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A CONSTRUCTIVE APPROACH FOR THE OBSERVABILITY OF COUPLED LINEAR SYSTEMS

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Abstract

We present a constructive method for the study of observability or partial observability of weakly coupled linear distributed systems or, more generally, of compactly perturbed systems. This approach allows us to avoid the usual indirect compactness–uniqueness arguments in many cases. It is based on two ideas: a detailed study of the dependence of some constants on the initial data and on a generalization of an estimation method of A. Haraux in nonharmonic analysis.

1 A model problem

Let Ω be a bounded open domain of class C^4 in $\mathbb{R}^N$ with boundary Γ and with outward unit normal vector ν, and let a_{11}, a_{12}, a_{21}, a_{22} be given real numbers. We consider the following system:

$$\begin{cases} u_1'' - \Delta u_1 + a_{11}u_1 + a_{12}u_2 = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \\ u_2'' + \Delta^2 u_2 + a_{21}u_1 + a_{22}u_2 = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \\ u_1 = u_2 = \Delta u_2 = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \\ u_1(0) = u_{10} \quad \text{and} \quad u_1'(0) = u_{11} \quad \text{in} \quad \Omega, \\ u_2(0) = u_{20} \quad \text{and} \quad u_2'(0) = u_{21} \quad \text{in} \quad \Omega. \end{cases} \tag{1}$$

This problem is well posed: for every given initial data

$$x_0 := (u_{10}, u_{20}, u_{11}, u_{21})$$

in the Hilbert space

$$H := H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega),$$

the system (1) has a unique solution satisfying

$$x := (u_1, u_2, u_1', u_2') \in C(\mathbb{R}; H).$$

We are going to study the possibility of the equivalence

$$\int_I \int_\Gamma |\partial_\nu u_1|^2 + |\partial_\nu u_2|^2 \, d\Gamma \, dt \sim \|x_0\|_H^2 \tag{2}$$

where I is some given interval. (Here and in the sequel all intervals are assumed to have a finite, strictly positive length.) We have the following results provided I is sufficiently long:

- The estimate (2) holds *true* if all coefficients a_{ij} vanish (uncoupled case). Indeed, this follows from earlier results of Lasiecka and Triggiani [9], Lions [11] in one direction, and of Ho [3], Lions [12], Zuazua [14], Bardos, Lebeau and Rauch [1], Lebeau [10] in the other direction.
- The estimate (2) also holds *true* if all coefficients a_{ij} are sufficiently close to zero, by a perturbation argument of Lions [13].
- In fact, the estimate (2) holds *true* for *almost all* choices of the coefficients a_{ij}. This can be established by different methods:
  - if Ω is a ball, then there is a direct computational proof using some special properties of the eigenfunctions of the Laplacian operator in a ball and applying an Ingham type theorem for vector-valued functions [4], [5], [6];

- in the general case this result was first proved by an indirect compactness–uniqueness argument in a joint work of the authors with Zuazua [8];
- finally, a constructive approach led to the same result but with explicit constants. This method will be outlined below; complete proofs and various other applications are given in [7].

- The estimate (2) does *not* hold true for *some* choices of the coefficients a_{ij}, at least if Ω is a ball: explicit counterexamples can be given by using the special structure of the solutions in this case. We will also outline this construction later.

Remark. The problem (1), (2) is just an example. Applying the same approach one can also study

- more general expressions of the type

$$\int_{I_1}\int_{\Gamma_1} |\partial_\nu u_1|^2 \; d\Gamma \; dt + \int_{I_2}\int_{\Gamma_2} |\partial_\nu u_2|^2 \; d\Gamma \; dt$$

  on the left-hand side of (2), where Γ_1, Γ_2 are only parts of Γ and I_1, I_2 are different intervals: one observes u_1 and u_2 at different places and at different times;

- more than two equations;

- many other systems with either boundary or internal observation;

- partial observability problems; we refer to [7] for more details.

2 Description of the method

In this section we outline the proof of the following

Theorem 1 *Let Ω belong to a ball of radius R. There exist countably many hypersurfaces in $\mathbb{R}^4$ such that if (A, B, C, D) does not belong to any of them, then the solutions of* (1) *satisfy the estimates*

$$C_1\|x_0\|_H^2 \le \int_I\int_\Gamma |\partial_\nu u_1|^2 + |\partial_\nu u_2|^2 \; d\Gamma \; dt \le C_2\|x_0\|_H^2$$

for every bounded interval I of length $|I| > 2R$. The constants $C_1, C_2 > 0$ depend on A, B, C, D and on the length of I, but not on the particular choice of the initial data.

