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MATHEMATICAL JUSTIFICATION OF A COMPRESSIBLE TWO-PHASE AVERAGED SYSTEM FOR IDEAL GAS WITHOUT CONDUCTION: A SEMI-DISCRETE APPROACH AND NUMERICAL ILLUSTRATIONS. *

Pierre Gonin--Joubert [1, 2]

Abstract. In this paper, we propose a space-discrete approach to compressible Navier-Stokes homogenization for ideal gaz without heat-conductivity, in dimension 1. This point of view motivates the definition of an efficient moving-mesh numerical scheme, which allows to visualize the homogenization, as well as the closeness of the models with and without conductivity.

Résumé. Dans ce papier, nous proposons une approche semi-discrète pour l'homogénéisation dans les équations de Navier-Stokes compressible pour des gaz parfaits, en l'absence de conductivité thermique et en dimension 1. Ce point de vue motive la définition d'un schéma numérique à mailles mobiles, qui permet d'illustrer l'homogénéisation d'une part, et la proximité du modèle avec celui en présence de conductivité d'autre part.

Introduction

Several models have been proposed by physicists in order to describe the behaviour of a compressible fluid mixture. Some of them are commonly known as Baer-Nunziato models, and their dynamics look like Navier-Stokes equations, with the addition of new relaxation terms. See for example [2,14]. A strategy for mathematically justifying such models is homogenization (see [1,9]). We start with a *mesoscopic* mixture of two Newtonian fluids $+$ and $-$, verifying the compressible Navier-Stokes equations. The fluids partitioning the space ($\Omega = \Omega_+ \sqcup \Omega_-$), the lenght of each connected space of $\Omega_\pm$ being expected to be of the order of ε. Then passing to the limit, we obtain a *macroscopic* mixture in which the two fluids are merged.

In the barotropic case (when the pressure p depends only of the density ρ) and in dimension 1, compressible Navier-Stokes equations can be written as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho) = \partial_x(\mu \partial_x u). \end{cases}$$

where u is the velocity and μ the viscosity.

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For now, assume the fluids $\pm$ have the same viscosity and pressure law. Then the fact that we model a mixture is only seen via the initial conditions. More precisely, if $(\rho_{0,\pm}, u_{0,\pm})$ are the initial condition, we denote for any $x \in \Omega$,

$$\rho_0(x) = \begin{cases} \rho_{0,+}(x) & \text{if } x \in \Omega_{0,+} \\ \rho_{0,-}(x) & \text{if } x \in \Omega_{0,-} \end{cases}, \quad u_0(x) = \begin{cases} u_{0,+}(x) & \text{if } x \in \Omega_{0,+} \\ u_{0,-}(x) & \text{if } x \in \Omega_{0,-}. \end{cases}$$

Thus, in this case, the problem is reduced to the study of the Navier-Stokes equations with largely oscillating initial data. The behaviour of such solutions is analysed in [11, 18]. Roughly speaking, oscillations on ρ will be conserved over time, whereas oscillations on u will immediatly disappear because of the viscous term. It does not therefore seem restrictive to choose u_0 continuous. To be more precise, we can consider

$$\rho_0 \in L^\infty(\Omega), \qquad u_0 \in H^1(\Omega).$$

Thus, a relevant framework for homogenization of Navier-Stokes is *"à la Hoff" solutions*, a class of intermediate solutions between weak and strong ([10, 13]).

In this framework, some results have also been obtained when the viscosity μ depends on ρ, in order to consider a mixture of two fluids with different viscosities ([6, 7]). Finally, the averaged system is obtain in a more general case in [4], where the authors introduce a *color function* c in order to follow both fluids over time. They studied the new Cauchy problem

$$\begin{cases} \partial_t c + u\partial_x c = 0, \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(c,\rho) = \partial_x(\mu(c)\partial_x u), \end{cases}$$

where

$$\mu(c) = c\mu_+ + (1-c)\mu_-, \qquad p(c,\rho) = cp_+(\rho) + (1-c)p_-(\rho).$$

The function c is simply the indicator function of Ω_+. In particular, $c(t,x) \in \{0,1\}$. For ideal gas, Navier-Stokes equations are the following:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (1.2) \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \partial_x(\mu\partial_x u), & (1.3) \\ \partial_t(\rho E) + \partial_x(\rho E u) + \partial_x(pu) = \partial_x(\mu\partial_x(u^2/2)) + \partial_x(\kappa\partial_x\theta), & (1.4) \end{cases}$$

where θ is the temperature, $e = \mathfrak{c}_\mathrm{v}\,\theta$ the internal energy ($\mathfrak{c}_\mathrm{v} > 0$), $\kappa > 0$ the heat-conductivity coefficient, $E = u^2/2 + e$ the total energy. Note that we have

$$p = (\gamma - 1)\rho e = R\rho\theta$$

where $\gamma > 1$ is the adiabatic constant, and $R > 0$.

Adapting the approach described above for ideal gas is not straightforwart. The system of equations to be considered is (1.1)–(1.4), where

$$\partial_t c + u\partial_x c = 0 \qquad (1.1)$$

and with $f(c) = cf_+ + (1-c)f_-$ for $f \in \{\mu, \mathfrak{c}_\mathrm{v}, \gamma, R, \kappa\}$. If $\kappa > 0$, oscillations in the temperature should disappear thanks to diffusion. Note that if $\mathfrak{c}_\mathrm{v} = \mathfrak{c}_\mathrm{v}(c)$, where c is the color function, then E oscillates. But the analysis of (1) is based on the control of E in $L^2(0,T,H^1(\Omega))$ (see [16])... However, some ideas for homogenization have been obtained for a short time and low oscillations ([12]).

The Navier-Stokes equations for ideal gas and without conductivity ($\kappa = 0$) seems to be a more delicate problem, because of oscillations on the temperature. However, this Cauchy problem has surprising algebraic

properties recently highlighted in [17]. The magic quantity is the *Cauchy stress* $\sigma = \mu\partial_x u - p$. More precisely, some ideas of [17] allows to show in [3] the following result:

Definition 0.1. Let $T > 0$. Suppose $\kappa = 0$. We call a *"à la Hoff"* solution of (1.1)–(1.4) any quadruplet $(c, \rho, \theta, u) \in (L^\infty([0,T] \times \mathbb{T}))^3 \times L^2(0, T, H^1(\mathbb{T}))$ that also satisfies

$$\left.\begin{aligned} 0 \leq c(t,x) \leq 1 \quad & (2.1)\\ \text{for all } t \in [0,T], x \in \mathbb{T}, \quad \underline{\rho}^0 \leq \rho(t,x) \leq \overline{\rho}^0 \quad & (2.2)\\ \underline{\theta}^0 \leq \theta(t,x) \leq \overline{\theta}^0 \quad & (2.3)\end{aligned}\right\},$$

$$\sup_{0\leq t\leq T} \int_0^1 \sigma^2 + \int_0^T \int_0^1 (\partial_x \sigma)^2 \leq {C_1}^0, \tag{2.4}$$

$$\int_0^T \|\sigma\|_\infty^2(t)dt \leq {C_2}^0, \tag{2.5}$$

$$\sup_{0\leq t\leq T} \int_0^1 (\partial_x u)^2 \leq {C_3}^0, \tag{2.6}$$

$$\int_0^T \max_{0\leq j\leq J-1} \|\partial_x u\|_\infty^2(t)dt \leq {C_4}^0, \tag{2.7}$$

$$\sup_{0\leq t\leq T} \|u\|_\infty(t) \leq {C_5}^0, \tag{2.8}$$

$$\int_0^T \int_0^1 (\partial_t u)^2 \leq {C_6}^0, \tag{2.9}$$

$$\sup_{0\leq t\leq T} \min(1,t) \int_0^1 (\partial_x \sigma)^2 + \int_0^T \min(1,t) \int_0^1 (\partial_t \sigma)^2 \leq {C_7}^0, \tag{2.10}$$

for some $\overline{\rho}^0, \underline{\rho}^0, \overline{\theta}^0, \underline{\theta}^0, {C_1}^0, {C_2}^0, {C_3}^0, {C_4}^0, {C_5}^0, {C_6}^0, {C_7}^0 > 0$.

Remark 0.2. Equation (2.4) is called Hoff's first energy, and (2.2), (2.3), (2.5)–(2.9) can be seen as consequences of (2.4). Equation (2.10) is called Hoff's second energy. This is a finer time weight bound, important for obtaining compactness on σ.

Theorem 0.3. *Let $T > 0$. Consider $(c_0, \rho_0, \theta_0, u_0) \in (L^\infty(\mathbb{T}))^3 \times H^1(\mathbb{T})$ verifying*

$$\left.\begin{aligned} 0 \leq c_0(x) \leq 1 \quad & (3.1)\\ \textit{for all } x \in \mathbb{T}, \quad \underline{\rho_0} \leq \rho(x) \leq \overline{\rho_0} \quad & (3.2)\\ \underline{\theta_0} \leq \theta(x) \leq \overline{\theta_0} \quad & (3.3)\end{aligned}\right\}$$

for some constants $0 < \underline{\rho_0} < \overline{\rho_0}$ and $0 < \underline{\theta_0} < \overline{\theta_0}$. Then, the Cauchy problem (1.1)–(1.4) with initial condition $(c_0, \rho_0, \theta_0, u_0)$ has a unique "à la Hoff" solution on $[0,T]\times\mathbb{T}$. Moreover, the constants $\overline{\rho}^0, \underline{\rho}^0, \overline{\theta}^0, \underline{\theta}^0, {C_1}^0, {C_2}^0, {C_3}^0, {C_4}^0$, ${C_5}^0, {C_6}^0, {C_7}^0$ in Definition 0.1 depend only on $\overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, \underline{\theta_0}, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, \|u_0\|_{H^1}, T$.

