ESAIM: PROCEEDINGS AND SURVEYS, November 2025, Vol. 78, p.136-143
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TWO UNSTEADY SOLUTIONS OF A HOMOGENEOUS EQUILIBRIUM MODEL IN A POROUS MEDIUM

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Abstract. We present in this note two reference one-dimensional unsteady solutions of a HEM model, considering possible heat source terms, and focusing on a steady porous medium. The model is recalled first, then details on exact solutions are provided. Several Equations Of State are discussed.

Introduction

Reference solutions are useful, in order to better understand the main features in fluid models, but also to verify algorithms in CFD codes.

Some exact solutions are available for Euler equations, for instance, both in the steady and the unsteady framework, at least in a one-dimensional setting. Regular solutions in rarefaction waves have long been well known (see [14, 8] for instance), and are also necessary in order to build the one-dimensional unsteady solution of the Riemann problem when no source term arises. For the Homogeneous Equilibrium Model (HEM) [3, 4] or the Homogeneous Relaxation Model (HRM) [2, 3, 4], steady and unsteady numerical solutions can also be found when considering fluid flows in ducts with variable cross section (see for instance [7, 9, 10, 12]). However, few exact solutions are available when investigating flows with heat transfer, even in the steady case (see [5, 6, 11, 13]), and to our knowledge none in the unsteady case, when retaining the HEM model (or Euler equations) in a porous medium (or equivalently in a duct with variable cross section).

Hence we focus herein on two reference **unsteady** solutions of the HEM model, in a 1D framework, when the fluid flows in a **porous medium**, or when **a heat source term** arises. Using the notations ε for porosity, ρ for density, u for velocity, p for pressure and $e(p,\rho)$ for internal energy, the corresponding system can be written as

$$\begin{aligned} \partial_t(\varepsilon\rho) + \partial_x(\varepsilon\rho u) &= 0, \\ \partial_t(\varepsilon\rho u) + \partial_x(\varepsilon\rho u^2) + \varepsilon\partial_x p &= 0, \\ \partial_t(\varepsilon E) + \partial_x(\varepsilon u(E+p)) &= \varepsilon\phi(x,t), \\ \partial_t\varepsilon &= 0, \end{aligned} \tag{1}$$

with the total energy $E = \rho(\frac{1}{2}u^2 + e(p,\rho))$, the internal energy $e(p,\rho)$ given by a thermodynamic law, the porosity $\varepsilon(x)$ given by the user and the heating source term $\phi(x,t)$ also given by the user.

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Article published online by EDP Sciences and available at https://www.esaim-proc.org or https://doi.org/10.1051/proc/202578136

1. A self-similar unsteady solution without heat exchange.

Considering System (1) without heat exchange, it writes

$$\begin{aligned} \partial_t(\varepsilon\rho) + \partial_x(\varepsilon\rho u) &= 0, \\ \partial_t(\varepsilon\rho u) + \partial_x(\varepsilon\rho u^2) + \varepsilon\partial_x p &= 0, \\ \partial_t(\varepsilon E) + \partial_x(\varepsilon u(E+p)) &= 0, \\ \partial_t \varepsilon &= 0. \end{aligned} \tag{2}$$

We note $\tau = 1/\rho$ the specific volume and we define the entropy as the function verifying

$$c^2(p,\rho)\left(\frac{\partial s}{\partial p}\right)_\rho + \left(\frac{\partial s}{\partial \rho}\right)_p = 0, \tag{3}$$

with c the speed of sound defined by

$$(\rho c)^2 = \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} (p - \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_p). \tag{4}$$

It is assumed that the speed of sound is real for the mixture equation of state considered. By making the classical variable change from $(\rho, \rho u, E, \varepsilon)$ to (s, u, p, ε), the system becomes

$$\begin{aligned} &\partial_t s + u\partial_x s = 0, \\ &\partial_t u + u\partial_x u + \tau\partial_x p = 0, \\ &\partial_t p + u\partial_x p + \rho c^2 \partial_x u + \rho c^2 \frac{u}{\varepsilon}\partial_x \varepsilon = 0, \\ &\partial_t \varepsilon = 0. \end{aligned} \tag{5}$$