Note that the exceptional parameters form a set of measure zero in $\mathbb{R}^4$.

The proof consists of several steps.

Step 1: Direct inequality. Using the multiplier method we obtain in the usual way one half of the estimate (2):

$$\int_I \int_\Gamma |\partial_\nu u_1|^2 + |\partial_\nu u_2|^2 \ d\Gamma \ dt \le c_1 \|x_0\|_H^2.$$

Step 2: Inverse inequality for small a_{ij}'s. Using always the multiplier method, for I *long enough* we obtain the estimate

$$\|x_0\|_H^2 \le c_2 \int_I \int_\Gamma |\partial_\nu u_1|^2 + |\partial_\nu u_2|^2 \ d\Gamma \ dt + c_3 \|x_0\|_H^2$$

with the constant c_3 depending on the coupling parameters a_{ij}. If all these parameters are sufficiently small, then $c_3 < 1$ and the second half of the estimate (2) follows.

Otherwise we may have $c_3 \ge 1$, so that this estimate seems to be useless.

Let us note that even when $c_3 < 1$, the interval has to be chosen usually much longer than $2R$.

Step 3: Inverse inequality for special initial data. Now a crucial observation is that c_3 also depends on the initial data in a particular way, so that $c_3 < 1$ if the initial data satisfy a finite number of orthogonality conditions.

To make this precise, let us denote by Z_1, Z_2, ... the eigenspaces of $-\Delta$ in $H_0^1(\Omega)$. Then, given a bounded interval I of length $> 2R$ and arbitrary numbers a_{ij}, there exists a positive integer n such that the estimate (2) holds with some $c_3 < 1$ for all solutions of (1) whose initial data satisfy the extra condition

$$u_{10}, u_{20}, u_{11}, u_{21} \perp Z_1, \ldots, Z_n \quad \text{in} \quad L^2(\Omega). \tag{3}$$

(This property is related to the compactness of the coupling terms with respect to the energy. In fact we can take c_3 arbitrarily close to zero if we choose n large enough.)

Thus we are led to the following problem: *How to get rid of the extra assumption* (3)*?*

Step 4: A method of Haraux. Before proceeding further, let us recall an elementary (but very useful) result from nonharmonic analysis. Let λ_1, λ_2, ... be a sequence of pairwise distinct imaginary numbers, and consider the functions of the form

$$x(t) = \sum_{k=1}^{\infty} x_k e^{\lambda_k t}$$

with complex coefficients x_k. Assume that there exist an interval I and a positive integer n such that the equivalence

$$\int_I |x(t)|^2 \ dt \sim \sum_{k=1}^{\infty} |x_k|^2 \tag{4}$$

holds for all functions of the above form which also satisfy the extra condition

$$x_1 = \cdots = x_n = 0. \tag{5}$$

Then Haraux [2] proved by an elegant method that

$$\int_J |x(t)|^2 \, dt \sim \sum_{k=1}^{\infty} |x_k|^2$$

for every interval J *strictly longer* than I, *even if the condition* (5) *is not satisfied.*

Step 5: An abstract setting. It turns out that our problem is analogous to the problem studied by Haraux. Indeed, rewriting (1) in the form

$$x' = Ax, \quad x(0) = x_0$$

(see the definition of x and x_0 after the formulation of the system (1)), the solution of (1) can be written in the form

$$x(t) = \sum_{k=1}^{\infty} \sum_{l=1}^{m_k} x_{k,l} \sum_{j=0}^{l-1} \frac{t^j}{j!} e^{\lambda_k t} v_{k,l-j}.$$

Here $\{v_{k,l}\}$ is a Riesz basis in H, for each k the vectors $v_{k,1}, \ldots, v_{k,m_k}$ form a chain of generalized eigenvectors of A associated with the eigenvalue λ_k (similar to the Jordan decomposition of the matrixes), and the complex coefficients $x_{k,l}$ depend on the initial data. (Note that in the case $m_k \equiv 1$ the series is almost the same as that considered by Haraux, with the exception that here we have complex exponents λ_k, and vector coefficients $x_{k,1} v_{k,1}$ instead of complex numbers x_k.)