These bounds enable the mathematical justification of Navier-Stokes homogenization in this context. The authors demonstrate the following result:

Theorem 0.4. *Let $T>0$. Consider some sequence of "à la Hoff" solutions $(c^\varepsilon,\rho^\varepsilon,\theta^\varepsilon,u^\varepsilon)$ of (1.1)–(1.4) with initial conditions $(c_0^\varepsilon,\rho_0^\varepsilon,\theta_0^\varepsilon,u_0^\varepsilon)\in L^\infty(\mathbb{T})^3\times H^1(\mathbb{T})$, verifying (3.2)–(3.3) with bounds uniform in ε. Suppose moreover that there exists some $\alpha_{0,\pm},\rho_{0,\pm},\theta_{0,\pm}\in L^\infty(\mathbb{T})$, $u_0\in H^1(\mathbb{T})$ such that*

$$\text{for all } \beta\in C(\mathbb{R}^3),\quad \beta(c_0^\varepsilon,\rho_0^\varepsilon,\theta_0^\varepsilon)\underset{\varepsilon\to0}{\rightharpoonup}\alpha_{0,+}\beta(1,\rho_{0,+},\theta_{0,+})+\alpha_{0,-}\beta(0,\rho_{0,-},\theta_{0,-}) \text{ in } L^\infty([0,T]\times\mathbb{T})-\star,$$

$$u_0^\varepsilon\underset{\varepsilon\to0}{\rightharpoonup}u_0 \text{ in } H^1(\mathbb{T}).$$

then there exists some $\alpha_\pm$, $\rho_\pm$, $\theta_\pm\in L^\infty([0,T]\times\mathbb{T})$, $u\in L^2(0,T,H^1(\mathbb{T}))$ such that, up to the extraction of a subsequence,

$$\text{for all } \beta\in C(\mathbb{R}^3),\quad \beta(c^\varepsilon,\rho^\varepsilon,\theta^\varepsilon)\underset{\varepsilon\to0}{\rightharpoonup}\alpha_+\beta(1,\rho_+,\theta_+)+\alpha_-\beta(0,\rho_-,\theta_-) \text{ in } L^\infty([0,T]\times\mathbb{T})-\star,$$

$$u^\varepsilon\underset{\varepsilon\to0}{\rightharpoonup}u \text{ in } L^2(0,T,H^1(\mathbb{T})).$$

Finally, $(\alpha_\pm,\rho_\pm,\theta_\pm,u)$ verifies

$$\begin{cases}
\partial_t\alpha_\pm+u\partial_x\alpha_\pm=\dfrac{\alpha_+\alpha_-}{\alpha_-\mu_++\alpha_+\mu_-}(\sigma_\mp-\sigma_\pm) & (4.1)\\
\partial_t(\alpha_\pm\rho_\pm)+\partial_x(\alpha_\pm\rho_\pm u)=0 & (4.2)\\
\partial_t(\rho u)+\partial_x(\rho u^2)=\partial_x\sigma \text{ with } \rho=\alpha_+\rho_++\alpha_-\rho_- & (4.3)\\
\partial_t(\alpha_\pm\rho_\pm\,\mathfrak{c}_{v\pm}\,\theta_\pm)+\partial_x(\alpha_\pm\rho_\pm\,\mathfrak{c}_{v\pm}\,\theta_\pm u)=\dfrac{\alpha_+\alpha_-}{\alpha_-\mu_++\alpha_+\mu_-}\sigma(\sigma_\mp-\sigma_\pm)+\alpha_\pm\sigma\partial_x u & (4.4)
\end{cases}$$

where $\sigma_\pm=\mu_\pm\partial_x u-R_\pm\rho_\pm\theta_\pm$ and $\sigma=\mu_{eff}\partial_x u-p_{eff}$ with

$$\mu_{eff}=\frac{1}{\alpha_+/\mu_++\alpha_-/\mu_-},\qquad p_{eff}=\frac{\alpha_+R_+\rho_+\theta_+/\mu_++\alpha_-R_-\rho_-\theta_-/\mu_-}{\alpha_+/\mu_++\alpha_-/\mu_-}.\quad\square$$

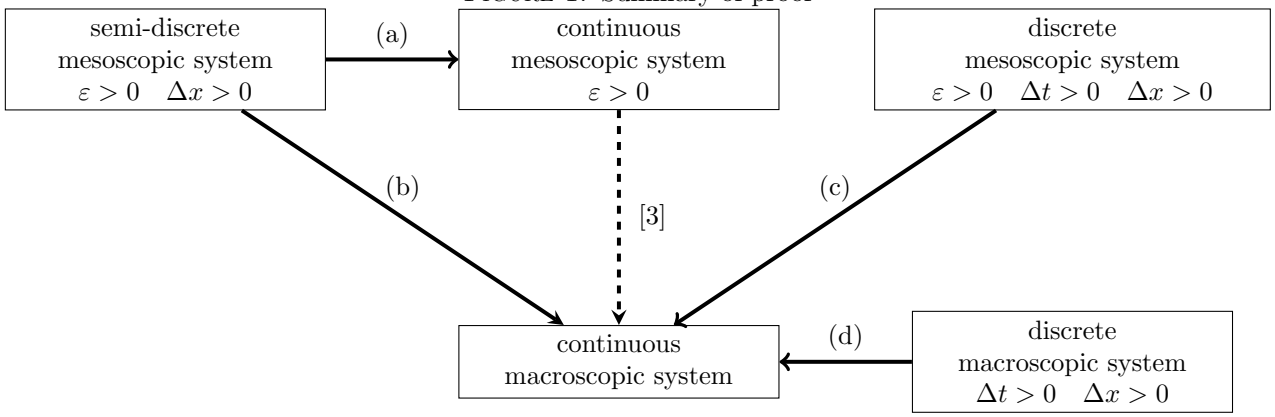
The purpose of this paper is to outline the approach of [3], studying a semi-discrete model (discrete in space, continuous in time). In the spirit of [5] that adapted [4] in a semi-discrete framework, this paper provides elements for a semi-discrete analysis of Navier-Stokes homogenization for ideal gas. This point of view allows for example to obtain better results for the existence at large time of solution to Navier-Stokes with temperature and with heat-conductivity (see [8]). In the present case, it makes it easier to show the existence of solutions to the continuous mesoscopic system (the arrow (a) in Figure 1), and would be an other method to obtain homogenization. In this paper we will give a detailed proof of the convergence of the non-conservative approximation of the Navier-Stokes equations (33.1)–(33.4) obtained via the discrete model to the Navier-Stokes equations, using some quantitative estimates (Lemma 1.24 and Proposition 1.25).

The semi-discrete approach seems physically meaningfull: pure mobile fluid cells are considered, with a length $\Delta x\simeq\varepsilon$. We then make then cell size tend towards 0 (it is the arrow (b)). Moreover, the semi-discrete approach is a first step for a totally discrete approach, and the obtention of a totally numerical scheme. Writing on one hand a scheme for the *mesoscopic* equations and on the other hand for the *macroscopic* equations allows the visualization of the homogeneization (via (c) and (d)). These numerical schemes are presented in this paper. They enable to obtain the different illustrations of [3], which will be recalled.

Finally, in a brief third part, we will adapt the schemes to deal with the case with conductivity (which still resists to mathematical analysis). We illustrate some convergences when $\kappa\to0$, which for now are only conjectures.

In this problem, the use of a Eulerian numerical scheme seems not relevant. Indeed, discretization of the transport equation (1.1), for example by an upwind scheme, leads to the appearance of numerical diffusion

FIGURE 1. Summary of proof



which prevents the approximation of the color function from remaining equal to 0 or 1, which makes no physical sense since it is an indicator function. More generally, discontinuities for density and temperature are crucial in the modelization, thus we would like to keep them in the discretization.

One idea could be to accept some numerical diffusion but control them. It will force us to consider some discretisation with a larger number of cells, which seems unreasonable even in dimension 1.

Writing the system choosing a pseudo Lagrangian formulation, we obtain

$$D_t c = 0, \qquad D_t \rho = -\rho \partial_x u, \qquad D_t \theta = -\frac{\sigma \partial_x u}{\rho \mathfrak{c}_{\mathrm{v}}},$$

where $D_t = \partial_t + u\partial_x$ is the Lagrangian derivative.

Thus, roughly speaking, the estimates (2.1)–(2.10) give that c, ρ, θ are continuous in time along the characteristics of u. Hence the motivation of using a scheme with mobile interfaces, in order to conserve the discontinuities without paying too high a price in term of computing costs. Remark that an other approach is to consider a fixed-mesh scheme, rewriting the Navier-Stokes system in Lagrangian variables (see [8]).

1. A SEMI-DISCRETE APPROACH

1.1. **Heuristic**

Let $T > 0$. Starting from (c, ρ, θ, u) some *"à la Hoff"* solution of (1.1)–(1.4) on $[0, T] \times \mathbb{T}$, we suppose that there exists some $(x_{0,j+1/2})_{j=0,\cdots J-1}$ such that

$$0 \leq x_{0,1/2} < x_{0,3/2} < \cdots < x_{0,J-1/2} \leq 1$$

and that for all $j \in \{0, \cdots J-1\}$, c is constant on $]x_{0,j-1/2}, x_{0,j+1/2}[$. We denote this constant c_j. Roughly speaking, the positions $x_{0,j+1/2}$ spatially separate the different fluids. The color number c_j indicates which fluid occupies the portion of space $]x_{0,j-1/2}, x_{0,j+1/2}[$. The length of each portion is expected to be in the order of $1/J$, thus we suppose moreover that there exists some $L_0 > 0$ such that

$$\text{for all } j \in \{0, \cdots, J-1\}, \quad \frac{1}{L_0 J} \leq x_{0,j+1/2} - x_{0,j-1/2} \leq \frac{L_0}{J}. \tag{5}$$

The sequence $(x_{0,j+1/2})_j$ will be the initial mesh of our scheme. We consider a Lagrangian scheme, thus each position must follow the flow of velocity u. For a fixed j, we define then $t \mapsto x_{j+1/2}(t)$ on $[0,T]$ via

$$\begin{cases} \dfrac{d}{dt} x_{j+1/2}(t) = u(t, x_{j+1/2}(t)), \\ x_{j+1/2}(0) = x_{0,j+1/2}. \end{cases}$$

We denote $u_{j+1/2}(t) := u(t, x_{j+1/2}(t))$ and $\Delta x_j(t) := x_{j+1/2}(t) - x_{j-1/2}(t)$. Remark that if f is some function defined on $[0,T] \times \mathbb{T}$,

$$\begin{aligned} \frac{d}{dt} \int_{x_{j-1/2}(t)}^{x_{j+1/2}(t)} f(t,x)\, dx &= u(t, x_{j+1/2}) f(t, x_{j+1/2}(t)) - u(t, x_{j-1/2}) f(t, x_{j-1/2}(t)) + \int_{x_{j-1/2}(t)}^{x_{j+1/2}(t)} \partial_t f(t,x)\, dx \\ &= \int_{x_{j-1/2}(t)}^{x_{j+1/2}(t)} \left(\partial_t f(t,x) + \partial_x (fu)(t,x)\right)\, dx. \end{aligned}$$

