Omega) \cap H_o^1(\Omega)) \times L^2(\Omega)$. But, then difficulties related to the absence of some compactness properties will arise and the approach becomes more complicated.*

References

- [1] V. Barbu, Optimal Control of Variational Inequalities, *Research Notes in Mathematics* 100, Pitman, Boston, 1984.
- [2] V. Barbu - D. Tiba, Optimal control of abstract variational inequalities, *M. Amouroux and A. El Jai Ed.*, Pergamon Press, Oxford, 1990.
- [3] M. Bergounioux, Optimal Control of an Obstacle Problem, to appear.
- [4] M. Bergounioux, Optimal control of abstract elliptic variational inequalities with state constraints, to appear, *SIAM Journal on Control and Optimization*, Vol. 36, n° 1, Janvier 1998.
- [5] M. Bergounioux, D. Tiba, General Optimality Conditions for Constrained Convex Control Problems, *SIAM Journal on Control and Optimization*, vol. 34, no. 3, 1996.
- [6] A. Friedman, Optimal Control for Variational Inequalities, *SIAM Journal on Control and Optimization*, vol. 24, n° . 3, 1986, pp. 439-451.
- [7] F. Mignot, Contrôle dans les inéquations variationnelles elliptiques, *Journal of Functional Analysis*, vol. 22, no. 2, 1976.
- [8] F. Mignot, J.P. Puel, Optimal Control in Some Variational Inequalities, *SIAM Journal on Control and Optimization*, vol. 22, no. 3, 1984, pp. 466-476.
- [9] C. Saguez - A. Bermudez, Optimal Control of Variational Inequalities, *Control and Cybernetics* 14 (1985).

- [10] D. Tiba, Optimal Control of Nonsmooth Distributed Parameter Systems, *LNM 1459*, Springer Verlag, Berlin (1990).
- [11] Zheng-Xu He, State Constrained Control Problems Governed by Variational Inequalities, *SIAM Journal on Control and Optimization*, vol. 25, n° 5, 1987, pp. 1119-1144.