By denoting the variable vector $Y = (s, u, p, \varepsilon)$, the system is rewritten in matrix form

$$\partial_t \mathbf{Y} + \mathbf{A}\partial_x \mathbf{Y} = 0, \tag{6}$$

$$\text{with } \mathbf{A} = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & \tau & 0 \\ 0 & \rho c^2 & u & \rho c^2 \frac{u}{\varepsilon} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

We search for solutions depending only on $\xi = \frac{x}{t}$. In this case, the solutions $u(\xi)$, $s(\xi)$ and $p(\xi)$ must verify, $\forall t > 0$,

$$\begin{cases} \frac{1}{t}(u-\xi)s'(\xi) = 0, \\ \frac{1}{t}(u-\xi)u'(\xi) + \frac{1}{t}\tau p'(\xi) = 0, \\ \frac{1}{t}(u-\xi)p'(\xi) + \frac{1}{t}\rho c^2 u'(\xi) = -\rho c^2 u \frac{\varepsilon'(x)}{\varepsilon(x)}. \end{cases} \tag{8}$$

In the following, we exclude the degenerate case $\varepsilon(x) = \varepsilon_0$. The third equation (multiplied by x) imposes a particular form to the porosity $\varepsilon(x)$, such that

$$\varepsilon(x) = \left(\frac{x}{x_0}\right)^\alpha, \text{ with a constant } \alpha \in \mathbb{R}^*. \tag{9}$$

Remark 1.1. Specific values for α.
We can highlight some specific values for α. The specific value $\alpha = 1$ corresponds to a radial solution corresponding to $\varepsilon(x) \propto 2\pi x$, while $\alpha = 2$ corresponds to a spherical solution where $\varepsilon(x) \propto 4\pi x^2$. □

The first equation imposes, for $u(\xi) \neq \xi$, a constant entropy

$$s(\xi) = s_0. \tag{10}$$

The solutions $(u(\xi), p(\xi))$ must be such that

$$\begin{cases} p'(\xi) = \rho(\xi - u(\xi))u'(\xi), \\ \rho c^2 u'(\xi) + (u - \xi)p'(\xi) = -\rho c^2 u(\xi)\dfrac{\alpha}{\xi}. \end{cases} \tag{11}$$

Excluding the void case ($\rho = 0$), the equations combine into

$$\left[c^2 - (u - \xi)^2\right] u'(\xi) + \alpha \frac{uc^2(\xi)}{\xi} = 0. \tag{12}$$

Remark 1.2.

- For the particular case $\alpha = 0$, we recover the trivial solutions $u(\xi) = u_0$ and $p(\xi) = p_0$.
  Moreover, for $u'(\xi) \neq 0$, the solutions verify

$$c^2 - (u - \xi)^2 \Leftrightarrow \xi = u \pm c, \tag{13}$$

  which correspond to the two characteristics associated with the two Genuinely Non Linear fields (GNL) for Euler equations $\lambda = u - c$ and $\lambda = u + c$.
- In the **general case** $\alpha \neq 0$, the constant solution $(u, p) = (u_0, p_0)$ is admissible **if and only if** $u_0 = 0$.

□

In the case $\alpha \neq -1$, we look for velocity solutions of the form

$$u(\xi) = a\xi, \tag{14}$$

with an unknown constant a. Using Equation (12), we have

$$a\xi((1 + \alpha)c^2 - (a - 1)^2\xi^2) = 0. \tag{15}$$

For any EOS, we note

$$f^{EOS}(c, s_0) = \int \frac{dp}{(\rho c)(s_0, p)}. \tag{16}$$

For $a \neq 0$, we deduce from Equation (15)

$$c(\xi) = \pm \frac{(1 - a)\xi}{\sqrt{1 + \alpha}}. \tag{17}$$

- If $c = \frac{(a-1)\xi}{\sqrt{1+\alpha}}$, using the first Equation from (11) and Equation (16), an equation linking unknowns a and c is obtained

$$a\xi\sqrt{1 + \alpha} + f^{EOS}(c(\xi), s_0) = 0. \tag{18}$$

- If $c = \frac{(1-a)\xi}{\sqrt{1+\alpha}}$, the relation is

$$a\xi\sqrt{1+\alpha} - f^{EOS}(c(\xi), s_0) = 0. \tag{19}$$

Summary: A one-dimensional self-similar regular solution $(s(\xi), u(\xi), c(\xi))$ of Equations (2), with $\xi(x,t) = \frac{x}{t+t_0}, t_0 > 0$ and $\varepsilon(x) = \left(\frac{x}{x_0}\right)^{\alpha} \in [0,1], x_0 \in \mathbb{R}^{+*}$, is given by