Then we define $\rho_j(t)$ and $E_j(t)$ by

$$\rho_j(t) = \frac{1}{\Delta x_j(t)} \int_{x_{j-1/2}(t)}^{x_{j+1/2}(t)} \rho(t,x)\, dx, \qquad \rho_j(t) E_j(t) = \frac{1}{\Delta x_j} \int_{x_{j-1/2}(t)}^{x_{j+1/2}(t)} \rho E(t,x) dx,$$

which gives, denoting $\sigma_{j+1/2}(t) = \sigma(t, x_{j+1/2}(t))$,

$$\frac{d}{dt} c_j = 0, \tag{6.1}$$

$$\frac{d}{dt} (\rho_j \Delta x_j) = 0, \tag{6.2}$$

$$\frac{d}{dt} \left(\rho_j E_j \Delta x_j\right) = \rho_j \Delta x_j \frac{d}{dt} E_j = \sigma_{j+1/2} u_{j+1/2} - \sigma_{j-1/2} u_{j-1/2}, \tag{6.3}$$

denoting $x_j(t) := (x_{j+1/2}(t) + x_{j-1/2}(t))/2$ then $\Delta x_{j+1/2}(t) := x_{j+1}(t) - x_j(t)$. Equation (6.1) reflects the fact that each portion of fluid $]x_{j-1/2}, x_{j+1/2}[$ is occupied by the same fluid at each time. Moreover, the total mass of each portion is conserved (by (6.2)) and (6.3) expresses the exchange of energy between each portion. We define $\rho_{j+1/2}(t)$, the average density at the position $x_{j+1/2}(t)$ by

$$\rho_{j+1/2}(t) \Delta x_{j+1/2}(t) = \frac{\rho_j(t) \Delta x_j(t) + \rho_{j+1}(t) \Delta x_{j+1}(t)}{2}$$

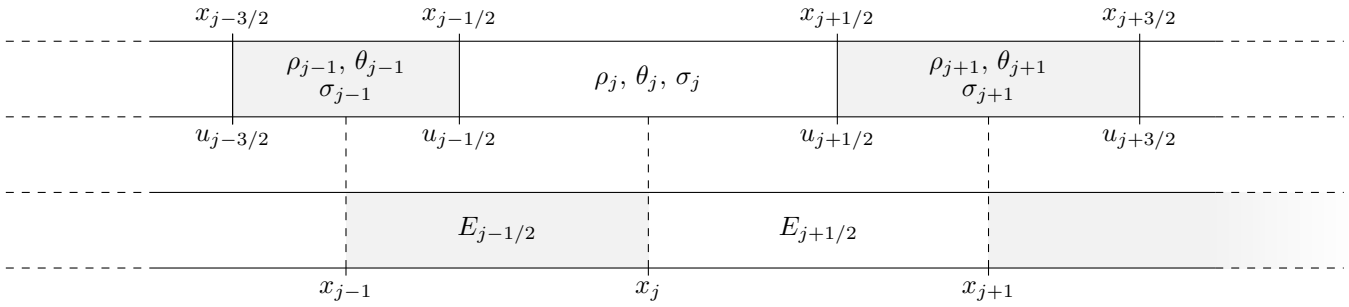
thus reproducing precedent computations, we get the equation of moments

$$\rho_{j+1/2} \Delta x_{j+1/2} \frac{d}{dt} u_{j+1/2} = \sigma_{j+1} - \sigma_j. \tag{6.4}$$

where $\sigma_j(t) = \sigma(t, x_j(t))$.

Note that we have too many unknowns in relation to the number of equations. The difficulty is then to find some relations between $E_j, \sigma_{j+1/2}$ and $u_{j+1/2}, \theta_j, \theta_{j+1/2}$. We remark some conflicts: we defined ρ_j and E_j as above because it seems clear that they are cell-specific data. Similarly, we would define the internal energy $e = \mathfrak{c}_{\mathrm{v}}\, \theta$ into the mesh. But the pressure $p = (\gamma - 1)\rho e = R\rho\theta$ should be defined on the interface... some choices have to be done, but respecting equations (6.1)–(6.4).

FIGURE 2. Semi-discretization choice



1.2. **Definition of the semi-discrete system**

Let $T > 0$. Let $\mu_\pm, \mathfrak{c}_{\mathrm{v}\pm} > 0$, $\gamma_\pm > 1$, $R_\pm = (\gamma_\pm - 1)\,\mathfrak{c}_{\mathrm{v}\pm}$ and $p_\pm = p_\pm(\rho, \theta) = R_\pm \rho\theta$. For $f \in \{\mu, \mathfrak{c}_{\mathrm{v}}, \gamma, R, p\}$, we define $f(c)$ by

$$f(c) = cf_+ + (1-c)f_-$$

and we denote in the following, for $j \in \{0, \cdots J-1\}$, $f_j := f(c_j)$. We adopt the definition $\Delta g_j := g_{j+1/2} - g_{j-1/2}$, $\Delta g_{j+1/2} := g_{j+1} - g_j$.

Let $(c_{0,j}, u_{0,j+1/2}, x_{0,j+1/2}, \rho_{0,j}, \theta_{0,j})_{j=0,\cdots J-1} \subset \mathbb{R}^5$.
We consider on $[0,T]$ the system, for $j \in \{0, \cdots J-1\}$,

$$\begin{cases} \dfrac{d}{dt} c_j = 0 & (7.1) \\ \dfrac{d}{dt}(\rho_j \Delta x_j) = 0 & (7.2) \\ \rho_{j+1/2} \Delta x_{j+1/2} \dfrac{d}{dt} u_{j+1/2} = \Delta \sigma_{j+1/2} & (7.3) \\ \rho_j \Delta x_j\, \mathfrak{c}_{\mathrm{v}j}\, \dfrac{d}{dt}\theta_j = \sigma_j \Delta u_j & (7.4) \end{cases}$$

solve for $(c_j, \rho_j, \theta_j, u_{j+1/2})$ and with the initial condition $(c_{0,j}, \rho_{0,j}, \theta_{0,j}, u_{0,j+1/2})$, where for all $t \in [0,T]$,

$$x_{j+1/2}(t) = x_{0,j+1/2} + \int_0^t u_{j+1/2}(s)ds, \tag{7.5}$$

$$x_j = \frac{x_{j-1/2} + x_{j+1/2}}{2}.$$

Moreover, $\rho_{j+1/2}$ is defined by

$$\rho_{j+1/2} \Delta x_{j+1/2} = \frac{\rho_j \Delta x_j + \rho_{j+1} \Delta x_{j+1}}{2} \tag{7.6}$$

and σ_j by

$$\sigma_j = \mu_j \frac{\Delta u_j}{\Delta x_j} - p_j(\rho_j, \theta_j). \tag{7.7}$$

Remark 1.1. It is not immediately clear that the system (7.1)–(7.4) is consistent with the principle of conservation of energy. However, for $j \in \{0, \cdots J-1\}$, considering $e_{j+1/2}$ and $E_{j+1/2}$ defined by

$$\rho_{j+1/2} \Delta x_{j+1/2} e_{j+1/2} = \frac{\rho_j \Delta x_j\, \mathfrak{c}_{\mathrm{v}j}\, \theta_j}{2} + \frac{\rho_{j+1} \Delta x_{j+1}\, \mathfrak{c}_{\mathrm{v}j+1}\, \theta_{j+1}}{2}, \qquad E_{j+1/2} = \frac{u_{j+1/2}^2}{2} + e_{j+1/2},$$

we get

$$\begin{aligned}\frac{d}{dt}\rho_{j+1/2}\Delta x_{j+1/2}E_{j+1/2} &= \rho_{j+1/2}\Delta x_{j+1/2}u_{j+1/2}\frac{d}{dt}u_{j+1/2} + \frac{\rho_j\Delta x_j\,\mathfrak{c}_{\mathrm{v}j}\,\frac{d}{dt}\theta_j}{2} + \frac{\rho_{j+1}\Delta x_{j+1}\,\mathfrak{c}_{\mathrm{v}j+1}\,\frac{d}{dt}\theta_{j+1}}{2}\\ &= u_{j+1/2}\Delta\sigma_{j+1/2} + \frac{\sigma_j\Delta u_j}{2} + \frac{\sigma_{j+1}\Delta u_{j+1}}{2}\\ &= \Delta(\sigma u)_{j+1/2} \end{aligned} \tag{7.8}$$

where

$$u_j = \frac{u_{j-1/2}+u_{j+1/2}}{2}. \tag{7.9}$$

It might seem more natural to replace (7.4) with (7.8) to define the semi-discrete system. However, this approach does not allow us to find θ_j from $E_{j+1/2}$ in the case where J is even.

Let's consider the following assumptions:

H1 The sequence $(x_{0,j+1/2})_{j=0,\cdots J-1}$ verifies $0 \le x_{0,1/2} < x_{0,3/2} < \cdots < x_{0,J-1/2} < 1$ and there exists some $L_0 > 0$ such that

$$\max_{0\le j\le J-1}\Delta x_{0,j} \le L_0 \min_{0\le j\le J-1}\Delta x_{0,j}.$$

H2 For any $j \in \{0,\cdots J-1\}$, $\quad c_{0,j} \in \{0,1\}$.

H3 There exists some $0 < \underline{\rho_0} < \overline{\rho_0}$, $0 < \underline{\theta_0} < \overline{\theta_0}$ such that, for any $j \in \{0,\cdots J-1\}$,

$$\underline{\rho_0} \le \rho_{0,j} \le \overline{\rho_0}, \qquad\qquad \underline{\theta_0} \le \theta_{0,j} \le \overline{\theta_0}.$$

H4 There exists some constants $\varepsilon_0 > 0$, $C_0 > 0$ such that

$$\left|\sum_{j=0}^{J-1}\rho_{0,j+1/2}\Delta x_{0,j+1/2}u_{0,j+1/2}\right| \le \varepsilon_0, \tag{8}$$

$$\sum_{j=0}^{J-1}\frac{(\Delta u_{0,j})^2}{\Delta x_{0,j}} \le C_0. \tag{9}$$

Remark 1.2. The assumption H1 is linked to (5): indeed it can be rewritten as

$$\text{for all } k,j \in \{0,\cdots,J-1\}, \quad \Delta x_{0,k} \le L_0\Delta x_{0,j}.$$

Thus summing on j (resp. k) at k fixed (resp. j) we get

$$\text{for all } l \in \{0,\cdots J-1\}, \quad \frac{1}{L_0 J} \le \Delta x_{0,l} \le \frac{L_0}{J}.$$

Remark moreover that $L_0 \ge 1$.

