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A GENERIC FRAMEWORK TO DERIVE SYSTEMS OF CONSERVATION LAWS WITH SOURCE TERMS AND ITS APPLICATION TO HEAT CONDUCTION IN FLUID FLOWS: AN ALTERNATIVE TO THE METHOD OF MOMENTS IN KINETIC THEORY OF GASES? *

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Abstract. A generic framework to derive systems of conservation laws through the Stationary Action Principle is proposed. The equations are expressed in Eulerian coordinates while the variation of the action assumes an underlying Lagrangian description thus avoiding the need for any Lin constraint. The resulting models admit a supplementary conservation equation for the evolution of the Hamiltonian. We then use the newly developed framework to derive a hyperbolic model that includes heat conduction in the compressible fluid dynamics equations through the introduction of a new variable called the thermal impulse. The resulting model has already been obtained previously through the Stationary Action Principle but its derivation relied on several non-standard assumptions. Our new framework allows not only to lift these assumptions, but also to recover them as a consequence of the Stationary Action Principle. Finally, a comparison with a model including heat conduction derived through the kinetic theory of gases is conducted.

Résumé. Un cadre générique permettant de dériver des systèmes de lois de conservation à travers le Principe d'Action Stationnaire est proposé. Les systèmes sont exprimés en coordonnées eulériennes tandis que la variation de l'action repose sur une description lagrangienne sous-jacente, évitant ainsi la nécessité d'une contrainte de Lin. Les modèles résultants admettent une équation de conservation supplémentaire pour l'évolution du Hamiltonien. Nous utilisons ensuite le cadre nouvellement développé pour dériver un modèle hyperbolique incluant la conduction thermique dans les équations de la dynamique des fluides compressibles grâce à l'introduction d'une nouvelle variable appelée impulsion thermique. Le modèle résultant a déjà été obtenu précédemment par le Principe d'Action Stationnaire, mais sa dérivation reposait sur plusieurs hypothèses inhabituelles. Notre nouveau cadre permet non seulement de lever ces hypothèses, mais aussi de les retrouver en tant que conséquence du Principe d'Action Stationnaire. Nous finissons par une comparaison avec un modèle de conduction thermique dérivé de la théorie cinétique.

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Introduction

The Euler-Fourier equations allows for heat-conduction in inviscid fluid flows and are obtained as the vanishing viscosity limit of the Navier-Stokes-Fourier system [27]. The governing equations read as follows

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{0}, \tag{2}$$

$$\partial_t (\rho E) + \nabla \cdot ((\rho E + p)\boldsymbol{u}) = -\nabla \cdot \boldsymbol{q}. \tag{3}$$

The notations used here are standard. The fluid's density is denoted ρ, its velocity by $\boldsymbol{u}$, its specific total and internal energies are E and $e = E - \frac{1}{2}|\boldsymbol{u}|^2$ respectively, and the pressure p is a thermodynamic function of the density and specific internal energy only. Finally, $\boldsymbol{q} = -\lambda \nabla T$ denotes the heat flux following Fourier's law. Here T is the fluid's temperature and $\lambda \geq 0$ is the thermal conductivity which is a function solely of the thermodynamic state. The system admits the following entropy equation

$$\partial_t (\rho s) + \nabla \cdot \left(\rho s \boldsymbol{u} + \frac{\boldsymbol{q}}{T}\right) = -\frac{\boldsymbol{q} \cdot \nabla T}{T^2} \geq 0, \tag{4}$$

where the specific entropy s satisfies the following Gibbs identity $\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}(1/\rho)$. In particular, the entropy equation shows that the heat conduction model consists in a reversible part, given by the entropy flux $\nabla \cdot (\boldsymbol{q}/T)$, which describes entropy exchanges among the fluid particles, and an irreversible part described by the entropy production term $-\boldsymbol{q} \cdot \nabla T/T^2$.

Although Fourier's law for the heat flux allows for an accurate modelling of heat conduction, it has been reproached for relying on a local thermodynamic equilibrium assumption [20]. From a theoretical point a view, this means that the model assumes that the thermodynamic fluxes adapt instantaneously to the thermodynamic forces. A consequence of this assumption is that perturbations propagate at infinite speed therefore violating the principle of causality. Finally, this also leads to numerical difficulties as the standard explicit numerical discretizations require that the time step scales quadratically with the grid size for stability constraints, whereas phenomena propagating with finite speed typically only require a linear scaling.

At the microscopic level, heat is related to the kinetic energy of the molecules, while heat flux corresponds to a change in the velocity distribution and is therefore subjected to inertia, which, when neglected, leads to Fourier's law. By accounting for inertia, Cattaneo and Vernotte independently proposed an alternative expression for the heat flux leading to a modified heat equation [5, 6, 29]. This new equation corresponds to the telegraphers' equation; as it is hyperbolic, heat propagates at finite speed through waves. This is the so-called second sound which corresponds to thermal waves propagating at a finite speed, lower than the one of acoustic waves and which can be observed in liquid helium and some solids at low temperature.

Although the Cattaneo-Vernotte equations represent a qualitative improvement of the parabolic heat equation, they do not allow for accurate predictions of the second sound [20]. Moreover, their coupling with the equations of fluid dynamics proved to be challenging [8].

As the lifting of the local thermodynamic equilibrium assumption leads to an evolution equation for the heat flux, its status should be raised to that of an internal variable of the system as proposed in the theory of extended thermodynamics [23]. A change of variables then allows to introduce the so-called thermal impulse, which is defined so that the entropy flux associated to heat conduction is the derivative of the internal energy with respect to this new variable [26]. Hence, the equations are reformulated so that the resulting model is compatible with continuum mechanics and with the second law of thermodynamics [24]. More recently, a derivation using Hamilton's Stationary Action Principle of the model has been proposed and its hyperbolicity established [8]. The model reads

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{5a}$$

$$\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{\Pi}) = \boldsymbol{0}, \tag{5b}$$

$$\partial_t (\rho E_{\text{ext}}) + \nabla \cdot ((\rho E_{\text{ext}} \mathbf{Id} + \boldsymbol{\Pi}) \boldsymbol{u} + \boldsymbol{q}) = 0, \tag{5c}$$

$$\partial_t \boldsymbol{j} + \nabla (\boldsymbol{j} \cdot \boldsymbol{u} + T) + (\nabla \wedge \boldsymbol{j}) \wedge \boldsymbol{u} = -\tfrac{1}{\tau} \boldsymbol{j}, \tag{5d}$$

with an extended total energy given by $\rho E_{\text{ext}} = \frac{1}{2}\rho |\boldsymbol{u}|^2 + \rho e + \frac{1}{2}\alpha(\rho) |\boldsymbol{j}|^2$ and $\boldsymbol{\Pi} = (p + \frac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho)) |\boldsymbol{j}|^2)\mathbf{Id} + \alpha(\rho)\boldsymbol{j} \otimes \boldsymbol{j}$, the associated pressure tensor and where $\alpha > 0$ is a positive function of the thermodynamic state related to the thermal conductivity. Indeed, in this model, the heat flux has its own evolution equation as it is given by $\boldsymbol{q} = \alpha(\rho)T\boldsymbol{j}$, instead of Fourier's law. The system also admits an entropy equation and the dissipation yields a signed entropy production as we have

$$\partial_t (\rho s) + \nabla \cdot (\rho s \boldsymbol{u} + \alpha(\rho)\boldsymbol{j}) = \tfrac{\alpha(\rho)}{\tau T} |\boldsymbol{j}|^2 > 0. \tag{6}$$

Here s is the usual specific entropy and follows a classical Gibbs relation

$$\mathrm{d}s = \frac{1}{T}\mathrm{d}e + \frac{p}{T}\mathrm{d}\left(\frac{1}{\rho}\right). \tag{7}$$

As shown in [8], in the regime $\tau \ll 1$, a formal expansion in powers of τ of the thermal impulse, $\boldsymbol{j} = \boldsymbol{j}_0 + \tau \boldsymbol{j}_1 + \dots$, can be injected into (5d). Identifying terms with the same magnitude in τ then suggests that $\boldsymbol{j} = -\tau\nabla T + \mathcal{O}(\tau^2)$, and thus that the heat flux writes

$$\boldsymbol{q} = -\tau\alpha(\rho)T\nabla T + \mathcal{O}(\tau^2). \tag{8}$$

Formally, one then recovers the Fourier law by setting $\tau = \lambda/(\alpha(\rho)T)$ with λ the thermal conductivity, and when $\mathcal{O}(\tau^2)$ terms are discarded, one recovers the Euler-Fourier system. This approach therefore allows to separate the reversible part and the irreversible part of heat conduction into the hyperbolic fluxes and the dissipative source term respectively.

The derivation, inspired by the works of Green and Naghdi [16, 17], required to introduction of new assumptions. In particular, the evolution equation for the thermal impulse — a variable related to the heat flux — is not obtained as an Euler-Lagrange equation but results from the *a priori* assumptions used in the variational principle (see also in Section 2.1).