$$\begin{cases} s(\xi) = s_0, \\ u(\xi) = a\xi, \\ c(\xi) = \dfrac{|(1-a)\xi|}{\sqrt{1+\alpha}}, \end{cases} \tag{20}$$

with $a \in \mathbb{R}$, such that Equation (18) is satisfied when $(a-1)\xi > 0$, or fulfilling Equation (19) when $(1-a)\xi > 0$.

Equation (18) (or Equation (19)), together with Equation (17), which must be valid for all ξ, leads to a **compatibility** condition on a. We consider two specific EOS in the following remarks.

Remark 1.3. Perfect gas EOS : We consider a perfect gas thermodynamics respecting $p = (\gamma-1)\rho e, \gamma > 1$. We can then write

$$\begin{cases} s = \ln(p\rho^{-\gamma}), \\ c^2(p,s) = \gamma e^{s/\gamma} p^{\frac{\gamma-1}{\gamma}}. \end{cases} \tag{21}$$

We have, using $s = s_0$,

$$\int \frac{dp}{(\rho c)(s_0, p)} = \frac{2c(s_0,p)}{\gamma - 1}. \tag{22}$$

- If $c(\xi) = \frac{(a-1)\xi}{\sqrt{1+\alpha}}$, from Equation (18), we get

$$\xi(a\sqrt{1+\alpha} + \frac{2}{\gamma-1}\frac{(a-1)}{\sqrt{1+\alpha}}) = 0. \tag{23}$$

Thus

$$a = \frac{2}{\gamma + 1 + \alpha(\gamma-1)}. \tag{24}$$

When the porosity is uniform ($\varepsilon = \varepsilon_0$ i.e. $\alpha = 0$), we retrieve the standard value $a = \frac{2}{\gamma+1}$. Moreover Equation (18) corresponds to the Riemann invariant associated with the 1-rarefaction wave in the Riemann problem [14, 5].

- If $c(\xi) = \frac{(1-a)\xi}{\sqrt{1+\alpha}}$, we get the same result $a = \dfrac{2}{\gamma + 1 + \alpha(\gamma-1)}$, with a similar significance for Equation (19) that is a Riemann invariant of the 3-wave, when $\varepsilon = \varepsilon_0$.

□

Remark 1.4. Stiffened gas EOS : Considering a stiffened gas EOS

$$p + \gamma\pi = (\gamma-1)\rho e, \tag{25}$$

with $\pi > 0$ and $\gamma > 1$, we get once again

$$f^{EOS}(c, s_0) = \frac{2c}{\gamma - 1}. \tag{26}$$

It leads to the same solution for the constant a as in Equation (24). □

Numerical application for a radial case: We consider the solution for a choice $\alpha = 1$ on a domain $[0, x_L]$. A perfect gas of constant $\gamma = 1.2$ is chosen. In this case, for $T = 1$ s, $x_L = 2$ m and $u_L = ax_L/T$ with $a = 1/\gamma$, the solutions are represented on Figure 1. The initial (respectively boundary) conditions are deduced from the analytical solution given in Equation (20), setting $t = 0$ (respectively $x = x_\Gamma$, x_Γ being a boundary).