Remark 1.3. Assumptions H2 and H3 are discrete versions of (3.1)–(3.3). Moreover, H4 is a discrete version of "$u_0 \in H^1(\mathbb{T})$". See the proof of (1.8) for more details. Note that in the continuous case, we can assume by Galilean invariance

$$\int_0^1 \rho_0 u_0 = 0.$$

Inspired by definition 0.1, let's consider

Definition 1.4. Let $T > 0$. We call *"à la Hoff"* solution of (7.1)–(7.5) any quintuplet $(c_j, u_{j+1/2}, x_{j+1/2}, \rho_j, \theta_j)_{j=0}^{J-1} \in (L^\infty([0,T]))^5$ that also satisfies

$$\left.\begin{aligned} 0 \le c_j(t) \le 1 \qquad\qquad\qquad\qquad & (10.1)\\ \text{for all } t \in [0,T], j \in \{0,\cdots,J-1\}, \quad \underline{\rho} \le \rho_j(t) \le \overline{\rho} \qquad & (10.2)\\ \underline{\theta} \le \theta_j(t) \le \overline{\theta} \qquad\qquad\qquad & (10.3)\end{aligned}\right\},$$

for all $t \in [0,T]$,
$$\max_{0\le j\le J-1} \Delta x_j(t) \le L \min_{0\le j\le J-1} \Delta x_j(t), \tag{10.4}$$

$$\sup_{0\le t\le T} \sum_{j=0}^{J-1} \Delta x_j \sigma_j^2 + \sum_{j=0}^{J-1} \int_0^T \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} \le C_1, \tag{10.5}$$

$$\int_0^T \max_{0\le j\le J-1} \sigma_j^2 \le C_2, \tag{10.6}$$

$$\sup_{0\le t\le T} \sum_{j=0}^{J-1} \frac{(\Delta u_j)^2}{\Delta x_j} \le C_3, \tag{10.7}$$

$$\int_0^T \max_{0\le j\le J-1} \left(\frac{\Delta u_j}{\Delta x_j}\right)^2 \le C_4, \tag{10.8}$$

$$\sup_{0\le t\le T} \max_{0\le j\le J-1} |u_{j+1/2}| \le C_5, \tag{10.9}$$

$$\int_0^T \sum_{j=0}^{J-1} \Delta x_{j+1/2} |u'_{j+1/2}|^2 \le C_6, \tag{10.10}$$

$$\sup_{0\le t\le T} \min(1,t) \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} + \int_0^T \min(1,t) \sum_{j=0}^{J-1} \Delta x_j |\sigma'_j|^2 \le C_7. \tag{10.11}$$

Remark 1.5. Equations (10.1)–(10.11) are the semi-discrete version of (2.1)–(2.10). More precisely, the estimate with for constant C_i in Definition 1.4 is the discrete version of C_i^0 in Definition 0.1, for $i \in \{1,\cdots 7\}$.

We can then show the following existence theorem:

Theorem 1.6. *Suppose that $(c_{0,j}, u_{0,j+1/2}, x_{0,j+1/2}, \rho_{0,j}, \theta_{0,j})_{j=0,\cdots J-1}$ verifies* H1–H4 *for some $L_0 > 0$, $0 < \underline{\rho_0} < \overline{\rho_0}$, $0 < \underline{\theta_0} < \overline{\theta_0}$. Then the Cauchy problem* (7.1)–(7.5) *with initial conditions $(c_{0,j}, u_{0,j+1/2}, x_{0,j+1/2}, \rho_{0,j}, \theta_{0,j})_{j=0,\cdots J-1}$ has a unique "à la Hoff" solution on $[0,+\infty[$. Moreover, for $T > 0$, the constants $\overline{\rho}, \underline{\rho}, \overline{\theta}, \underline{\theta}, C_1, C_2, C_3, C_4, C_5, C_6, C_7, L > 0$ in Definition 1.4 depend only on $\overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, \underline{\theta_0}, \mu_\pm, \gamma_\pm, \mathfrak{c}_{v\pm}, \varepsilon_0$, C_0, L_0, T.*

One of the steps in order to prove Theorem 1.6 is the

Lemma 1.7. *Suppose that $(c_{0,j}, u_{0,j+1/2}, x_{0,j+1/2}, \rho_{0,j}, \theta_{0,j})_{j=0,\cdots J-1}$ verifies* H1–H4.
Then there exists $T_{\max} > 0$ such that (7) *with initial conditions $(c_{0,j}, u_{0,j+1/2}, x_{0,j+1/2}, \rho_{0,j})$ has a unique maximal solution $(c_j, u_{j+1/2}, x_{j+1/2}, \rho_j)$ on $[0, T_{\max}[$. Moreover, either $T_{\max} = +\infty$, or $T_{\max} < +\infty$ and one of the following case happens:*

- $|X(t)| + |U(t)| \underset{t\to T_{\max}}{\longrightarrow} +\infty$
- $\liminf_{t\to T_{\max}} \min_j \Delta x_j = 0$
- $\liminf_{t\to T_{\max}} \min_j \theta_j = 0$.

Proof. Remark that system of equations (7.1)–(7.4) can in fact be largely decoupled. Of course, (7.1) is independant and gives

$$\text{for all } t \in [0,T],\ \text{for all } j \in \{0,\cdots,J-1\},\quad c_j(t) = c_{0,j}.$$

Moreover we obtain directly from (7.2) that

$$\text{for all } t \in [0,T],\ \text{for all } j \in \{0,\cdots,J-1\},\quad \rho_j(t) = \frac{\rho_{0,j}\Delta x_{0,j}}{\Delta x_j(t)}. \tag{11}$$

Then, from (7.3), there exists some smooth function F such that,
denoting $X = (x_{j+1/2})_{j=0,\cdots J-1}$, $U = (u_{j+1/2})_{j=0,\cdots J-1}$ and $\Theta = (\theta_j)_{j=0,\cdots J-1}$,

$$\frac{d}{dt}(X,U,\Theta) = F(X,U,\Theta) \tag{12}$$

with

$$F(X,U,\Theta)_j = \begin{pmatrix} u_{j+1/2} \\ \dfrac{1}{\rho_{0,j+1/2}\Delta x_{0,j+1/2}}\left(\mu_{0,j+1}\dfrac{\Delta u_{j+1}}{\Delta x_{j+1}} - \mu_{0,j}\dfrac{\Delta u_j}{\Delta x_j} - R_{0,j+1}\rho_{j+1}\theta_{j+1} + R_{0,j}\rho_j\theta_j\right) \\ \dfrac{1}{\rho_{0,j}\Delta x_{0,j}\,\mathfrak{c}_{\mathrm{v}0,j}}\left(\mu_{0,j}\dfrac{(\Delta u_j)^2}{\Delta x_j} - R_{0,j}\rho_j\theta_j\Delta u_j\right) \end{pmatrix}$$

where $F : D \to \mathbb{R}$, with

$$D = \left\{(X,U,\Theta) \in (\mathbb{R}^J)^3,\quad x_{1/2} < x_{3/2} < \cdots < x_{J-1/2},\quad \min_j \theta_j > 0\right\}$$

is an open set of $(\mathbb{R}^J)^3$. Thus by the Cauchy-Lipshitz theorem, there exists some $T_{\max} > 0$ and a unique maximal solution of (12) $(X,U,\Theta) : [0,T_{\max}[\to D$. Moreover, using H1, H3 and (11), we observe that for all $j \in \{0,\cdots J-1\}$ and for all $t \in [0,T_{\max}[$, $\rho_j(t) > 0$. □

To prove the theorem 1.6, we want to obtain uniform bounds in J to get the global existence of a solution to the Cauchy problem, i.e. $T_{\max} = +\infty$. Note that in the following, (x_j) and $(x_{j+1/2})$ are extended for $j \in \mathbb{Z}$ by

$$\text{for all } k \in \mathbb{Z},\ \text{for all } j \in \{0,\cdots J-1\},\quad x_{j+kJ} = k + x_j,\quad x_{j+1/2+kJ} = k + x_{j+1/2}.$$

Moreover, for $f \in \{c,\rho,\theta\}$ and $g \in \{u,E\}$, (f_j) and $(g_{j+1/2})$ are extended for $j \in \mathbb{Z}$ by

$$\text{for all } k \in \mathbb{Z},\ \text{for all } j \in \{0,\cdots J-1\},\quad f_{j+kJ} = f_j,\quad g_{j+1/2+kJ} = g_{j+1/2}. \tag{13}$$

1.3. **A priori estimates**

In this part, we adapt a priori estimates obtained in [3] in the continuous case (see Theorem 0.3). We start by finding an upper bound $\overline{\rho}$ on the density (proposition 1.11), which is a very classical idea in the study of barotropic Navier-Stokes (see for example [18]). A lower bound $\underline{\theta}$ on the temperature is similarly easy to get, with or without conductivity (see [8] in a semi-discrete view). However, to go further, the methods differ depending on whether the case is with or without conductivity. In the abscence of oscillations and with heat-conductivity, the idea is to prove an H^1 regularity on the total energy. In our case (without conductivity), the key is to consider the effective flux (see [17] with no oscillations, and [3] with oscillations): it is our proposition 1.15. In the following, we fix $J \in \mathbb{N}^*$ and $0 < T_0 \leq T_{\max}$, $T_0 < +\infty$.