Another popular method to derive and analyse the governing equations of gas dynamics is the kinetic theory [19]. Starting from a statistical description at the molecular level, it allows for a rigorous derivation of the Navier-Stokes equations and identification of the transport coefficients in the context of diluted gases [7]. More recently, a kinetic derivation of the diffuse interface Cahn-Hilliard equations has been proposed [12, 13]. These derivations rely on a perturbative approach by performing an asymptotic expansion of the particle distribution near equilibrium. An alternative strategy is the method of moments which allows for the derivation of first-order systems with a larger number of equations (given by the number of selected moments). However, the method of moments in itself is not closed and several closure methods have been developed [15, 21, 22, 25]. Interestingly, when considering Grad's closure method [15] applied to the Bhatnagar-Gross-Krook (BGK) model [1], one can derive the following 8-moment system, which can be considered as a minimal model for heat conduction in fluid flows:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{9a}$$

$$\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{0}, \tag{9b}$$

$$\partial_t (\rho E) + \nabla \cdot ((\rho E + p)\boldsymbol{u} + \boldsymbol{q}) = 0, \tag{9c}$$

$$\mathrm{D}_t \boldsymbol{q} + \tfrac{2\gamma-1}{\gamma} (\nabla \cdot \boldsymbol{u}) \boldsymbol{q} + \tfrac{\gamma-1}{\gamma} (\nabla \boldsymbol{u}) \boldsymbol{q} + \tfrac{2\gamma-1}{\gamma} (\boldsymbol{q} \cdot \nabla) \boldsymbol{u} + \tfrac{\gamma}{\gamma-1} p \nabla \left(\tfrac{p}{\rho}\right) = -\tfrac{1}{\tau} \boldsymbol{q}. \tag{9d}$$

Here the notation $\mathrm{D}_t = \partial_t + (\boldsymbol{u} \cdot \nabla)$ indicates the material derivative. As for the previous system, in the limit $\tau \ll 1$, we obtain up to second order terms the Euler-Fourier system with

$$\boldsymbol{q} = -\tau \tfrac{\gamma}{\gamma-1} \rho r^2 T \nabla T + \mathcal{O}\left(\tau^2\right), \tag{10}$$

Although different, model (9) has striking similarities with the previous one (5). Both are first order systems, which recover the Euler and Euler-Fourier system as asymptotic limits at zeroth and first order in τ respectively. In each system, a new variable related to heat flux is added, and is then raised to the status of internal variable. It has to be remarked that the dissipative source terms of $\boldsymbol{q}$ in model (9) and $\boldsymbol{j}$ in model (5) are identical. Moreover, although the dissipation for model (5) is added a posteriori, the dissipative term for the moment model (9) is a direct consequence of the BGK collision / relaxation operator used in the derivation from the mesoscopic level.

In this contribution, we propose a new derivation of the hyperbolic model for heat conduction in fluid flows (5), in particular it does not rely on the introduction of any supplementary assumptions and the evolution equation for the thermal impulse is a direct consequence of the Euler-Lagrange equations. To do so, a new generic framework for the derivation of systems of conservation laws through the Stationary Action Principle is proposed. As the Stationary Action Principle allows only for the derivation of non-dissipative systems, the source terms of model (5) need to be added a posteriori. Hence, we propose a comparison of the obtained model with a model derived from kinetic theory. The strong similarities between the models shed some light on the structure of the dissipation that is proposed for model (5) as the dissipative structure of the kinetic model (9) is a direct heritage of the microscopic description. The outline of this paper is the following. In Section 1, the generic framework for the derivation of systems of conservation laws through Stationary Action Principle is presented, in Section 2, we discuss the derivation of model (9) proposed in [8] and propose an original derivation in which previous assumptions are lifted. Section 3 concerns the kinetic model (9) while the subsequent sections concern the conclusion, appendices and acknowledgements.

1. A generic framework for deriving systems through Stationary Action Principle

We consider a Lagrangian density which depends on the fluid velocity $\boldsymbol{u}$, its density ρ, a set of transported variables $\mathscr{B} = \{b_i,\, i = 1\dots, n\}$, and a set of independent variables $\Xi = \{\xi_j,\, j = 1\dots, m\}$ and their derivatives. Here, derivatives in time of the independent variables will be expressed through their material derivative as it is more suitable to obtain a frame invariant expression of the generic system. The general form of the Lagrangian density that we consider therefore writes

$$\mathscr{L} \equiv \mathscr{L}\left(\boldsymbol{u}, \rho, (b_i)_i, (\mathrm{D}_t \xi_j, \nabla \xi_j, \xi_j)_j\right). \tag{11}$$

We propose a framework that allows for the derivation of the governing equations in Eulerian coordinates. However an underlying Lagrangian description is assumed. In particular, this avoids the introduction of the so-called Lin's constraint without which the derivation of the Euler equations in a fully Eulerian description would lead to a momentum equation valid only for irrotational flows [2]. We therefore introduce the fluid particle trajectory $\boldsymbol{\Phi}$, which maps the Lagrangian coordinate $\boldsymbol{X} \in \mathscr{V}$ of a fluid particle at time t_0 to its Eulerian position $\boldsymbol{x} \in \mathscr{V}(t)$ at time $t \in [t_0,\, t_1]$. The Eulerian velocity field $\boldsymbol{u}$ is then defined by

$$\boldsymbol{u}(t, \boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}}{\partial t}(t, \boldsymbol{X}), \quad \boldsymbol{x} = \boldsymbol{\Phi}(t, \boldsymbol{X}). \tag{12}$$

The density ρ is assumed to be conserved along the fluid particle trajectories,

$$\rho(t, \boldsymbol{\Phi}(t, \boldsymbol{X})) = \det\left(\partial_{\boldsymbol{X}} \boldsymbol{\Phi}(t, \boldsymbol{X})\right) \rho(t_0, \boldsymbol{X}), \tag{13}$$

while the transported variables $b_i \in \mathscr{B}$ are constant along fluid particle trajectories

$$b_i(t, \boldsymbol{\Phi}(t, \boldsymbol{X})) = b_i(t_0, \boldsymbol{X}), \quad j = 1, \ldots, n. \tag{14}$$

Having introduced the fluid particle trajectories, the variations of the action will be performed using a family of virtual motions $\left(\boldsymbol{\Phi}(t, \boldsymbol{X}, \varepsilon)\right)_\varepsilon$. Lagrangian and Eulerian perturbations $\delta\boldsymbol{\Phi}^{\mathrm{L}}$ and $\delta\boldsymbol{\Phi}$ are then introduced such that

$$\boldsymbol{\Phi}(t, \boldsymbol{X}, \varepsilon) = \boldsymbol{\Phi}(t, \boldsymbol{X}) + \varepsilon \delta\boldsymbol{\Phi}^{\mathrm{L}}(t, \boldsymbol{X}) + \mathcal{O}\left(\varepsilon^2\right), \qquad \delta\boldsymbol{\Phi}(t, \boldsymbol{x}) = \delta\boldsymbol{\Phi}^{\mathrm{L}}(t, \boldsymbol{X}). \tag{15}$$

The Stationary Action Principle requires these perturbations to vanish at the boundaries

$$\delta\boldsymbol{\Phi}^{\mathrm{L}}(t, \boldsymbol{X}) = \boldsymbol{0} \quad \forall (t, \boldsymbol{X}) \in \partial\left([t_0,\, t_1] \times \mathscr{V}\right), \tag{16}$$

$$\delta\boldsymbol{\Phi}(t, \boldsymbol{x}) = \boldsymbol{0} \quad \forall (t, \boldsymbol{x}) \in \partial\Omega, \tag{17}$$

where Ω denotes the space-time domain, i.e. $\Omega = \{(t, \boldsymbol{x}) \mid t_0 \le t \le t_1,\, \boldsymbol{x} \in \mathscr{V}(t)\}$.

Perturbations of variables ρ, $\boldsymbol{u}$ and $b_i \in \mathscr{B}$ will be expressed in terms of perturbations $\delta\boldsymbol{\Phi}$. The independent variables in Ξ will have their own virtual motions $\xi(t, \boldsymbol{x}, \varepsilon)$ and expressed through perturbations $\delta\xi(t, \boldsymbol{x})$. As shown in the proof given in Section 5.1, the variations $\delta(\nabla\xi)$ and $\delta(\mathrm{D}_t\xi)$ are given by

$$\delta\left(\nabla\xi_j\right) = \nabla\left(\delta\xi_j\right), \tag{18a}$$

$$\delta\left(\mathrm{D}_t\xi_j\right) = \mathrm{D}_t\left(\delta\xi_j\right) + \left(\nabla\xi_j\right)^{\intercal} \delta\boldsymbol{u}. \tag{18b}$$

These perturbations also vanish at the boundaries. For these variables, we only introduce Eulerian perturbations as they are not necessarily related to any Lagrangian representation. The Stationary Action Principle then reads

$$\delta\mathscr{A} = \delta\left[\iint_\Omega \mathscr{L}\right] = 0 \quad \forall \delta\boldsymbol{\Phi},\, \delta\xi_j, \tag{19}$$

and leads to a system of governing PDEs. Application of Noether's theorem leads to the conservation of a new quantity called Hamiltonian which corresponds to the total energy. Derivation of the generic system of governing PDEs and of the conservation law for the Hamiltonian are provided in Theorems 1.1 and 1.2, respectively. The proofs of these results can be found in Sections 5.1 and 5.2. Vector calculus notations and conventions used are also recalled in Section 5.