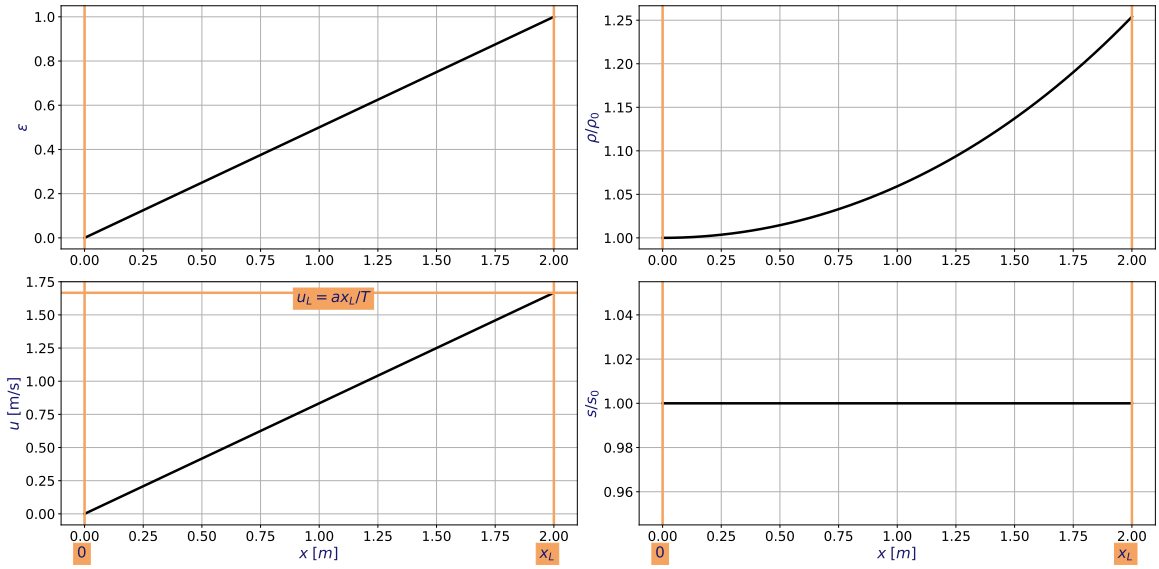

FIGURE 1. Analytical Solutions for $\alpha = 1$, $\gamma = 1.2$ and $x_L = 2.0$ m at $T = 1.0$ s.

2. A SELF-SIMILAR UNSTEADY SOLUTION IN FREE MEDIUM ($\varepsilon = 1$) WITH A SPECIFIC HEAT PROFILE.

For this solution, we consider a free medium ($\varepsilon = 1$) and a specific heat source term noted $\phi(x, t)$. System (1) becomes

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0 \\ \partial_t E + \partial_x(u(E + p)) &= \phi(x, t) \end{aligned} \tag{27}$$

Using a change of variables from $(\rho, \rho u, E)$ to (u, p, s), the system writes

$$\begin{aligned}
&\partial_t u + u\partial_x u + \tau \partial_x p = 0 \\
&\partial_t p + u\partial_x p + \rho c^2 \partial_x u = \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \phi(x,t) \\
&\partial_t s + u\partial_x s = \left(\frac{\partial s}{\partial p}\right)_\rho \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \phi(x,t)
\end{aligned} \tag{28}$$

We look for solutions depending on

$$\xi = \frac{x^a}{(t+t_0)^b}, \tag{29}$$

i.e. by normalizing $\xi = \frac{x^\beta}{t+t_0}, \beta \in \mathbb{R}^*$, for variables $s(\xi)$, $u(\xi)$ and $p(\xi)$ where $t_0 > 0$ is a constant. The momentum equation shows that non-trivial solutions require $\beta = 1$. It constrains the heating profile to verify

$$\phi(x,t) = (t+t_0)^{-1} \psi(\xi), \tag{30}$$

with $\psi(\xi)$ to be given by the user.

The entropy equation gives

$$(u(\xi) - \xi) s'(\xi) = \left(\frac{\partial s}{\partial p}\right)_\rho \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \psi(\xi). \tag{31}$$

For this particular form of the heating source term, the variables must verify

$$\begin{bmatrix} u-\xi & 0 & 0 \\ 0 & u-\xi & \tau \\ 0 & \rho c^2 & u-\xi \end{bmatrix} \cdot \begin{bmatrix} s' \\ u' \\ p' \end{bmatrix} = \begin{pmatrix} \left(\frac{\partial s}{\partial p}\right)_\rho \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \psi(\xi) \\ 0 \\ \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \psi(\xi) \end{pmatrix} \tag{32}$$

Remark 2.1. Case $\psi = 0$.

For the particular case where the heating term is not considered, we find the classic framework. The solution $(s,u,p)(\xi) = (s_0, u_0, p_0)$ is then trivial. In the general case, a non-trivial solution (u,p) then imposes

$$(u-\xi)^2 - c^2 = (u - \xi + c)(u - \xi - c) = 0, \tag{33}$$

with $s'(\xi) = 0 \Leftrightarrow s(\xi) = s_0$.