We begin our analysis by showing a uniform bound with respect to J of mass and energy :

Lemma 1.8. *Let $T \in]0, T_0[$. There exists some $\overline{\mathcal{M}}, \overline{\mathcal{E}} > 0$ depending only of $\overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, C_0, \varepsilon_0, \mathfrak{c}_{\mathrm{v}\pm}$ such that*

$$\sup_{0\le t\le T} \sum_{j=0}^{J-1} \rho_j \Delta x_j(t) \le \overline{\mathcal{M}}, \qquad \sup_{0\le t\le T} \sum_{j=0}^{J-1} \rho_{j+1/2} \Delta x_{j+1/2} E_{j+1/2}(t) \le \overline{\mathcal{E}}.$$

Proof. Let $t \in [0, T]$. From the conservative form of (7.2), we get immediatly

$$\sum_{j=0}^{J-1} \rho_j \Delta x_j(t) = \sum_{j=0}^{J-1} \rho_{0,j} \Delta x_{0,j} \le \overline{\rho_0} \sum_{j=0}^{J-1} \Delta x_{0,j} \le \overline{\rho_0}. \tag{14}$$

Using now the conservative form of (7.8), we get

$$\sum_{j=0}^{J-1} \rho_{j+1/2} \Delta x_{j+1/2} E_{j+1/2}(t) = \sum_{j=0}^{J-1} \rho_{0,j+1/2} \Delta x_{0,j+1/2} E_{0,j+1/2} \le \mathfrak{c}_{\mathrm{v}\max} \overline{\rho_0}\overline{\theta_0} + \frac{1}{2} \sum_{j=0}^{J-1} \rho_{0,j+1/2} \Delta x_{0,j+1/2} u_{0,j+1/2}^2.$$

Let $j, k \in \{0, \cdots, J-1\}$. Writing now

$$u_{0,j+1/2} = u_{0,k+1/2} + \sum_{l=k}^{j-1} \Delta u_{0,l}$$

then multiplying by $\rho_{0,k+1/2} \Delta x_{0,k+1/2}$ and summing in k we get

$$\begin{aligned} &\left(\sum_{k=0}^{J-1} \rho_{0,k+1/2} \Delta x_{0,k+1/2} \right) u_{0,j+1/2} \\ &\qquad = \sum_{j=0}^{J-1} \rho_{0,k+1/2} \Delta x_{0,k+1/2} u_{0,k+1/2} + \sum_{k=0}^{J-1} \rho_{0,k+1/2} \Delta x_{0,k+1/2} \sum_{l=k}^{j-1} \Delta u_{0,l}. \end{aligned} \tag{15}$$

Note that, reproducing the reasoning of (14), we have

$$\underline{\rho_0} \le \sum_{k=0}^{J-1} \rho_{0,k+1/2} \Delta x_{0,k+1/2} \le \overline{\rho_0}. \tag{16}$$

From (15) and (16) we get by Hölder's inequality, using (8)–(9),

$$\underline{\rho_0} |u_{0,j+1/2}| \le \varepsilon_0 + \overline{\rho_0} \sum_{l=0}^{J-1} |\Delta u_{0,l}| \le \varepsilon_0 + \overline{\rho_0} \left(\sum_{l=0}^{J-1} \frac{|\Delta u_{0,l}|^2}{\Delta x_{0,l}} \right)^{1/2} \le \varepsilon_0 + \overline{\rho_0} C_0^{1/2}$$

hence by (16)

$$\sum_{j=0}^{J-1} \rho_{0,j+1/2} \Delta x_{0,j+1/2} u_{0,j+1/2}^2 \le \overline{\rho_0} u_{0,j+1/2}^2 \le \overline{\rho_0} \left(\frac{\varepsilon_0 + \overline{\rho_0} C_0^{1/2}}{\underline{\rho_0}} \right)^2.$$

□

We continue with the trivial but crucial

Lemma 1.9. *Let $T \in]0, T_0[$. For any smooth function $\beta :]0, +\infty[\to \mathbb{R}$, for all $j \in \{0, \cdots, J-1\}$, the following renormalization property holds on $[0, T]$:*

$$\frac{d}{dt}\beta(\rho_j) = -\rho_j \beta'(\rho_j)\frac{\Delta u_j}{\Delta x_j}.$$

Proof. Let $j \in \{0, \cdots, J-1\}$. From (7.2) and (7.5), we get on $[0, T]$

$$0 = \frac{d}{dt}(\rho_j \Delta x_j) = \Delta x_j \frac{d}{dt}\rho_j + \rho_j \Delta u_j,$$

hence

$$\frac{d}{dt}\beta(\rho_j) = \beta'(\rho_j)\frac{d}{dt}\rho_j = -\rho_j \beta'(\rho_j)\frac{\Delta u_j}{\Delta x_j}.$$

□

The classical way to obtain an upper bound for ρ is to show the

Lemma 1.10. *Let $T \in]0, T_0[$. There is such a $C_{\overline{\rho},1} = C_{\overline{\rho},1}(T_0, \mu_\pm, \gamma_\pm, \overline{\mathcal{M}}, \overline{\mathcal{E}}) > 0$ such that*

$$\sup_{0 \le t \le T} \max_{0 \le j \le J-1} \left| \int_0^t \sigma_j \right| \le C_{\overline{\rho},1}. \tag{17}$$

Proof. Let $j, l \in \{0, \cdots, J-1\}$. Summing (7.3) between l and $j-1$ we get on $[0, T]$

$$\frac{d}{dt}\sum_{k=l}^{j-1} \rho_{k+1/2}\Delta x_{k+1/2} u_{k+1/2} = \sigma_j - \sigma_l, \tag{18}$$

then multiplying (18) by Δx_l and suming in l, using $\sum_{l=0}^{J-1} \Delta x_l = 1$ we get

$$\sum_{l=0}^{J-1} \Delta x_l \frac{d}{dt}\sum_{k=l}^{j-1} \rho_{k+1/2}\Delta x_{k+1/2} u_{k+1/2} = \sigma_j - \sum_{l=0}^{J-1} \sigma_l \Delta x_l. \tag{19}$$

But using (7.5),

$$\begin{aligned}\sum_{l=0}^{J-1} \Delta x_l \frac{d}{dt}\sum_{k=l}^{j-1} \rho_{k+1/2}\Delta x_{k+1/2} u_{k+1/2} = \frac{d}{dt}\sum_{l=0}^{J-1} \Delta x_l \sum_{k=l}^{j-1} \rho_{k+1/2}\Delta x_{k+1/2} u_{k+1/2} \\ - \sum_{l=0}^{J-1} \Delta u_l \sum_{k=l}^{j-1} \rho_{k+1/2}\Delta x_{k+1/2} u_{k+1/2}\end{aligned}$$

Then, using the IPPs formula

$$\sum_{l=0}^{J-1} g_l \Delta f_l = -\sum_{l=0}^{J-1} (\Delta g_{l+1/2}) f_{l+1/2} \tag{20}$$

which is the discrete version of

$$\int_0^1 f g' = -\int_0^1 g' f$$

due to periodicity, we get

$$\sum_{l=0}^{J-1}\Delta u_l\sum_{k=l}^{j-1}\rho_{k+1/2}\Delta x_{k+1/2}u_{k+1/2}=\sum_{l=0}^{J-1}\rho_{l+1/2}\Delta x_{l+1/2}u_{l+1/2}^2.$$

Thus, (19) rewrites

$$\frac{d}{dt}\sum_{l=0}^{J-1}\Delta x_l\sum_{k=l}^{j-1}\rho_{k+1/2}\Delta x_{k+1/2}u_{k+1/2}-\sum_{l=0}^{J-1}\rho_{l+1/2}\Delta x_{l+1/2}u_{l+1/2}^2=\sigma_j-\sum_{l=0}^{J-1}\sigma_l\Delta x_l.$$

Then, integrating in time this last equation,

$$\begin{aligned}\int_0^t\sigma_j=&\sum_{l=0}^{J-1}\Delta x_l\sum_{k=l}^{j-1}\rho_{k+1/2}\Delta x_{k+1/2}u_{k+1/2}-\sum_{l=0}^{J-1}\Delta x_{0,l}\sum_{k=l}^{j-1}\rho_{0,k+1/2}\Delta x_{0,k+1/2}u_{0,k+1/2}\\&-\sum_{l=0}^{J-1}\int_0^t\rho_{l+1/2}\Delta x_{l+1/2}u_{l+1/2}^2+\sum_{l=0}^{J-1}\int_0^t\sigma_l\Delta x_l=:\sum_{i=1}^4 I_i.\end{aligned}$$

Let us control each term I_i to obtain (17):

$$\begin{aligned}|I_1|&\le\left|\sum_{l=0}^{J-1}\Delta x_l\sum_{k=l}^{j-1}\rho_{k+1/2}\Delta x_{k+1/2}u_{k+1/2}\right|\le\sum_{l=0}^{J-1}\Delta x_l\sum_{k=l}^{j-1}\rho_{k+1/2}\Delta x_{k+1/2}|u_{k+1/2}|\\&\le\sum_{k=0}^{J-1}\rho_{k+1/2}\Delta x_{k+1/2}|u_{k+1/2}|\\&\le\left(\sum_{j=0}^{J-1}\rho_{k+1/2}\Delta x_{k+1/2}\right)^{1/2}\left(\rho_{k+1/2}\Delta x_{k+1/2}u_{k+1/2}^2\right)^{1/2}\le(2\overline{\mathcal{ME}})^{1/2}\end{aligned}$$

where we use Hölder's inequality, and Lemma 1.8. Similarly,

$$|I_2|\le\left|\sum_{l=0}^{J-1}\Delta x_{0,l}\sum_{k=l}^{j-1}\rho_{0,k+1/2}\Delta x_{0,k+1/2}u_{0,k+1/2}\right|\le(2\overline{\mathcal{ME}})^{1/2}.$$

Moreover, thanks to Lemma 1.8,

$$|I_3|\le\left|\sum_{l=0}^{J-1}\int_0^t\rho_{l+1/2}\Delta x_{l+1/2}u_{l+1/2}^2\right|\le 2T_0\overline{\mathcal{E}}.$$

Finally, using (7.7) and Lemma 1.8,

$$\begin{aligned}|I_4|\le\left|\sum_{l=0}^{J-1}\int_0^t\sigma_l\Delta x_l\right|&\le\left|\sum_{l=0}^{J-1}\mu(c_l)\int_0^t\Delta u_l\right|+\sum_{l=0}^{J-1}\int_0^t(\gamma(c_l)-1)\rho_l e_l\Delta x_l\\&\le\left|\sum_{l=0}^{J-1}\mu(c_l)(\Delta x_l-\Delta x_{0,l})\right|+\sum_{l=0}^{J-1}(\gamma(c_l)-1)\int_0^t\rho_l e_l\Delta x_l\\&\le 2\mu_{\max}+T_0(\gamma_{\max}-1)\overline{\mathcal{E}}.\end{aligned}$$