Theorem 1.1 (Euler-Lagrange equations)**.** *Consider the Lagrangian density (11). Then, the Stationary Action Principle (19) yields the following system of equations*

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{20a}$$

$$\partial_t (\rho b_i) + \nabla \cdot (\rho b_i \boldsymbol{u}) = 0, \quad i = 1, \dots, n, \tag{20b}$$

$$\partial_t \mathscr{M}_j + \nabla \cdot (\mathscr{M}_j \boldsymbol{u}) - \frac{\partial \mathscr{L}}{\partial \xi_j} + \nabla \cdot \boldsymbol{D}_j = 0, \quad j = 1, \dots, m, \tag{20c}$$

$$\partial_t \boldsymbol{K} + \nabla \cdot (\boldsymbol{K} \otimes \boldsymbol{u}) - \nabla \mathscr{L}^\star - \nabla \cdot (\sum_{j=1}^{m} \nabla \xi_j \otimes \boldsymbol{D}_j) = \boldsymbol{0}, \tag{20d}$$

with the following definitions

$$\boldsymbol{K}^\intercal = \frac{\partial \mathscr{L}}{\partial \boldsymbol{u}}, \quad \mathscr{L}^\star = \rho \frac{\partial \mathscr{L}}{\partial \rho} - \mathscr{L}, \quad \mathscr{M}_j = \frac{\partial \mathscr{L}}{\partial (\mathrm{D}_t \xi_j)}, \quad \boldsymbol{D}_j^\intercal = \frac{\partial \mathscr{L}}{\partial (\nabla \xi_j)}. \tag{21}$$

Theorem 1.2 (Hamiltonian evolution)**.** *Consider the Lagrangian density (11) and define the associated Hamiltonian $\mathcal{H}$ by its Legendre transform with respect to $\boldsymbol{u}$ and the material derivatives of the independent variables,*

$$\mathcal{H} = \boldsymbol{K}^\intercal \boldsymbol{u} + \sum_{j=1}^{m} \mathscr{M}_j \mathrm{D}_t \xi_j - \mathscr{L}. \tag{22}$$

Then system (20) obtained by the Stationary Action Principle (19) admits the following supplementary conservation law

$$\partial_t \mathcal{H} + \nabla \cdot \left((\mathcal{H} - \mathscr{L}^\star) \boldsymbol{u} + \sum_{j=1}^{m} \boldsymbol{D}_j \partial_t \xi_j \right) = 0. \tag{23}$$

2. Application the hyperbolic approximation of the Euler-Fourier system

2.1. Derivation of an hyperbolic approximation of the Euler-Fourier model

The original derivation by means of the Stationary Action Principle of (5) proposed in [8] is based on the works of Green and Naghdi [16, 17]. It relies on the introduction of a new variable $\boldsymbol{j}$, called thermal impulse, and a new potential called thermal displacement ϕ such that

$$\mathrm{D}_t \phi + T = 0, \tag{24a}$$

$$\nabla \phi = \boldsymbol{j}. \tag{24b}$$

In particular, from these assumptions, an evolution equation for the thermal impulse $\boldsymbol{j}$ can be obtained by taking the material derivative of (24b), which results in

$$\mathrm{D}_t \boldsymbol{j} + (\nabla \boldsymbol{u}) \boldsymbol{j} + \nabla T = \boldsymbol{0}, \tag{25a}$$

or alternatively

$$\partial_t \boldsymbol{j} + \nabla (\boldsymbol{j} \cdot \boldsymbol{u} + T) + (\nabla \wedge \boldsymbol{j}) \wedge \boldsymbol{u} = \boldsymbol{0}. \tag{25b}$$

Notice that the last term in (25b) is zero since $\nabla \wedge \boldsymbol{j} = \boldsymbol{0}$, and thus the equation admits a conservative formulation although this formulation does not preserve the Galilean invariance of the system. The conservative formulation can be directly obtained by applying the partial time derivative instead of the material derivative to (24b) as in [8]. The evolution equation for the thermal impulse is then coupled with the fluid equations through the

Stationary Action Principle. In order to achieve this goal, the authors assume that the thermal impulse carries an energy density $\frac{1}{2}\alpha(\rho)\left|\boldsymbol{j}\right|^2$ and consider the following Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\rho\left|\boldsymbol{u}\right|^2 - \tfrac{1}{2}\alpha(\rho)\left|\nabla\phi\right|^2 - \rho f(\rho, -\mathrm{D}_t\phi), \tag{26}$$

with f the free energy a function of density ρ and temperature T, which is obtained as the Legendre transform of the internal energy e with respect to the specific entropy s. Assuming ϕ to be an independent variable, application of theorem 1.1 to (26) with $\mathscr{B} = \emptyset$ and $\Xi = \{\phi\}$, states that the Stationary Action Principle yields the following set of equations

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{27a}$$
$$\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{u} + \alpha(\rho)\nabla\phi) = 0, \tag{27b}$$
$$\partial_t(\rho\boldsymbol{u}) + \nabla\cdot\left(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \left(p + \tfrac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho))\left|\nabla\phi\right|^2\right)\mathbf{Id} + \alpha(\rho)\nabla\phi\otimes\nabla\phi\right) = \mathbf{0}. \tag{27c}$$

The thermal gradient displacements are then replaced by the thermal impulse (24b) and its evolution equation (25b) is added to the system so as to obtain the following first-order system

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{28a}$$
$$\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{u} + \alpha(\rho)\boldsymbol{j}) = 0, \tag{28b}$$
$$\partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{\Pi}) = \mathbf{0}, \tag{28c}$$
$$\partial_t\boldsymbol{j} + \nabla(\boldsymbol{j}\cdot\boldsymbol{u} + T) + (\nabla\wedge\boldsymbol{j})\wedge\boldsymbol{u} = \mathbf{0}, \tag{28d}$$

where the pressure tensor writes $\boldsymbol{\Pi} = \left(p + \frac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho))\left|\boldsymbol{j}\right|^2\right)\mathbf{Id} + \alpha(\rho)\boldsymbol{j}\otimes\boldsymbol{j}$. This system must also be adjoined with the constraint $\nabla\wedge\boldsymbol{j}(t=0,\boldsymbol{x}) = \mathbf{0}$ on the initial condition in order to be compatible with (24b). The Hamiltonian of the system then corresponds to the extended total energy $\mathcal{H} = \rho E_{\mathrm{ext}} = \frac{1}{2}\rho\left|\boldsymbol{u}\right|^2 + \frac{1}{2}\alpha(\rho)\left|\boldsymbol{j}\right|^2 + \rho e(\rho, s)$ and theorem 1.2 yields

$$\partial_t(\rho E_{\mathrm{ext}}) + \nabla\cdot(\rho E_{\mathrm{ext}}\boldsymbol{u} + \boldsymbol{\Pi}\boldsymbol{u} + \boldsymbol{q}) = 0, \tag{29}$$

where the heat flux $\boldsymbol{q}$ is given by $\boldsymbol{q} = \alpha(\rho)T(\rho,s)\boldsymbol{j}$. The Hamiltonian is convex provided that $\rho \mapsto 1/\alpha(\rho)$ is concave as shown in [8]. Under this condition, it could be used as mathematical entropy, but as is customary for systems derived through the Stationary Action Principle, we exchange the roles of the entropy equation and the Hamiltonian's equation (29). The entropy equation should therefore be used to select admissible solutions, and source terms should yield a signed entropy production. Adding the dissipative source term then yields the hyperbolic approximation (5) of the Euler-Fourier system (3) presented in [8].

This derivation, originally proposed in [8], required the introduction of the thermal displacement potential ϕ. Integration of (24a) along fluid particle trajectories leads to the definition of the thermal displacement potential in Lagrangian coordinates

$$\phi(t,\boldsymbol{x}) = -\int_{t_0}^{t} T(t,\boldsymbol{X})\mathrm{d}t, \qquad \boldsymbol{X} \text{ such that } \boldsymbol{x} = \boldsymbol{\Phi}(t,\boldsymbol{X}). \tag{30}$$

It would then seem that this new potential entails memory effects. Variables which depend on the whole history are standard in the theory of Rational Thermodynamics and are often accompanied with the notion of fading memory [20]. If we assume that the dissipation on $\boldsymbol{j}$ added to obtain (5) originates from the following dissipation on ϕ

$$\mathrm{D}_t\phi + T = -\tfrac{1}{\tau}\phi, \tag{31}$$

we get that it corresponds to the case of fading memory with $\tau \to 0$ being the amnesic limit.

However, the use of this new potential as independent variable in the variational principle as well as the use the free energy instead of the internal energy for the definition of the Lagrangian are non-standard. Moreover, the equation on the thermal impulse (25b) does not come from the Stationary Action Principle but is rather an assumption expressed through (24b) and (24a). The Stationary Action Principle then merely acts as a coupling mechanism between this equation and the fluid equations. In the next section, we propose an alternative derivation of the hyperbolic Euler-Fourier model in which these assumptions are lifted and the entire system is derived at once.

2.2. An alternative derivation of the hyperbolic Euler-Fourier model: lifting the thermal displacement assumption

The Stationary Action Principle allows one to derive systems governing reversible processes. For such systems, the entropy production is zero, however entropy exchanges among fluid particles are allowed. Our approach is as follows: instead of only identifying the energies that should appear in the Lagrangian, we also provide the corresponding entropy exchange rates. A similar approach has been used in [9] for the derivation of the Korteweg tensor for capillary fluids.

First, we introduce the thermal impulse variable $\boldsymbol{j}$ to which we associate an energy density $\frac{1}{2}\alpha(\rho)\,|\boldsymbol{j}|^2$. Next, we assume that for a fluid particle $\mathscr{P}$, the entropy exchange rate through its boundary $\partial\mathscr{P}$ is $-\alpha(\rho)\boldsymbol{j}\cdot\boldsymbol{n}$ with $\boldsymbol{n}$ the outward unit normal to $\partial\mathscr{P}$. This new energy contribution will be accounted for through the Lagrangian while the associated entropy exchange rate will join the set of constraints of the variational principle. The entropy balance equation writes

$$\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{u} + \alpha(\rho)\boldsymbol{j}) = 0. \tag{32}$$

We enforce this equation by means of a Lagrange multiplier. With a slight abuse of notation, we denote the Lagrangian multiplier with ϕ, i.e. the same symbol employed for the thermal displacement defined in (24). The reason of this choice will be discussed at the end of the Section. The action then writes as

$$\mathscr{A} = \iint_\Omega \left\{\tfrac{1}{2}\rho\,|\boldsymbol{u}|^2 + \tfrac{1}{2}\alpha(\rho)\,|\boldsymbol{j}|^2 - \rho e(\rho,s)\right\} + \iint_\Omega \phi\left\{\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{u} + \alpha(\rho)\boldsymbol{j})\right\}. \tag{33}$$

Notice that the sign of the term quadratic in $\boldsymbol{j}$ indicates that it is accounted for in the kinetic energy part of the Lagrangian whereas in (26) it was accounted for in the potential energy.