□

We consider **the general case** $\psi(\xi) \neq 0$. We can note that the profile $(u(\xi), p(\xi)) = (u_0, p_0)$ is no longer a solution of System (32). The first two equations can be rewritten as:

$$p'(\xi) = -\rho(\xi)(u(\xi) - \xi) u'(\xi), \tag{34}$$

and

$$\begin{aligned}
&\rho c^2(\xi) u'(\xi) - \rho(\xi)(u(\xi)-\xi)^2 u'(\xi) = \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau \psi(\xi), \\
\Leftrightarrow &(c^2(\xi) - (u(\xi)-\xi)^2) u'(\xi) = \left(\frac{\partial e}{\partial p}\right)_\rho^{-1} \tau^2(\xi) \psi(\xi).
\end{aligned} \tag{35}$$

Equation (35) imposes $c^2 - (u-\xi)^2 \neq 0$. The solutions must verify

$$\begin{cases} s'(\xi) = \left(\dfrac{\partial s}{\partial p}\right)_\rho \left(\dfrac{\partial e}{\partial p}\right)_\rho^{-1} \tau(\xi) \dfrac{\psi(\xi)}{(u(\xi)-\xi)}, \\ u'(\xi) = \dfrac{1}{c^2-(u-\xi)^2} \left(\dfrac{\partial e}{\partial p}\right)_\rho^{-1} \tau^2(\xi)\psi(\xi), \\ p'(\xi) = \dfrac{u-\xi}{(u-\xi)^2-c^2} \left(\dfrac{\partial e}{\partial p}\right)_\rho^{-1} \tau(\xi)\psi(\xi) \end{cases} \tag{36}$$

Summary: A one-dimensional self-similar regular solution $(s(\xi), u(\xi), p(\xi))$ of Equations (27), with a heat source given by Equation (30), is obtained through Equations (36).

For an arbitrary $\psi(\xi)$, the integration of Equations (36) is not trivial. It can be done numerically using an implicit Euler method, proceeding backwards, and using a very dense mesh of the ξ axis, see [9] for the numerical method. In particular, the specific case with perfect gas EOS of [1] can be mentioned, where the considered heating is in the form of $\phi = \frac{A}{27}\frac{\gamma-2}{\gamma-1}t^{-1}\xi^4, A \in \mathbb{R}$ and fits within the framework proposed here.

Remark 2.2. Perfect gas EOS. Considering the perfect gas thermodynamics law defined in (21), $\left(\frac{\partial s}{\partial p}\right)_\rho = \frac{1}{p}$ and $\left(\frac{\partial e}{\partial p}\right)_\rho = \frac{\tau}{\gamma-1}$, so we have

$$\begin{bmatrix} u-\xi & 0 & 0 \\ 0 & u-\xi & \tau \\ 0 & \gamma p & u-\xi \end{bmatrix} \cdot \begin{bmatrix} s' \\ u' \\ p' \end{bmatrix} = \begin{pmatrix} (\gamma-1)\psi(\xi)p^{-1} \\ 0 \\ (\gamma-1)\psi(\xi) \end{pmatrix} \tag{37}$$

When $\psi = 0$, some manipulations of Equations (32) lead to

$$(u-\xi)(2(u-\xi)' + (\gamma-1)u') = 0. \tag{38}$$

We recover the results from Section 1 for a constant porosity $\varepsilon = 1$. In this case, the velocity writes

$$u(\xi) - u_0 = \frac{2}{\gamma+1}(\xi - \xi_0), \tag{39}$$

leading to the profile of the sound velocity with $c(\xi) = \pm(u(\xi) - \xi)$. □

Conclusion

We have presented two analytical solutions of the HEM model in a 1D framework when the fluid flows through a porous medium or when a heating profile is applied to the fluid. For the first solution with porosity, exact solutions are given for a perfect gas and a stiffened gas EOS but this can be extended to more complex EOS. The second solution with heating is valid for any EOS. Obviously, it would be interesting to numerically test the two proposed solutions and compare them with solutions obtained from numerical schemes.

We refer the readers who are not familiar with classical exact solutions to references [5, 8, 14], among others for pure one-dimensional exact solutions of Euler equations, when $\alpha = 0$ (or equivalently when the porosity is uniform $(\epsilon(x) = \epsilon_0)$) and without any heat source term, focusing on a perfect gas EOS.

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