□

Proposition 1.11. *Let $T \in]0, T_0[$. There exists some constant $\overline{\rho} = \overline{\rho}(T_0, \overline{\rho_0}, \mu_\pm, \gamma_\pm, \overline{\mathcal{M}}, \overline{\mathcal{E}}) > 0$ such that*

$$\sup_{0\le t\le T} \max_{0\le j\le J-1} \rho_j(t) \le \overline{\rho}.$$

Proof. Let $j \in \{0, \cdots, J-1\}$. Using Lemma 1.9 then (7.7), we get on $[0, T]$

$$\frac{d}{dt} \ln \rho_j = -\frac{\Delta u_j}{\Delta x_j} = -\frac{\sigma_j}{\mu_j} - \frac{p_j}{\mu_j}.$$

Hence integrating in time then using Lemma 17 we obtain, for $t \in [0, T]$,

$$\begin{aligned} \ln \rho_j(t) &= \ln \rho_{0,j} - \frac{1}{\mu_j} \int_0^t \sigma_j - \frac{1}{\mu_j} \int_0^t p_j \\ &\le \ln \rho_{0,j} + \frac{1}{\mu_{\min}} \left| \int_0^t \sigma_j \right| \le \ln \overline{\rho_0} + \frac{C_{\overline{\rho},1}}{\mu_{\min}}, \end{aligned}$$

and finally, taking the exponential,

$$\rho_j(t) \le \overline{\rho_0} e^{C_{\overline{\rho},1}/\mu_{\min}} =: \overline{\rho}.$$

□

Proposition 1.12. *Let $T \in]0, T_0[$. There exists some constant $\underline{\theta} = \underline{\theta}(T_0, \overline{\rho_0}, \overline{\theta_0}, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, \overline{\mathcal{E}}) > 0$ such that*

$$\inf_{0\le t\le T} \min_{0\le j\le J-1} \theta_j(t) \ge \underline{\theta}.$$

Proof. Let $j \in \{0, \cdots, J-1\}$. On $[0, T]$ we get from (7.7)

$$\mu_j \rho_j \Delta x_j \frac{d}{dt} e_j = \sigma_j \mu_j \Delta u_j = \Delta x_j \sigma_j^2 + \Delta x_j \sigma_j p_j = \Delta x_j (\sigma_j + p_j/2)^2 - \Delta x_j p_j^2/4 \ge -\Delta x_j p_j^2/4$$

hence (remember that $p_j = \mathfrak{c}_{\mathrm{v}\,j}(\gamma_j - 1)\rho_j\theta_j$)

$$\frac{d}{dt}\theta_j \ge -\frac{(\gamma_{\max} - 1)^2\, \mathfrak{c}_{\mathrm{v}\max}\, \overline{\rho}\, \theta_j^2}{4\mu_{\min}}.$$

Then, dividing by $-\theta_j^2$, we get

$$\frac{d}{dt}\frac{1}{\theta_j} \le \frac{(\gamma_{\max} - 1)^2\, \mathfrak{c}_{\mathrm{v}\max}\, \overline{\rho}}{4\mu_{\min}} =: C_{\underline{\theta},1}.$$

And integrating in time, we get for $t \in [0, T]$,

$$\frac{1}{\theta_j}(t) \le \frac{1}{\theta_{0,j}} + tC_{\underline{\theta},1} \le \frac{1}{\underline{\theta_0}} + T_0 C_{\underline{\theta},1}.$$

This states the result, taking

$$\underline{\theta} = \frac{1}{1/\underline{\theta_0} + T_0 C_{\underline{\theta},1}}.$$

□

The continuation of the argument is to find bounds on sigma, following the idea of [17].

Lemma 1.13. *Let $T \in]0, T_0[$. For all $j \in \{0, \cdots J-1\}$, σ_j satisfies on $[0,T]$ the equation*

$$\frac{d}{dt}(\Delta x_j \sigma_j) - \mu_j \Delta \left(\frac{\Delta \sigma}{\rho \Delta x} \right)_j = -(\gamma_j - 1)\sigma_j \Delta u_j. \tag{21}$$

Proof. Let $j \in \{0, \cdots, J-1\}$. Starting from

$$\Delta x_j \sigma_j = \mu_j \Delta u_j - (\gamma_j - 1)\rho_j \Delta x_j e_j$$

due to (7.7), then deriving both sides of the equality, we get from (7.2)

$$\frac{d}{dt}(\Delta x_j \sigma_j) = \mu_j \Delta \frac{d}{dt} u_j - (\gamma_j - 1)\rho_j \Delta x_j \frac{d}{dt} e_j.$$

On the other hand, one has

$$\frac{d}{dt} u_{j+1/2} = \frac{\Delta \sigma_{j+1/2}}{\rho_{j+1/2} \Delta x_{j+1/2}} \tag{7.3}$$

and

$$\rho_j \Delta x_j \frac{d}{dt} e_j = \sigma_j \Delta u_j, \tag{7.8}$$

hence (21). □

Lemma 1.14. *Let $T \in]0, T_0[$. There exists some $C_{1,1} = C_{1,1}(\mu_\pm, \gamma_\pm, \mathfrak{c}_{v\pm}, \overline{\rho_0}, \overline{\theta_0}, C_0, \overline{\mathcal{E}}) > 0$ such that*

$$\sup_{0 \le t \le T} \sum_{j=0}^{J-1} \Delta x_j |\sigma_j| \le C_{1,1}.$$

Proof. Let $j \in \{0, \cdots, J-1\}$. We will denote $f^+ = \max(0, f)$. Multiplying (21) by σ_j^+, we get

$$\sigma_j^+ \frac{d}{dt}(\Delta x_j \sigma_j) - \mu_j \sigma_j^+ \Delta \left(\frac{\Delta \sigma}{\rho \Delta x} \right)_j = -(\gamma_j - 1)(\sigma_j^+)^2 \Delta u_j. \tag{22}$$

On the other hand, thanks to (7.5),

$$\frac{d}{dt}(\Delta x_j (\sigma_j^+)^2) = 2\Delta x_j \sigma_j^+ \frac{d}{dt}\sigma_j + \Delta u_j (\sigma_j^+)^2 \tag{23}$$

and

$$\frac{d}{dt}(\Delta x_j (\sigma_j^+)^2) = \frac{d}{dt}(\Delta x_j \sigma_j^+ \sigma_j) = \sigma_j^+ \frac{d}{dt}(\Delta x_j \sigma_j) + \Delta x_j \sigma_j^+ \frac{d}{dt}\sigma_j \tag{24}$$

thus, substracting (23)/2 from (24) we get :

$$\frac{1}{2}\frac{d}{dt}(\Delta x_j (\sigma_j^+)^2) = \sigma_j^+ \frac{d}{dt}(\Delta x_j \sigma_j) - \frac{1}{2}\Delta u_j (\sigma_j^+)^2$$

and (22) gives

$$\frac{1}{2}\frac{d}{dt}(\Delta x_j (\sigma_j^+)^2) - \mu_j \sigma_j^+ \Delta \left(\frac{\Delta \sigma}{\rho \Delta x} \right)_j = -\left(\gamma_j - \frac{1}{2} \right)(\sigma_j^+)^2 \Delta u_j.$$

As $p_j > 0$ (because $\rho_j > 0$ and $\theta_j > 0$), we remark from (7.7) that

$$\sigma_j \ge 0 \Rightarrow \Delta u_j > 0,$$

thus

$$\frac{1}{2}\frac{d}{dt}(\Delta x_j(\sigma_j^+)^2) - \mu_j\sigma_j^+\Delta\left(\frac{\Delta\sigma}{\rho\Delta x}\right)_j \leq 0,$$

since $\gamma_j > 1 > 1/2$. Dividing by μ_j then summing in j and using (20) we get

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\frac{\Delta x_j(\sigma_j^+)^2}{\mu_j} + \sum_{j=0}^{J-1}\frac{\Delta(\sigma^+)_{j+1/2}\Delta\sigma_{j+1/2}}{\rho_{j+1/2}\Delta x_{j+1/2}} \leq 0.$$

Because $y \mapsto y^+$ is nondecreasing, we have $\Delta(\sigma^+)_{j+1/2}\Delta\sigma_{j+1/2} \geq 0$, so finally

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\frac{\Delta x_j(\sigma_j^+)^2}{\mu_j} \leq 0,$$

thus

$$\sum_{j=0}^{J-1}\Delta x_j(\sigma_j^+)^2 \leq \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_j(\sigma_j^+)^2}{\mu_j} \leq \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_{0,j}(\sigma_{0,j}^+)^2}{\mu_{0,j}} \leq \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_{0,j}\sigma_{0,j}^2}{\mu_{0,j}}$$

then by Young's inequality, using (7.7) and (8)

$$\sum_{j=0}^{J-1}\Delta x_j(\sigma_j^+)^2 \leq 2\mu_{\max}\sum_{j=0}^{J-1}\mu_{0,j}\frac{(\Delta u_{0,j})^2}{\Delta x_{0,j}} + 2\mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_{0,j}R_{0,j}^2\rho_{0,j}^2\theta_{0,j}^2}{\mu_{0,j}} \leq C_{1,2},$$

where

$$C_{1,2} := 2\mu_{\max}^2 C_0 + 2\frac{\mu_{\max}}{\mu_{\min}}R_{\max}^2\overline{\rho_0}^2\overline{\theta_0}^2.$$