Through an integration by parts in the second integral, we can recover an equivalent Lagrangian density which is more convenient to work with under the framework presented in the previous section. Hence, we get[1]

$$\mathscr{L} = \tfrac{1}{2}\rho\,|\boldsymbol{u}|^2 + \tfrac{1}{2}\alpha(\rho)\,|\boldsymbol{j}|^2 - \rho e(\rho,s) - \rho s \mathrm{D}_t\phi - \alpha(\rho)\boldsymbol{j}\cdot\nabla\phi. \tag{34}$$

Here ρ and $\boldsymbol{u}$ satisfy the usual constraints, the set of transported variables is empty, $\mathscr{B} = \emptyset$, while the set of independent variables is now $\Xi = \{s,\,\phi,\,j_1,\ldots,j_d\}$. Note that the specific entropy is no longer a transported variable as is custom in the derivation of fluid equations by means of Stationary Action Principle as entropy conservation is enforced by means of a Lagrange multiplier. Application of Theorem 1.1 leads to the following

[1]Note that during the integration by parts, a boundary term appears. This term may be dropped as it does not contribute to the variational principle as all variations vanish at the boundaries.

system of equations

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{35a}$$
$$\partial_t (\rho s) + \nabla \cdot (\rho s \boldsymbol{u} + \alpha(\rho) \boldsymbol{j}) = 0, \tag{35b}$$
$$\partial_t (\rho \boldsymbol{u}) + \nabla \cdot \left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \widehat{\boldsymbol{\Pi}}\right) = \boldsymbol{0}, \tag{35c}$$
$$\mathrm{D}_t \phi + T = 0, \tag{35d}$$
$$\nabla \phi - \boldsymbol{j} = \boldsymbol{0}. \tag{35e}$$

First, we notice that eqs. (35d) and (35e) are exactly (24a) and (24b), however they are now Euler-Lagrange equations resulting from the Stationary Action Principle rather than *a priori* assumptions. In order to satisfy the imposed entropy flux, the Lagrange multiplier ϕ takes the value of the thermal displacement introduced in (24) introduced in [16, 17] and used in [8]. The pressure tensor $\widehat{\boldsymbol{\Pi}}$ now writes

$$\widehat{\boldsymbol{\Pi}} := \left(p + \tfrac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho))\boldsymbol{j} \cdot (2\nabla\phi - \boldsymbol{j})\right) \mathbf{Id} + \alpha(\rho) \nabla\phi \otimes \boldsymbol{j} \tag{36}$$

and owing to (35e), we have that $\widehat{\boldsymbol{\Pi}} = \boldsymbol{\Pi}$. Once again, equations (35d) and (35e) can be replaced by the evolution equation (25b) for $\boldsymbol{j}$, resulting in a first order system that is identical to (28). Similarly, application of Theorem 1.2 allows to recover the equation for the Hamiltonian (29). Moreover, notice that

$$\mathscr{L} = \tfrac{1}{2}\rho \left|\boldsymbol{u}\right|^2 \underbrace{-\rho\left[e(\rho, s) - (-\mathrm{D}_t\phi)\, s\right]}_{= -\rho f(\rho, -\mathrm{D}_t \phi)} + \underbrace{\tfrac{1}{2}\alpha(\rho)\left|\boldsymbol{j}\right|^2 - \alpha(\rho)\boldsymbol{j} \cdot \nabla\phi}_{= -\frac{1}{2}\alpha(\rho)\left|\nabla\phi\right|^2}, \tag{37}$$

so that the use of the free energy in the Lagrangian is also justified. Therefore, in this new derivation, all previous assumptions have been lifted and identifying only the entropy production rate allowed us not only to recover an identical model but also to recover the previous assumptions as a result of Hamilton's principle.

We end this section that by showing that the new system admits a Gibbs equation in the sense of extended thermodynamics. Indeed rewriting the total energy E_{ext} and pressure tensor $\boldsymbol{\Pi}$ as

$$E_{\mathrm{ext}} = \tfrac{1}{2}\left|\boldsymbol{u}\right|^2 + e_{\mathrm{ext}}(\rho, s, \boldsymbol{j}), \tag{38a}$$
$$\boldsymbol{\Pi} = p_{\mathrm{ext}}(\rho, s, \boldsymbol{j})\mathbf{Id} + \alpha(\rho)\boldsymbol{j} \otimes \boldsymbol{j}, \tag{38b}$$

where $e_{\mathrm{ext}} = e(\rho, s) + \frac{1}{2}\rho^{-1}\alpha(\rho)\left|\boldsymbol{j}\right|^2$ and $p_{\mathrm{ext}} = p + \frac{1}{2}(\rho\alpha'(\rho) - \alpha(\rho)\left|\boldsymbol{j}\right|^2$, we obtain the following Gibbs relation

$$\mathrm{d}e_{\mathrm{ext}} = T\mathrm{d}s - p_{\mathrm{ext}}\mathrm{d}(\tfrac{1}{\rho}) + \rho^{-1}\alpha(\rho)\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{j}. \tag{39}$$

In particular, the augmented pressure appears as the natural pressure in the extended thermodynamic framework. Similarly, the augmented free energy can be defined as $f_{\mathrm{ext}} = e_{\mathrm{ext}} - Ts$, as a function of T, ρ and $\boldsymbol{j}$

3. Comparison with a model derived from kinetic theory

Comparisons between models derived through the kinetic theory and the extended thermodynamics theory have been proposed in previous works, in particular for Grad's 13-moment model which includes a variable for the heat flux but also the stress tensor [23]. As the models derived from the kinetic theory through the method of moments can suffer from mathematical shortcomings, such as lack of global hyperbolicity or lack of entropy equation, some regularization processes have been proposed [4, 28]. In this section, we will not apply such techniques, and we will restrict ourselves to recalling the basic steps for the derivation of the kinetic model (9) and will compare it with the previous model. The kinetic theory provides the macroscopic properties of a fluid based on its microscopic description through a number density function f such that $f(t, \boldsymbol{x}, \boldsymbol{v})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{v}$ indicates the number of molecules at $(t, \boldsymbol{x})$ with velocity $\boldsymbol{v}$. In the case of a gas composed of molecules with mass m with no internal degrees of freedom and such that the gas has mean velocity $\boldsymbol{u}$, density ρ and is in thermal equilibrium at temperature T, the velocity distribution is given by the equilibrium distribution $\mathbb{M}$ called Maxwellian, which writes

$$\mathbb{M}(\rho, T, \boldsymbol{u}) = \frac{\rho}{m}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{1}{2}\frac{m\,(\boldsymbol{v}-\boldsymbol{u})^2}{k_B T}\right). \tag{40}$$

This expression for the Maxwellian is directly related to Boltzmann's collision integral for binary elastic collisions between hard spheres for which it cancels out the collision operator and the entropy production term [7]. When there is a deviation of the particles from thermal equilibrium, the Maxwellian distribution no longer holds and one has to solve the Boltzmann equation. A standard simplification of the Boltzmann equation is to replace the particle collision operator with the so-called BGK operator [1], which corresponds to a relaxation towards thermal equilibrium with a characteristic relaxation time $\tau > 0$. The kinetic equation then writes

$$\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f = \frac{\mathbb{M}(f) - f}{\tau}, \tag{41}$$

where, with a slight abuse of notation, $\mathbb{M}(f)$ refers to the Maxwellian distribution (40) taken at the density ρ, temperature T and velocity $\boldsymbol{u}$ defined by f through its moments. Denoting $\langle\bullet\rangle$ the integration operator over all velocities $\boldsymbol{v} \in \mathbb{R}^d$, the 8 first moments of f are given by

$$m\langle f\rangle = \rho, \tag{42a}$$
$$m\langle \boldsymbol{v} f\rangle = \rho\boldsymbol{u}, \tag{42b}$$
$$m\langle \tfrac{1}{2}\left|\boldsymbol{v}\right|^2 f\rangle = \rho E = \tfrac{1}{2}\rho\left|\boldsymbol{u}\right|^2 + \frac{3}{2}\rho\frac{k_B}{m}T, \tag{42c}$$
$$m\langle \tfrac{1}{2}\boldsymbol{v}\left|\boldsymbol{v}\right|^2 f\rangle = \rho\boldsymbol{u}\,(E+p) - \boldsymbol{\tau}\boldsymbol{u} + \boldsymbol{q}. \tag{42d}$$

In particular, $\mathbb{M}(f)$ depends only on the first moments of f. In practice, it also useful to introduce the centred moments, setting $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}$, they write

$$m\langle \boldsymbol{c} f\rangle = 0, \tag{43a}$$
$$m\langle \tfrac{1}{2}\left|\boldsymbol{c}\right|^2 f\rangle = \frac{3}{2}\frac{k_B}{m}\rho T, \tag{43b}$$
$$m\langle \tfrac{1}{2}\boldsymbol{c}\left|\boldsymbol{c}\right|^2 f\rangle = \boldsymbol{q}. \tag{43c}$$

In particular, the kinetic description leads to the equation of state of a perfect gas with an adiabatic index $\gamma = 5/3$, i.e.

$$p = r\rho T, \quad e = \frac{r}{\gamma-1}T = \frac{p}{(\gamma-1)\rho}, \quad c^2 = \frac{\gamma p}{\rho}, \quad r = \frac{k_B}{m}, \quad \gamma = \frac{5}{3}, \tag{44}$$

with e the specific internal energy, c the speed of sound, and r the specific gas constant.