Furthermore, introducing a *seminorm* by the formula

$$|x| := \|\partial_\nu u_1\|_{L^2(\Gamma)} + \|\partial_\nu u_2\|_{L^2(\Gamma)},$$

the estimate (2) can be rewritten in the form

$$\int_I |x(t)|^2 \, dt \sim \sum_{k,l}^{\infty} |x_{k,l}|^2, \tag{6}$$

very similar to (4).

Step 6: Extending the method of Haraux. It follows from step 3 that if I is sufficiently long, then there exists an integer n such that the estimate (6) holds true for all solutions satisfying the extra condition

$$x_{k,l} = 0 \quad \text{for} \quad k = 1, \ldots, n \quad \text{and} \quad l = 1, \ldots, m_k. \tag{7}$$

We can get rid of this condition by adapting the method of Haraux if the *seminorm* $| \ |$ is in fact a *norm* in each of the eigenspaces of A:

$$Av = \lambda v \quad \text{and} \quad |v| = 0 \quad \text{imply that} \quad v = 0. \tag{8}$$

The proof also uses the relations

$$|\lambda_k| \to \infty$$

and

$$|\Re \lambda_k| < \text{ constant}$$

which are satisfied in the present case.

Step 7: Unique continuation. For almost every choice of the coefficients a_{ij}, the eigenvectors of A have a simple structure and the property (8) reduces to Carleman's classical unique continuation theorem: if a function $z \in H^2(\Omega)$ satisfies for some real number γ the equations

$$\begin{cases} -\Delta z = \gamma z \quad \text{in} \quad \Omega, \\ z = \partial_\nu z = 0 \quad \text{on} \quad \Gamma, \end{cases}$$

then $z = 0$ in Ω. This completes the proof of theorem 1.

3 A counterexample

In this last section we show that in the case of the unit disc

$$\Omega = \{y \in \mathbb{R}^2 \ : \ |y| < 1\}$$

there are effectively exceptional values of the coefficients a_{ij} such that the estimate (2) does not hold. More precisely, we outline the proof of the

Proposition 1 *Let Ω be the unit disc of $\mathbb{R}^2$. There exist countably many two-dimensional surfaces in $\mathbb{R}^4$ such that if (A, B, C, D) belongs to one of them, then for some nonzero initial data the solution of* (1) *satisfies the equality*

$$\partial_\nu u_1 = \partial_\nu u_2 = 0 \quad \textit{on} \quad \mathbb{R} \times \Gamma.$$

Hence the first inequality in the estimate of theorem 1 cannot hold.

For the proof, fix the coefficients a_{ij} (to be chosen later) and fix three different positive roots $c_1 < c_2 < c_3$ of the Bessel function J_0 arbitrarily. Then a simple computation shows that there exist three complex numbers λ_1, λ_2, λ_3

and three vectors β_1, β_2, β_3 in $\mathbb{C}^2$ such that for every choice of the complex numbers δ_1, δ_2, δ_3 the function defined by

$$(u_1, u_2)(r, \varphi, t) := \sum_{k=1}^{3} \delta_k J_0(c_k r)\beta_k \cos(\sqrt{\lambda_k} t)$$

(we use polar coordinates) solves (1) with suitable initial data.

Now *we can choose* the numbers a_{ij} such that λ_1, λ_2, and λ_3 coincide. Denoting their common value by λ, then we have

$$(\partial_\nu u_1, \partial_\nu u_2)(1, \varphi, t) := \Big(\sum_{k=1}^{3} \delta_k c_k J_0'(c_k)\beta_k\Big) \cos(\sqrt{\lambda} t).$$

Since the three vectors β_1, β_2, β_3 cannot be linearly independent in the two-dimensional space $\mathbb{C}^2$ and since $c_k J_0'(c_k) \neq 0$ for $k = 1, 2, 3$, we can choose the complex numbers δ_1, δ_2, δ_3 such that not all of them are zero, but

$$\sum_{k=1}^{3} \delta_k c_k J_0'(c_k)\beta_k = 0.$$

The the corresponding solution of (1) is different from zero but

$$\partial_\nu u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R},$$

so that the estimate (2) cannot hold.

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