Using the equality $|\sigma_j| = 2\sigma_j^+ - \sigma_j$ we get from Young's inequality, (7.7) and (25)

$$\begin{aligned}
\sum_{j=0}^{J-1}\Delta x_j|\sigma_j| &\leq \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_j|\sigma_j|}{\mu_j}\\
&\leq 2\frac{\mu_{\max}}{\mu_{\min}}\sum_{j=0}^{J-1}\Delta x_j\sigma_j^+ - \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_j\sigma_j}{\mu_j}\\
&\leq 2\frac{\mu_{\max}}{\mu_{\min}}\left(\sum_{j=0}^{J-1}\Delta x_j(\sigma_j^+)^2\right)^{1/2}\left(\sum_{j=0}^{J-1}\Delta x_j\right)^{1/2} - \mu_{\max}\sum_{j=0}^{J-1}\Delta u_j + \mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_j p_j}{\mu_j}\\
&\leq 2\frac{\mu_{\max}}{\mu_{\min}}{C_{1,2}}^{1/2} + 2\frac{\mu_{\max}}{\mu_{\min}}(\gamma_{\max}-1)\overline{\mathcal{E}} =: C_{1,1}.
\end{aligned}$$

□

Proposition 1.15. *Let $T \in]0, T_0[$. There exists some $C_1 = C_1(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{v\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that*

$$\sup_{0\leq t\leq T}\sum_{j=0}^{J-1}\Delta x_j\sigma_j^2 + \sum_{j=0}^{J-1}\int_0^T\frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} \leq C_1. \tag{10.5}$$

Proof. Let $j \in \{0, \cdots J-1\}$. Multiplying (21) by σ_j, the same computation as in Lemma 1.14 gives

$$\frac{1}{2}\frac{d}{dt}(\Delta x_j \sigma_j^2) - \mu_j \sigma_j \Delta\left(\frac{\Delta\sigma}{\rho\Delta x}\right)_j = -\left(\gamma_j - \frac{1}{2}\right)\sigma_j^2 \Delta u_j.$$

Dividing by μ_j then summing over j and using the IPP identity (20), then (7.7) and Lemma 1.14, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1}\frac{\Delta x_j\sigma_j^2}{\mu_j} + \sum_{j=0}^{J-1}\frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} &= -\sum_{j=0}^{J-1}\frac{1}{\mu_j}\left(\gamma_j - \frac{1}{2}\right)\sigma_j^2\Delta u_j \\
&= -\sum_{j=0}^{J-1}\frac{1}{\mu_j^2}\left(\gamma_j - \frac{1}{2}\right)\sigma_j^3\Delta x_j - \sum_{j=0}^{J-1}\frac{1}{\mu_j^2}\left(\gamma_j - \frac{1}{2}\right)\sigma_j^2 p_j \Delta x_j \\
&\le -\sum_{j=0}^{J-1}\frac{1}{\mu_j^2}\left(\gamma_j - \frac{1}{2}\right)\sigma_j^3\Delta x_j \\
&\le \max_i \sigma_i^2 \sum_{j=0}^{J-1}\frac{1}{\mu_j^2}\left(\gamma_j - \frac{1}{2}\right)|\sigma_j|\Delta x_j \\
&\le C_{1,1}\left(\gamma_{\max} - \frac{1}{2}\right)\frac{1}{\mu_{\min}^2}\max_i \sigma_i^2. \qquad (25)
\end{aligned}$$

Let $i, l \in \{0, \cdots, J-1\}$. We have

$$\sigma_i^2 = \sigma_l^2 + \sum_{k=l}^{i-1}\Delta(\sigma^2)_{k+1/2} = \sigma_l^2 + 2\sum_{k=l}^{i-1}\sigma_{k+1/2}\Delta\sigma_{k+1/2}.$$

Multiplying by Δx_l and summing in l we get using Hölder's inequality

$$\begin{aligned}
\sigma_i^2 &= \sum_{l=0}^{J-1}\Delta x_l\sigma_l^2 + 2\sum_{l=0}^{J-1}\Delta x_l\sum_{k=l}^{i-1}\sigma_{k+1/2}\Delta\sigma_{k+1/2} \\
&\le \sum_{l=0}^{J-1}\Delta x_l\sigma_l^2 + 2\sum_{k=0}^{J-1}|\sigma_{k+1/2}\Delta\sigma_{k+1/2}| \\
&\le \sum_{l=0}^{J-1}\Delta x_l\sigma_l^2 + 2\left(\sum_{k=0}^{J-1}\rho_{k+1/2}\Delta x_{k+1/2}\sigma_{k+1/2}^2\right)^{1/2}\left(\sum_{k=0}^{J-1}\frac{(\Delta\sigma_{k+1/2})^2}{\rho_{k+1/2}\Delta x_{k+1/2}}\right)^{1/2}.
\end{aligned}$$

Moreover, from (7.6),

$$\begin{aligned}
\rho_{k+1/2}\Delta x_{k+1/2}\sigma_{k+1/2}^2 &= \frac{\rho_k\Delta x_k + \rho_{k+1}\Delta x_{k+1}}{2}\left(\frac{\sigma_k + \sigma_{k+1}}{2}\right)^2 \\
&\le \frac{\rho_k\Delta x_k + \rho_{k+1}\Delta x_{k+1}}{2}\frac{\sigma_k^2 + \sigma_{k+1}^2}{2} \\
&\le \frac{\rho_k\Delta x_k\sigma_k^2 + \rho_{k+1}\Delta x_{k+1}\sigma_{k+1}^2}{4} + \frac{\rho_k\Delta x_k\sigma_{k+1}^2 + \rho_{k+1}\Delta x_{k+1}\sigma_k^2}{4}. \qquad (26)
\end{aligned}$$

On the other hand, from H1–H3,

$$\rho_k\Delta x_k = \rho_{0,k}\Delta x_{0,k} \le \overline{\rho_0}\Delta x_{0,k} \le L_0\overline{\rho_0}\Delta x_{0,k+1} \le L_0\frac{\overline{\rho_0}}{\underline{\rho_0}}\rho_{0,k+1}\Delta x_{0,k+1} = L_0\frac{\overline{\rho_0}}{\underline{\rho_0}}\rho_{k+1}\Delta x_{k+1},$$

and similarly,

$$\rho_{k+1}\Delta x_{k+1} \le L_0\frac{\overline{\rho_0}}{\underline{\rho_0}}\rho_k\Delta x_k.$$

Thus

$$\rho_{k+1/2}\Delta x_{k+1/2}\sigma_{k+1/2}^2 \le \frac{\underline{\rho_0} + L\overline{\rho_0}}{2\underline{\rho_0}}\frac{\rho_k\Delta x_k\sigma_k^2 + \rho_{k+1}\Delta x_{k+1}\sigma_{k+1}^2}{2}$$

and by (26) and Lemma 1.14

$$\begin{aligned}\sigma_i^2 &\le \sum_{l=0}^{J-1}\Delta x_l\sigma_l^2 + 2\sqrt{\frac{\underline{\rho_0} + L_0\overline{\rho_0}}{2\underline{\rho_0}}}\left(\sum_{k=0}^{J-1}\rho_k\Delta x_k\sigma_k^2\right)^{1/2}\left(\sum_{k=0}^{J-1}\frac{(\Delta\sigma_{k+1/2})^2}{\rho_{k+1/2}\Delta x_{k+1/2}}\right)^{1/2}\\ &\le C_{1,3}\sum_{l=0}^{J-1}\Delta x_l\sigma_l^2 + \frac{\mu_{\min}^2}{(2\gamma_{\max}-1)C_{1,1}}\sum_{k=0}^{J-1}\frac{(\Delta\sigma_{k+1/2})^2}{\rho_{k+1/2}\Delta x_{k+1/2}}\end{aligned} \tag{27}$$

denoting

$$C_{1,3} := 1 + \frac{1}{\mu_{\min}^2}\left(\gamma_{\max} - \frac{1}{2}\right)C_{1,1}\overline{\rho}\left(1 + L_0\frac{\overline{\rho_0}}{\underline{\rho_0}}\right).$$

Thus we get from (25)

$$\frac{d}{dt}\sum_{j=0}^{J-1}\frac{\Delta x_j\sigma_j^2}{\mu_j} + \sum_{j=0}^{J-1}\frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} \le 2C_{1,3}\sum_{j=0}^{J-1}\Delta x_j\sigma_j^2 \le 2C_{1,3}\mu_{\max}\sum_{j=0}^{J-1}\frac{\Delta x_j\sigma_j^2}{\mu_j}.$$

Thanks to the Grönwall's lemma, we can now close this last inequation and obtain from (25)

$$\begin{aligned}\sup_{0\le t\le T}\sum_{j=0}^{J-1}\frac{\Delta x_j\sigma_j^2}{\mu_j} + \int_0^T\sum_{j=0}^{J-1}\frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} &\le \sum_{j=0}^{J-1}\frac{\Delta x_{0,j}\sigma_{0,j}^2}{\mu_{0,j}}\exp(2C_{1,3}\mu_{\max}T_0)\\ &\le \frac{C_{1,2}}{\mu_{\min}}\exp(2C_{1,3}\mu_{\max}T_0)\end{aligned} \tag{28}$$

hence finally

$$\sup_{0\le t\le T}\sum_{j=0}^{J-1}\Delta x_j\sigma_j^2 + \int_0^T\sum_{j=0}^{J-1}\frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} \le \max(\mu_{\max},\overline{\rho})\frac{C_{1,2}}{\mu_{\min}}\exp(2C_{1,3}\mu_{\max}T_0).$$

□

Corollary 1.16. *Let $T \in]0, T_0[$. There exists $C_2 = C_2(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that*

$$\int_0^T \max_{0\le j\le J-1}\sigma_j^2 \le C_2. \tag{10.6}$$

Proof. Let $j \in \{0, \cdots, J-1\}$. Combining (27) and (28), we get

$$\int_0^T \sigma_j^2(t)dt \leq \left(C_{1,3} T_0 \frac{\mu_{\max}}{\mu_{\min}} + \frac{\mu_{\min}}{(2\gamma_{\max} - 1)C_{1,1}} \right) C_{1,2} \exp(2C_{1,3}\mu_{\max} T_0). \tag{29}$$

□

From this last result we get easily an upper bound on p, then some bounds on $\Delta u_j / \Delta x_j$.