Next, we look for the distribution function f as a perturbation of the Maxwellian of the following form

$$f = \mathbb{M}(f)\left(1 + \boldsymbol{A}^{\intercal}\boldsymbol{c} + B|\boldsymbol{c}|^2 + |\boldsymbol{c}|^2\boldsymbol{D}^{\intercal}\boldsymbol{c}\right). \tag{45}$$

Solving this implicit equation for f leads to an expression that is entirely defined by its first moments and which writes

$$f = \mathbb{M}(\rho, T, \boldsymbol{u})\left(1 - \frac{\rho}{p^2}\left(1 - \frac{\gamma-1}{2\gamma}\frac{\rho}{p}\left|\boldsymbol{c}\right|^2\right)\boldsymbol{q}^{\intercal}\boldsymbol{c}\right) \tag{46}$$

where the heat flux $\boldsymbol{q} = m\langle \frac{1}{2}\boldsymbol{c}|\boldsymbol{c}|^2 f\rangle$ corresponds to a 3^{rd} order moment. The kinetic equation (41) for the obtained distribution (46) then yields to following moment equations

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{47a}$$

$$\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{0}, \tag{47b}$$

$$\mathrm{D}_t p + \gamma p \nabla \cdot \boldsymbol{u} + (\gamma - 1)\nabla \cdot \boldsymbol{q} = 0, \tag{47c}$$

$$\mathrm{D}_t \boldsymbol{q} + \tfrac{2\gamma-1}{\gamma}\left(\nabla \cdot \boldsymbol{u}\right)\boldsymbol{q} + \tfrac{\gamma-1}{\gamma}\left(\nabla \boldsymbol{u}\right)\boldsymbol{q} + \tfrac{2\gamma-1}{\gamma}\left(\boldsymbol{q}\cdot\nabla\right)\boldsymbol{u} + \tfrac{\gamma}{\gamma-1} p \nabla\left(\tfrac{p}{\rho}\right) = -\tfrac{1}{\tau}\boldsymbol{q}, \tag{47d}$$

which up to a change of variables are identical to (9). Note that in this model, the relaxation source term on the right-hand side of (47d) has not been added a posteriori but is a direct consequence of the BGK assumption for the relaxation operator. In the regime $\tau \ll 1$, a formal expansion of the heat flux in powers of τ, i.e. $\boldsymbol{q} = \boldsymbol{q}_0 + \tau \boldsymbol{q}_1 + \ldots$, yields

$$\boldsymbol{q} = -\tau \tfrac{\gamma}{\gamma-1} r^2 \rho T \nabla T + \mathcal{O}\left(\tau^2\right), \tag{48}$$

and we recover the Euler-Fourier system up to $\mathcal{O}\left(\tau^2\right)$ terms. Note that using higher-order moment models allows to recover the full Navier-Stokes system, however doing so with the BGK collision operator (41) will result in an expressions of the viscosity and the thermal conductivity corresponding to a Prandtl number Pr equal to one while for gases, one usually has Pr < 1. The use of more complicated collision operators such as the ES-BGK operator [3, 18] allows for a better treatment of the transport coefficients. Comparing model (9) with model (5), we observe their similar structure. In both cases, the heat flux is raised to the status of independent variable with its own evolution equation (through the equation of the thermal impulse for (5)) and with an identical dissipative source term which allows to recover Fourier's law in the asymptotic regime. However, the models also present some substantial differences. Indeed, the kinetic models are either only valid near equilibrium (Chapman-Enskog method), or strongly depend on the closure technique (moment method). On the other hand, the variational model (5) does not rely on such assumptions and, as a consequence, the model potentially involves higher-order terms, which are not present in the kinetic model (9). In particular, the momentum equation (5b) has a pressure tensor proportional to $\boldsymbol{q}\otimes\boldsymbol{q}$ which does not appear in the kinetic model. Although in the regime $\tau \ll 1$, or equivalently for low values of heat flux, this term might be negligible, it does contribute to the well-posedness of the system. System (5) is symmetric hyperbolic [8], while the kinetic model (9) is only conditionally hyperbolic. Indeed as is shown in the appendix (Section 6), in the one dimensional case, system (9) has complex eigenvalues when

$$|q| > \frac{1}{4\sqrt{6}}\sqrt{\left(7\sqrt{70} - 55\right)\rho c^3}. \tag{49}$$

Furthermore, the variational model admits an entropy equation, which gives a criterion for selecting shock-type solutions. It also yields the admissibility of the dissipative source term as it leads to a signed entropy production. This is not the case for the kinetic model. Although the dissipative source terms of model (9) are directly inherited from the kinetic relaxation operator, the system does not posses an entropic structure. Indeed, it it is well known that the solution of the Boltzmann equation satisfies the second principle through the well-known H-theorem[2] which yields the following entropy inequality

$$\partial_t \left(\rho s_{\text{kin}}\right) + \nabla \cdot \left(\rho s_{\text{kin}} \boldsymbol{u} + \boldsymbol{\sigma}_{\text{kin}}\right) \geq 0, \tag{50}$$

where the kinetic entropy and entropy flux are given by

$$\rho s_{\text{kin}} = -k_B \int_{\boldsymbol{v}} f\left(\ln(f) - 1\right) \mathrm{d}\boldsymbol{v}, \quad \boldsymbol{\sigma}_{\text{kin}} = -k_B \int_{\boldsymbol{v}} \boldsymbol{c} f\left(\ln(f) - 1\right) \mathrm{d}\boldsymbol{v}. \tag{51}$$

[2]The fact that we have used the BGK relaxation operator instead of the Boltzmann collision operator does not alter this result.

Such a signed conservation equation on the entropy is preserved for the asymptotic perturbation of a Maxwellian distribution[3] – that is in the parabolic asymptotic limit. For moment methods, some closure techniques allow to directly transfer the kinetic entropy equation to the moment model but their realizability is not always guaranteed [25], i.e. for a given set of moments, there might be no distribution function which admits said moments. Grad's closure does not suffer from any realizability issues as for any ρ, p, $\boldsymbol{u}$ and $\boldsymbol{q}$, the approximate distribution is given by (46). However, this approximate distribution does not guarantee an entropy equation for the moment equations. Furthermore, for large values of $\boldsymbol{q}$, the positivity of f in (46) may be lost. In this case, the integrals in (51) are no longer defined. For flows close to equilibrium, a reasonable heuristics is to replace $\ln(f)$ by its expansion in $\boldsymbol{q}$, doing so we obtain

$$s_{\text{approx}} = s - \frac{\gamma - 1}{2\gamma p^2 T} |\boldsymbol{q}|^2 \tag{52a}$$

$$\boldsymbol{\sigma}_{\text{approx}} = \frac{\boldsymbol{q}}{T}, \tag{52b}$$

where we have neglected all $\mathcal{O}(|\boldsymbol{q}|^3)$ terms. While it can be shown that the relaxation source term $-\boldsymbol{q}/\tau$ in (9) yields a positive entropy production for s_{approx}, unfortunately, because of the aforementioned approximations, there is no conservation law associated with this entropy. The variational model on the other hand, has a coherent thermodynamic structure for which the same relaxation source term – up to a change of variables between $\boldsymbol{j}$ and $\boldsymbol{q}$ – is always dissipative. An interesting open question is to clarify if such a structure can be attained through a kinetic theory of gases approach relying on a more complex collision operator and to what extent such an approach can be compared to the system obtained through the Stationary Action Principle.

4. Conclusions

In this work, we have provided a generic framework for the derivation of conservation laws through the Stationary Action Principle. The framework has then been applied to the derivation of a variational model, which is the hyperbolic approximation of the Euler-Fourier model presented in [8]. In particular we have proposed an alternative derivation, which provides new insights on the assumptions used by the authors. We then have proposed a comparison with a model obtained from kinetic theory through the method of moments. The similarities between the models allow for justifying the dissipative source term that was added to the variational model. Indeed, although the Stationary Action Principle allows for the consistent derivation of systems of conservation laws, it is only applicable to non-dissipative systems and any dissipation can only be added a posteriori. Moreover, the variational principle offers close to no guidance in choosing the correct dissipation to add to the system whereas, for the kinetic model, the dissipation on the moment equations is directly inherited from the microscopic description. However, the kinetic models coming from the method of moments suffer from mathematical shortcomings such as the loss of hyperbolicity and the lack of an associated entropic structure. Although regularization processes have been developed and many works to identify moment closures, which lead to hyperbolic systems, have been proposed in the literature, the Stationary Action Principle appears as a possible alternative modelling technique and the present contribution paves an interesting way for a more detailed comparison of both approaches.

5. Appendix: Proofs of theorems 1.1 and 1.2

This section is devoted to the proofs of theorems 1.1 and 1.2, to do so we first define some notations and recall some vector calculus identities.

[3]The macroscopic entropy, for both Euler and Navier-Stokes systems of equations, is the kinetic entropy taken at equilibrium and is the opposite of a mathematical entropy [11, 14]

For any scalar field b, its differential is denoted $\frac{\partial b}{\partial \boldsymbol{x}}$ and is a row vector; its gradient is denoted ∇b and is the column vector defined as the transpose of the differential. More generally, derivation of a scalar field with respect to a vector field yields a row vector e.g. $\frac{\partial \mathscr{L}}{\partial \boldsymbol{u}} = (\frac{\partial \mathscr{L}}{\partial \boldsymbol{u}_1}, \dots, \frac{\partial \mathscr{L}}{\partial \boldsymbol{u}_d})$.
For any vector field $\boldsymbol{B}$, we denote its Jacobian matrix by $\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}} = (\frac{\partial \boldsymbol{B}_i}{\partial \boldsymbol{x}_j})_{i,j}$. Its gradient is the transpose of the Jacobian matrix $\nabla \boldsymbol{B} = (\frac{\partial \boldsymbol{B}_j}{\partial \boldsymbol{x}_i})_{i,j}$. In particular, as the material derivative of a vector field is taken component-wise $\mathrm{D}_t \boldsymbol{B} = (\mathrm{D}_t \boldsymbol{B}_i)_i$, we have $\mathrm{D}_t \boldsymbol{B} = \partial_t \boldsymbol{B} + \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}} \boldsymbol{u}$.
The tensor product of two vector fields $\boldsymbol{A}$, $\boldsymbol{B}$ is denoted $\boldsymbol{A} \otimes \boldsymbol{B}$ and is given by the matrix $(\boldsymbol{A}_i \boldsymbol{B}_j)_{ij}$. The divergence of a matrix is the column vector obtained by taking the divergence of each row vector i.e. $(\nabla \cdot \boldsymbol{M})_i = \sum_j \frac{\partial \boldsymbol{M}_{ij}}{\partial \boldsymbol{x}_j}$. The following proposition is a direct consequence of these definitions.

Proposition 5.1. *Consider a scalar field b and two vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$, we have the following vector calculus identities*

$$\nabla \cdot (\boldsymbol{A} \otimes \boldsymbol{B}) = (\nabla \cdot \boldsymbol{B})\, \boldsymbol{A} + (\partial_{\boldsymbol{x}} \boldsymbol{A})\, \boldsymbol{B} = (\nabla \cdot \boldsymbol{B})\, \boldsymbol{A} + (\nabla \boldsymbol{A})^{\mathsf{T}}\, \boldsymbol{B} \tag{53a}$$

$$\mathrm{D}_t (\nabla b) = \nabla (\mathrm{D}_t b) - (\nabla \boldsymbol{u})\, \nabla b \tag{53b}$$

$$\nabla (\boldsymbol{A}^{\mathsf{T}} \boldsymbol{B}) = (\nabla \boldsymbol{A})\, \boldsymbol{B} + (\nabla \boldsymbol{B})\, \boldsymbol{A} \tag{53c}$$

5.1. Proof of theorem 1.1

Proof. Equations (20a) and (20b) are constraints which are expressed through variations of the trajectory $\delta \boldsymbol{\Phi}$. It is therefore necessary to show that variations of the action lead to (20c) and (20d). For simplicity, we denote the set of independent variables of the Lagrangian by $\mathrm{Var} = \{\boldsymbol{u}, \rho, (b_i)_i, (\xi_j)_j, (\mathrm{D}_t \xi_j)_j, (\nabla \xi_j)_j\}$, we then have that

$$\delta \mathscr{A} = \sum_{\mathcal{X} \in \mathrm{Var}} \iint_\Omega \frac{\partial \mathscr{L}}{\partial \mathcal{X}} \delta \mathcal{X} = \sum_{\mathcal{X} \in \mathrm{Var}} \delta \mathscr{A}_{\mathcal{X}}. \tag{54}$$

Next, we express the variations $\delta \mathcal{X}$ in terms of the independent virtual displacements $\delta \boldsymbol{\Phi}$ and $\delta \xi_j$, $(j = 1, \dots, n)$. Recall that these variations are assumed to vanish at the boundaries i.e. $\delta \boldsymbol{\Phi}(t, \boldsymbol{x} = \boldsymbol{0}$ if $(t, \boldsymbol{x}) \in \partial \Omega$, and same goes for $\delta \xi_j$. The underlying Lagrangian description for variables $\boldsymbol{u}$, ρ and $(b_i)_i$ leads to

$$\delta \boldsymbol{u} = \mathrm{D}_t (\delta \boldsymbol{\Phi}) - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \delta \boldsymbol{\Phi}, \tag{55a}$$

$$\delta \rho = -\nabla \cdot (\rho \delta \boldsymbol{\Phi}), \tag{55b}$$

$$\delta b_i = -(\delta \boldsymbol{\Phi})^{\mathsf{T}} \nabla b_i, \quad i = 1, \dots, n. \tag{55c}$$

These expressions are quite classical and can be found in the literature [2,10]. For the independent variables, we commute the δ operator with space and time derivatives and use the Leibniz rule. Hence the following identities hold

$$\delta (\nabla \xi_j) = \nabla (\delta \xi_j), \tag{56a}$$

$$\delta (\mathrm{D}_t \xi_j) = \mathrm{D}_t (\delta \xi_j) + (\nabla \xi_j)^{\mathsf{T}} \delta \boldsymbol{u}. \tag{56b}$$

We now write the variation of the action with respect to the different variables. Starting with the variation with respect to the velocity field we get

$$\delta \mathscr{A}_{\boldsymbol{u}} = \iint_\Omega \boldsymbol{K}^{\mathsf{T}} \delta \boldsymbol{u} = -\iint_\Omega \delta \boldsymbol{\Phi}^{\mathsf{T}} \left\{ \partial_t \boldsymbol{K} + \nabla \cdot (\boldsymbol{K} \otimes \boldsymbol{u}) + (\nabla \boldsymbol{u})\, \boldsymbol{K} \right\}. \tag{57}$$

Variations with respect to the density write

$$\delta \mathscr{A}_\rho = \iint_\Omega \delta \boldsymbol{\Phi}^{\mathsf{T}} \left\{ \rho \nabla \left(\frac{\partial \mathscr{L}}{\partial \rho} \right) \right\} = \iint_\Omega \delta \boldsymbol{\Phi}^{\mathsf{T}} \left\{ \nabla \left(\rho \frac{\partial \mathscr{L}}{\partial \rho} \right) - \frac{\partial \mathscr{L}}{\partial \rho} \nabla \rho \right\}, \tag{58}$$

while variations with respect to the transported variables are

$$\delta \mathscr{A}_{b_i} = -\iint_\Omega \delta \mathbf{\Phi}^\mathsf{T} \frac{\partial \mathscr{L}}{\partial b_i} \nabla b_i, \quad i = 1, \ldots, n. \tag{59}$$

Next, turning towards the additional variables, for $j = 1, \ldots, m$, we get

$$\delta \mathscr{A}_{\xi_j} = \iint_\Omega \delta \xi_j \frac{\partial \mathscr{L}}{\partial \xi_j}, \tag{60}$$

$$\delta \mathscr{A}_{\nabla \xi_j} = -\iint_\Omega \delta \xi_j \nabla \cdot \boldsymbol{D}_j, \tag{61}$$

$$\delta \mathscr{A}_{\mathrm{D}_t \xi_j} = -\iint_\Omega \delta \xi_j \left\{ \partial_t \mathscr{M}_j + \nabla \cdot (\mathscr{M}_j \boldsymbol{u}) \right\} - \iint_\Omega \delta \mathbf{\Phi}^\mathsf{T} \left\{ \left[\partial_t \mathscr{M}_j + \nabla \cdot (\mathscr{M}_j \boldsymbol{u}) \right] \nabla \xi_j + \mathscr{M}_j \nabla \left(\mathrm{D}_t \xi_j \right) \right\}. \tag{62}$$

The complete variation of the action then writes

$$\begin{aligned} \delta \mathscr{A} = -\iint_\Omega \delta \mathbf{\Phi}^\mathsf{T} \Bigg\{ & \partial_t \boldsymbol{K} + \nabla \cdot (\boldsymbol{K} \otimes \boldsymbol{u}) - \nabla \left(\rho \frac{\partial \mathscr{L}}{\partial \rho} \right) \\ & + (\nabla \boldsymbol{u}) \boldsymbol{K} + (\nabla \rho) \frac{\partial \mathscr{L}}{\partial \rho} + \sum_{i=1}^n (\nabla b_i) \frac{\partial \mathscr{L}}{\partial b_i} + \sum_{j=1}^m \left[\nabla \left(\mathrm{D}_t \xi_j \right) \frac{\partial \mathscr{L}}{\partial \left(\mathrm{D}_t \xi_j \right)} + \left[\partial_t \mathscr{M}_j + \nabla \cdot (\mathscr{M}_j \boldsymbol{u}) \right] \nabla \xi_j \right] \Bigg\} \\ & - \iint_\Omega \sum_{j=1}^m \delta \xi_j \left\{ \partial_t \mathscr{M}_j + \nabla \cdot (\mathscr{M}_j \boldsymbol{u}) - \frac{\partial \mathscr{L}}{\partial \xi_j} + \nabla \cdot \boldsymbol{D}_j \right\}. \end{aligned} \tag{63}$$

According to the Stationary Action Principle we have $\delta \mathscr{A} = 0$ for any $\delta \mathbf{\Phi}$ and $\delta \xi_j$. Variations with respect to $\delta \xi_j$ yield (20c) while variations with respect to $\delta \mathbf{\Phi}$ lead to

$$\partial_t \boldsymbol{K} + \nabla \cdot (\boldsymbol{K} \otimes \boldsymbol{u}) - \nabla \left(\rho \frac{\partial \mathscr{L}}{\partial \rho} \right) + (\nabla \boldsymbol{u}) \boldsymbol{K} + \sum_{i=1}^n \frac{\partial \mathscr{L}}{\partial b_i} \nabla b_i + \frac{\partial \mathscr{L}}{\partial \rho} \nabla \rho + \sum_{j=1}^m \left[\frac{\partial \mathscr{L}}{\partial \left(\mathrm{D}_t \xi_j \right)} \nabla \left(\mathrm{D}_t \xi_j \right) + \left[\frac{\partial \mathscr{L}}{\partial \xi_j} - \nabla \cdot \boldsymbol{D}_j \right] \nabla \xi_j \right] = 0. \tag{64}$$

Using (20c) as well as the following identity

$$\nabla \mathscr{L} = (\nabla \boldsymbol{u}) \boldsymbol{K} + (\nabla \rho) \frac{\partial \mathscr{L}}{\partial \rho} + \sum_{i=1}^n (\nabla b_i) \frac{\partial \mathscr{L}}{\partial b_i} + \sum_{j=1}^m \left((\nabla \xi_j) \frac{\partial \mathscr{L}}{\partial \xi_j} + \nabla \left(\mathrm{D}_t \xi_j \right) \frac{\partial \mathscr{L}}{\partial \left(\mathrm{D}_t \xi_j \right)} + \nabla \left(\nabla \xi_j \right) \boldsymbol{D}_j \right) \tag{65}$$

allows one to write

$$\partial_t \boldsymbol{K} + \nabla \left(\boldsymbol{K} \otimes \boldsymbol{u} \right) - \nabla \left(\rho \frac{\partial \mathscr{L}}{\partial \rho} - \mathscr{L} \right) - \sum_{j=1}^m \left[\nabla \left(\nabla \xi_j \right) \boldsymbol{D}_j + \left(\nabla \cdot \boldsymbol{D}_j \right) \nabla \xi_j \right] = \mathbf{0}. \tag{66}$$

Since $\nabla \left(\nabla \xi_j \right)$ is a symmetric matrix, using (53a), we obtain equation (20d) which completes the proof. □

5.2. **Proof of theorem 1.2**

Proof. Using the conservation equation (20a) we get that eq. (23) is equivalent to

$$\rho \mathrm{D}_t \left(\frac{\mathcal{H}}{\rho}\right) - \nabla \cdot (\mathscr{L}^\star \boldsymbol{u}) + \sum_{j=1}^{m} \nabla \cdot (\boldsymbol{D}_j \partial_t \xi_j) = 0. \tag{67}$$

Using the definition of the Hamiltonian (22), it becomes

$$\rho \mathrm{D}_t \left(\frac{\mathscr{L}}{\rho}\right) = \rho \mathrm{D}_t \left(\frac{\boldsymbol{K}^\intercal \boldsymbol{u}}{\rho}\right) + \rho \sum_{j=1}^{m} \mathrm{D}_t \left(\frac{\mathscr{M}_j \mathrm{D}_t \xi_j}{\rho}\right) - \nabla \cdot (\mathscr{L}^\star \boldsymbol{u}) + \sum_{j=1}^{m} \nabla \cdot (\boldsymbol{D}_j \partial_t \xi_j)\,. \tag{68}$$

Using the Leibniz rule we also get that

$$\rho \mathrm{D}_t \left(\frac{\mathscr{L}}{\rho}\right) = \mathrm{D}_t \mathscr{L} - \mathscr{L} \frac{\mathrm{D}_t \rho}{\rho}. \tag{69}$$

Finally it is sufficient to prove that

$$\mathrm{D}_t \mathscr{L} = \mathscr{L} \frac{\mathrm{D}_t \rho}{\rho} + \rho \mathrm{D}_t \left(\frac{\boldsymbol{K}^\intercal \boldsymbol{u}}{\rho}\right) + \rho \sum_{j=1}^{m} \mathrm{D}_t \left(\frac{\mathscr{M}_j \mathrm{D}_t \xi_j}{\rho}\right) - \nabla \cdot (\mathscr{L}^\star \boldsymbol{u}) + \sum_{j=1}^{m} \nabla \cdot (\boldsymbol{D}_j \partial_t \xi_j)\,. \tag{70}$$

First, using the chain rule, we write

$$\mathrm{D}_t \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \rho} \mathrm{D}_t \rho + \frac{\partial \mathscr{L}}{\partial \boldsymbol{u}} \mathrm{D}_t \boldsymbol{u} + \sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial b_i} \mathrm{D}_t b_i + \sum_{j=1}^{m} \left[\frac{\partial \mathscr{L}}{\partial \xi_j} \mathrm{D}_t \xi_j + \frac{\partial \mathscr{L}}{\partial (\nabla \xi_j)} \mathrm{D}_t (\nabla \xi_j) + \frac{\partial \mathscr{L}}{\partial (\mathrm{D}_t \xi_j)} \mathrm{D}_t (\mathrm{D}_t \xi_j)\right]. \tag{71}$$

We then use the following identities

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{u}} \mathrm{D}_t \boldsymbol{u} = \boldsymbol{K}^\intercal \mathrm{D}_t \boldsymbol{u} = \mathrm{D}_t (\boldsymbol{K}^\intercal \boldsymbol{u}) - \boldsymbol{u}^\intercal \mathrm{D}_t \boldsymbol{K} = \rho \mathrm{D}_t \left(\frac{\boldsymbol{K}^\intercal \boldsymbol{u}}{\rho}\right) - \rho \boldsymbol{u}^\intercal \mathrm{D}_t \left(\frac{\boldsymbol{K}}{\rho}\right), \tag{72a}$$

$$\frac{\partial \mathscr{L}}{\partial (\mathrm{D}_t \xi_j)} \mathrm{D}_t (\mathrm{D}_t \xi_j) = \mathscr{M}_j \mathrm{D}_t (\mathrm{D}_t \xi_j) = \mathrm{D}_t (\mathscr{M}_j \mathrm{D}_t \xi_j) - \mathrm{D}_t \xi_j \mathrm{D}_t \mathscr{M}_j = \rho \mathrm{D}_t \left(\frac{\mathscr{M}_j \mathrm{D}_t \xi_j}{\rho}\right) - \rho \mathrm{D}_t \xi_j \mathrm{D}_t \left(\frac{\mathscr{M}_j}{\rho}\right). \tag{72b}$$

From left to right, the first equality is obtained using the definitions (21), the second equality is obtained using the Leibniz rule and the rightmost equality is obtained using eq. (20a). We also use a third identity which writes

$$\frac{\partial \mathscr{L}}{\partial \rho} \mathrm{D}_t \rho = \frac{\mathrm{D}_t \rho}{\rho} \left(\rho \frac{\partial \mathscr{L}}{\partial \rho} - \mathscr{L}\right) + \mathscr{L} \frac{\mathrm{D}_t \rho}{\rho} = -(\nabla \cdot \boldsymbol{u}) \mathscr{L}^\star + \mathscr{L} \frac{\mathrm{D}_t \rho}{\rho}. \tag{72c}$$

Using identities (72a), (72b) and (72c) as well as $\mathrm{D}_t b_i = 0$, (71) becomes

$$\begin{aligned} \mathrm{D}_t \mathscr{L} = \mathscr{L} \frac{\mathrm{D}_t \rho}{\rho} + \rho \mathrm{D}_t \left(\frac{\boldsymbol{K}^\intercal \boldsymbol{u}}{\rho}\right) - \rho \boldsymbol{u}^\intercal \mathrm{D}_t \left(\frac{\boldsymbol{K}}{\rho}\right) - (\nabla \cdot \boldsymbol{u}) \mathscr{L}^\star \\ + \sum_{j=1}^{m} \left[\frac{\partial \mathscr{L}}{\partial \xi_j} \mathrm{D}_t \xi_j + \boldsymbol{D}_j^\intercal \mathrm{D}_t (\nabla \xi_j) + \rho \mathrm{D}_t \left(\frac{\mathscr{M}_j \mathrm{D}_t \xi_j}{\rho}\right) - \rho \mathrm{D}_t \xi_j \mathrm{D}_t \left(\frac{\mathscr{M}_j}{\rho}\right)\right]. \end{aligned} \tag{73}$$

We then use the evolution equations for $\mathrm{D}_t\left(\frac{\boldsymbol{K}}{\rho}\right)$ and $\mathrm{D}_t\left(\frac{\mathscr{M}_j}{\rho}\right)$ to get

$$\begin{aligned}\mathrm{D}_t\mathscr{L} &= \mathscr{L}\frac{\mathrm{D}_t\rho}{\rho} + \rho\mathrm{D}_t\left(\frac{\boldsymbol{K}^\intercal\boldsymbol{u}}{\rho}\right) - \nabla\cdot\left(\mathscr{L}^\star\boldsymbol{u}\right) \\ &+ \sum_{j=1}^{m}\left[\left(\nabla\cdot\boldsymbol{D}_j\right)\mathrm{D}_t\xi_j + \boldsymbol{D}_j^\intercal\mathrm{D}_t\left(\nabla\xi_j\right) - \boldsymbol{u}^\intercal\nabla\cdot\left(\nabla\xi_j\otimes\boldsymbol{D}_j\right) + \rho\mathrm{D}_t\left(\frac{\mathscr{M}_j\mathrm{D}_t\xi_j}{\rho}\right)\right].\end{aligned} \tag{74}$$

The last part of this equation is treated using the following vectorial identities

$$\left(\nabla\cdot\boldsymbol{D}_j\right)\mathrm{D}_t\xi_j = \left(\nabla\cdot\boldsymbol{D}_j\right)\partial_t\xi_j + \left(\nabla\cdot\boldsymbol{D}_j\right)\boldsymbol{u}^\intercal\nabla\xi_j, \tag{75a}$$

$$\boldsymbol{D}_j^\intercal\mathrm{D}_t\left(\nabla\xi_j\right) = \boldsymbol{D}_j^\intercal\nabla\left(\partial_t\xi_j\right) + \boldsymbol{D}_j^\intercal\nabla\left(\boldsymbol{u}^\intercal\nabla\xi_j\right) - \boldsymbol{D}_j^\intercal\left(\nabla\boldsymbol{u}\right)\nabla\xi_j. \tag{75b}$$

Summing these yields

$$\begin{aligned}\left(\nabla\cdot\boldsymbol{D}_j\right)\mathrm{D}_t\xi_j + \boldsymbol{D}_j^\intercal\mathrm{D}_t\left(\nabla\xi_j\right) &= \nabla\cdot\left(\boldsymbol{D}_j\partial_t\xi_j\right) + \left(\nabla\cdot\boldsymbol{D}_j\right)\boldsymbol{u}^\intercal\nabla\xi_j + \boldsymbol{D}_j^\intercal\nabla\left(\boldsymbol{u}^\intercal\nabla\xi_j\right) - \boldsymbol{D}_j^\intercal\left(\nabla\boldsymbol{u}\right)\nabla\xi_j &(76)\\ &= \nabla\cdot\left(\boldsymbol{D}_j\partial_t\xi_j\right) + \left(\nabla\cdot\boldsymbol{D}_j\right)\boldsymbol{u}^\intercal\nabla\xi_j + \boldsymbol{D}_j^\intercal\nabla\left(\nabla\xi_j\right)\boldsymbol{u} \\ &= \nabla\cdot\left(\boldsymbol{D}_j\partial_t\xi_j\right) + \boldsymbol{u}^\intercal\nabla\cdot\left(\nabla\xi_j\otimes\boldsymbol{D}_j\right), &(77)\end{aligned}$$

so that finally,

$$\mathrm{D}_t\mathscr{L} = \mathscr{L}\frac{\mathrm{D}_t\rho}{\rho} + \rho\mathrm{D}_t\left(\frac{\boldsymbol{K}^\intercal\boldsymbol{u}}{\rho}\right) + \rho\sum_{j=1}^{m}\mathrm{D}_t\left(\frac{\mathscr{M}_j\mathrm{D}_t\xi_j}{\rho}\right) - \nabla\cdot\left(\mathscr{L}^\star\boldsymbol{u}\right) + \sum_{j=1}^{m}\nabla\cdot\left(\boldsymbol{D}_j\partial_t\xi_j\right), \tag{78}$$

which is the desired relation (70) and thus the proof is complete. □

6. Appendix: Hyperbolicity of the one-dimensional 8-moment Grad equations

In the one-dimensional setting, the equations can be written in quasi-linear form, introducing $\boldsymbol{W} = (\rho,\, u,\, p,\, q)^\intercal$, we obtain

$$\partial_t\boldsymbol{W} + \mathbb{A}(\boldsymbol{W})\partial_x\boldsymbol{W} = 0, \tag{79}$$

with

$$\mathbb{A}(\boldsymbol{W}) = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & 0 \\ 0 & \rho c^2 & u & \frac{2}{3} \\ -\frac{9}{10}c^4 & \frac{16}{5}q & \frac{3}{2}c^2 & u \end{bmatrix}. \tag{80}$$

For the system to be hyperbolic, $\mathbb{A}$ must have real eigenvalues and a complete set of real eigenvectors. Its eigenvalues λ are the roots of its characteristic polynomial, which writes

$$\Lambda^4 - 2\Lambda^2 - \tfrac{32}{15}\varkappa\Lambda + \tfrac{3}{5} = 0, \tag{81}$$

with $\Lambda = \frac{\lambda-u}{c}$ and $\varkappa = \frac{q}{\rho c^3}$. The nature of the roots of 4^{th} order polynomials can be directly determined by its coefficients through its discriminant and some additional quantities. Here they lead to the introduction of a critical value for the non-dimensional heat flux given by

$$\varkappa_{\text{crit}} = \tfrac{1}{4}\sqrt{\tfrac{1}{6}\left(7\sqrt{70} - 55\right)} \approx 0.192738. \tag{82}$$

When $|\varkappa| < \varkappa_{\text{crit}}$, we have that $\mathbb{A}$ has four real distinct eigenvalues and the system is strictly hyperbolic. However for a sufficiently high non-dimensional heat flux, in the sense that $|\varkappa| > \varkappa_{\text{crit}}$, we observe a loss of hyperbolicity as the system now has two complex conjugate eigenvalues. In the limit case $|\varkappa| = \varkappa_{\text{crit}}$, the system has two real simple eigenvalues and a real double eigenvalue, the nature of the eigenvectors not having been investigated, it is at least weakly hyperbolic.

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References

[1] P. L. Bhatnagar, E. P. Gross, and M. Krook. A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems. *Physical Review*, 94(3):511–525, May 1954.
[2] F. P. Bretherton. A note on Hamilton's principle for perfect fluids. *Journal of Fluid Mechanics*, 44(1):19–31, October 1970.
[3] S. Brull, V. Pavan, and J. Schneider. Derivation of a BGK model for mixtures. *European Journal of Mechanics - B/Fluids*, 33:74–86, May 2012.
[4] Z. Cai, Y. Fan, and R. Li. Globally Hyperbolic Regularization of Grad's Moment System. *Communications on Pure and Applied Mathematics*, 67(3):464–518, 2014.
[5] C. Cattaneo. Sulla conduzione del calore. *Atti del Seminario Matematico e Fisico della Università di Modena*, 3:83–101, 1948.
[6] C. Cattaneo. Sur une forme de I'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. *Comptes Rendus de l'Académie des Sciences*, 247:431–433, 1958.
[7] S. Chapman and T. G. Cowling. *The Mathematical Theory of Non-Uniform Gases an Account of the Kinetic Theory of Viscosity, Thermal Condition and Diffusion in Gases.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 3rd ed. prepared in co-operation with. edition, 1970.
[8] F. Dhaouadi and S. Gavrilyuk. An Eulerian hyperbolic model for heat transfer derived via Hamilton's principle: Analytical and numerical study. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 480(2283):20230440, February 2024.
[9] S. L. Gavrilyuk and S. M. Shugrin. Media with equations of state that depend on derivatives. *Journal of Applied Mechanics and Technical Physics*, 37(2):177–189, March 1996.
[10] S. Gavrilyuk. 'Uncertainty' principle in two fluid–mechanics. *ESAIM: Proceedings and Surveys*, 69:47–55, 2020.
[11] V. Giovangigli. *Multicomponent flow modeling.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston Inc., Boston, MA, 1999.
[12] V. Giovangigli. Kinetic derivation of diffuse-interface fluid models. *Physical Review E*, 102(1):012110, July 2020.
[13] V. Giovangigli. Kinetic derivation of Cahn-Hilliard fluid models. *Physical Review E*, 104(5):054109, November 2021.
[14] V. Giovangigli and M. Massot. Asymptotic stability of equilibrium states for multicomponent reactive flows. *Mathematical Models and Methods in Applied Sciences*, 08(02):251–297, 1998.
[15] H. Grad. On the kinetic theory of rarefied gases. *Communications on Pure and Applied Mathematics*, 2(4):331–407, 1949.
[16] A. E. Green and P. M. Naghdi. Thermoelasticity without energy dissipation. *Journal of Elasticity*, 31(3):189–208, June 1993.
[17] A. E. Green and P. M. Naghdi. A re-examination of the basic postulates of thermomechanics. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 432(1885):171–194, February 1991.
[18] L. H. Holway, Jr. New Statistical Models for Kinetic Theory: Methods of Construction. *The Physics of Fluids*, 9(9):1658–1673, September 1966.
[19] G. M. Kremer. *An Introduction to the Boltzmann Equation and Transport Processes in Gases*. Interaction of Mechanics and Mathematics. Springer, Berlin, Heidelberg, 2010.
[20] G. Lebon, D. Jou, and J. Casas-Vázquez. *Understanding Non-equilibrium Thermodynamics*. Springer Berlin, Heidelberg, 1st edition, December 2007.
[21] C. D. Levermore. Moment closure hierarchies for kinetic theories. *Journal of Statistical Physics*, 83(5):1021–1065, June 1996.
[22] R. McGraw. Description of Aerosol Dynamics by the Quadrature Method of Moments. *Aerosol Science and Technology*, 27(2):255–265, January 1997.
[23] I. Müller and T. Ruggeri. *Rational Extended Thermodynamics.* Springer Tracts in Natural Philosophy. Springer New York, YK, 2nd edition, December 2011.
[24] I. Peshkov, M. Pavelka, E. Romenski, and M. Grmela. Continuum Mechanics and Thermodynamics in the Hamilton and the Godunov-type Formulations. *Continuum Mechanics and Thermodynamics*, 30(6):1343–1378, November 2018.
[25] T. Pichard. Some recent advances on the method of moments in kinetic theory. In *SMAI 2021*, volume 75, pages 86–95, la Grande Motte, France, June 2021.

[26] E. Romenski, A. Resnyansky, and E. Toro. Conservative hyperbolic formulation for compressible two-phase flow with different phase pressures and temperatures. *Quarterly of Applied Mathematics*, 65:259–280, April 2007.

[27] D. Serre. Viscous System of Conservation Laws: Singular Limits. In Alberto Bressan, Gui-Qiang G. Chen, Marta Lewicka, and Dehua Wang, editors, *Nonlinear Conservation Laws and Applications*, pages 433–445, Boston, MA, 2011. Springer US.

[28] H. Struchtrup and M. Torrilhon. Regularization of Grad's 13 moment equations: Derivation and linear analysis. *Physics of Fluids*, 15(9):2668–2680, September 2003.

[29] P. Vernotte. Les paradoxes de la théorie continue de l'équation de la chaleur. *Comptes Rendus de l'Académie des Sciences*, 246:3154–3155, 1958.