Proposition 1.17. *Let $T \in]0, T_0[$. There exists some constant $\overline{p} = \overline{p}(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that*

$$\sup_{0 \leq t \leq T} \max_{0 \leq j \leq J-1} p_j(t) \leq \overline{p}.$$

Proof. Let $j \in \{0, \cdots J-1\}$. We have, on $[0, T]$,

$$\begin{aligned}
\mu_j \frac{d}{dt} p_j &= (\gamma_j - 1)\mu_j \frac{d}{dt}(\rho_j e_j) \\
&= \mu_j(\gamma_j - 1) e_j \frac{d}{dt}\rho_j + \mu_j(\gamma_j - 1)\rho_j \frac{d}{dt} e_j \\
&= -p_j \mu_j \frac{\Delta u_j}{\Delta x_j} + \sigma_j(\gamma_j - 1)\mu_j \frac{\Delta u_j}{\Delta x_j} \quad \text{using (7.2) and (7.4)} \\
&= ((\gamma_j - 1)\sigma_j - p_j)(\sigma_j + p_j) \\
&= \left(\frac{(\gamma_j - 2)^2}{4} + \gamma_j - 1 \right) \sigma_j^2 - \left(p_j - \frac{\gamma_j - 2}{2} \sigma_j \right)^2 \\
&\leq \left(\frac{(\gamma - 2)^2_{\max}}{4} + \gamma_{\max} - 1 \right) \max_j \sigma_j^2,
\end{aligned}$$

hence, integrating on $[0, T]$ and using Corollary 1.16,

$$p_j \leq \frac{\overline{p_0}}{\mu_{\min}} + \frac{1}{\mu_{\min}} \left(\frac{(\gamma - 2)^2_{\max}}{4} + \gamma_{\max} - 1 \right) C_2 =: \overline{p}.$$

□

Proposition 1.18. *There exists some constants C_3, C_4, $C_5 > 0$ depending only on $T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}$, $\underline{\rho_0}, \overline{\theta_0}, L_0$ such that*

$$\sup_{0 \leq t \leq T} \sum_{j=0}^{J-1} \frac{(\Delta u_j)^2}{\Delta x_j} \leq C_3, \tag{10.7}$$

$$\int_0^T \max_{0 \leq j \leq J-1} \left(\frac{\Delta u_j}{\Delta x_j} \right)^2 \leq C_4, \tag{10.8}$$

$$\sup_{0 \leq t \leq T} \max_{0 \leq j \leq J-1} |u_{j+1/2}| \leq C_5. \tag{10.9}$$

Proof. Let $j \in \{0, \cdots, J-1\}$. We have on $[0, T]$, from (7.7)

$$\left(\frac{\Delta u_j}{\Delta x_j} \right)^2 \leq 2 \frac{\sigma_j^2 + p_j^2}{\mu_{\min}^2}$$

hence from Proposition 1.15 and Proposition 1.17

$$\sum_{j=0}^{J-1} \frac{(\Delta u_j)^2}{\Delta x_j} \leq \frac{2}{\mu_{\min}^2} \sum_{j=0}^{J-1} \Delta x_j \sigma_j^2 + \frac{2}{\mu_{\min}^2} \sum_{j=0}^{J-1} \Delta x_j p_j^2 \leq \frac{2C_1 + 2\overline{p}^2}{\mu_{\min}^2}$$

and from Corollary 1.16 and Proposition 1.17

$$\int_0^T \left(\frac{\Delta u_j}{\Delta x_j} \right)^2 \leq \frac{2}{\mu_{\min}^2} \int_0^T \sigma_j^2 + \frac{2}{\mu_{\min}^2} \int_0^T p_j^2 \leq \frac{2C_2 + 2T_0\overline{p}^2}{\mu_{\min}^2}.$$

Moreover, by repeating the computations of the lemma 1.8, and using (8), (10.7) and

$$\sum_{k=0}^{J-1} \rho_{k+1/2} \Delta x_{k+1/2} u_{k+1/2} = \sum_{k=0}^{J-1} \rho_{0,k+1/2} \Delta x_{0,k+1/2} u_{0,k+1/2}$$

by the conservative form of (7.3), we get

$$|u_{j+1/2}| \leq \frac{\varepsilon_0 + \overline{\rho_0} {C_3}^{1/2}}{\underline{\rho_0}}.$$

□

Proposition 1.19. *There exists some $\underline{\rho} > 0$ and $\overline{\theta} > 0$ depending only on $T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0$ such that*

$$\inf_{0 \leq t \leq T} \min_{0 \leq j \leq J-1} \rho_j \geq \underline{\rho},$$

$$\sup_{0 \leq t \leq T} \max_{0 \leq j \leq J-1} \theta_j \leq \overline{\theta}.$$

Proof. Let $j \in \{0, \cdots, J-1\}$. Using the Lemma 1.9, we get on $[0,T]$

$$\frac{d}{dt} \ln \rho_j = -\frac{\Delta u_j}{\Delta x_j},$$

hence for $t \in [0,T]$,

$$\rho_j(t) = \rho_{0,j} \exp\left(-\int_0^t \frac{\Delta u_j}{\Delta x_j} \right) \geq \underline{\rho_0} \exp\left(-\int_0^t \max_j \left| \frac{\Delta u_j}{\Delta x_j} \right| \right) \geq \underline{\rho_0} \exp(-\sqrt{T}\sqrt{C_4}) =: \underline{\rho}$$

where we use (10.8). Moreover,

$$\theta_j = \frac{p_j}{R_j \rho_j} \leq \frac{\overline{p}}{R_{\min} \underline{\rho}} =: \overline{\theta}.$$

□

Proposition 1.20. *Let $T \in]0, T_0[$. There exists some $C_6 = C_6(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that*

$$\int_0^T \sum_{j=0}^{J-1} \Delta x_{j+1/2} |u'_{j+1/2}|^2 \leq C_6. \tag{30}$$

Proof. Let $j \in \{0, \cdots J-1\}$. Taking the square in $(7.3)^2$, we get, on $[0,T]$

$$\Delta x_{j+1/2}|u'_{j+1/2}|^2 = \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}\rho^2_{j+1/2}}. \tag{31}$$

Integrating on $[0,T]$ then summing in j we get using (10.5)

$$\int_0^T \sum_{j=0}^{J-1} \Delta x_{j+1/2}|u'_{j+1/2}|^2 = \int_0^T \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}\rho^2_{j+1/2}} \leq \frac{C_1}{\underline{\rho}^2.} \tag{32}$$

□

Proposition 1.21. *Let $T \in]0,T_0[$. There exists some $L = L(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that, for all $t \in [0,T]$,*

$$\max_{0\leq j\leq J-1} \Delta x_j(t) \leq L \min_{0\leq j\leq J-1} \Delta x_j(t). \tag{10.4}$$

Proof. Let $j,k \in \{0, \cdots J-1\}$. Then, for $t \in [0,T]$, we get from (7.2) and H1

$$\Delta x_j(t) = \frac{\rho_j \Delta x_j}{\rho_j}(t) = \frac{\rho_{0,j}\Delta x_{0,j}}{\rho_j(t)} \leq \frac{\overline{\rho_0}}{\underline{\rho}} L_0 \Delta x_{0,k} \leq \frac{\overline{\rho_0}}{\underline{\rho_0}\,\underline{\rho}} L_0 \rho_{0,k}\Delta x_{0,k} = \frac{\overline{\rho_0}}{\underline{\rho_0}\,\underline{\rho}} L_0 \rho_k \Delta x_k(t) \leq \frac{\overline{\rho_0}}{\underline{\rho_0}} \frac{\overline{\rho}}{\underline{\rho}} L_0 \Delta x_k(t).$$

□

Proposition 1.22. *Let $T \in]0,T_0[$. There exists some $C_7 = C_7(T_0, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0) > 0$ such that*

$$\sup_{0\leq t\leq T} \min(1,t) \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} + \int_0^t \min(1,s) \sum_{j=0}^{J-1} \Delta x_j (\sigma'_j)^2 \leq C_7.$$

Proof. Rewriting (21) as

$$\Delta x_j \sigma'_j - \mu_j \Delta\left(\frac{\Delta\sigma}{\rho\Delta x}\right)_j = -\gamma_j\sigma_j\Delta u_j$$

where we use (7.5), then multiplying by σ'_j/μ_j and summing, using (20) we get

$$\sum_{j=0}^{J-1} \Delta x_j \frac{(\sigma'_j)^2}{\mu_j} + \sum_{j=0}^{J-1} \frac{\Delta\sigma_{j+1/2}}{\rho_{j+1/2}\Delta x_{j+1/2}} \Delta\sigma'_{j+1/2} = -\sum_{j=0}^{J-1} \frac{\gamma_j}{\mu_j}\sigma_j\sigma'_j\Delta u_j$$

thus from (7.6), (7.2) and Young's inequality,

$$\begin{aligned}\sum_{j=0}^{J-1} \Delta x_j \frac{(\sigma'_j)^2}{\mu_j} + \frac{1}{2}\frac{d}{dt}\sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} &= -\sum_{j=0}^{J-1} \frac{\gamma_j}{\mu_j}\sigma_j\sigma'_j\Delta u_j \\ &\leq \frac{\gamma^2_{\max}}{2\mu_{\min}} \max_{0\leq j\leq J-1} \left|\frac{\Delta u_j}{\Delta x_j}\right|^2 \sum_{j=0}^{J-1} \Delta x_j\sigma_j^2 + \frac{1}{2}\sum_{j=0}^{J-1} \Delta x_j \frac{(\sigma'_j)^2}{\mu_j}\end{aligned}$$

and from (10.5)

$$\sum_{j=0}^{J-1} \Delta x_j \frac{(\sigma'_j)^2}{\mu_j} + \frac{d}{dt}\sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} \leq \frac{\gamma^2_{\max}}{\mu_{\min}} \max_{0\leq j\leq J-1} \left|\frac{\Delta u_j}{\Delta x_j}\right|^2 \sum_{j=0}^{J-1} \Delta x_j\sigma_j^2 \leq \frac{\gamma^2_{\max}}{\mu_{\min}} C_1 \max_{0\leq j\leq J-1} \left|\frac{\Delta u_j}{\Delta x_j}\right|^2.$$

Then, multiplying by $\min(1,t)$ and integrating with respect to t, using the equality

$$\int_0^t \min(1,s) f'(s) ds = \min(1,t) f(t) - \int_0^{\min(1,t)} f(s) ds$$

we get from (10.8)

$$\begin{aligned}
\min(1,t) \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} + \int_0^t \min(1,s) \sum_{j=0}^{J-1} \Delta x_j \frac{(\sigma_j')^2}{\mu_j} \\
&\leq \int_0^{\min(1,t)} \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\rho_{j+1/2}\Delta x_{j+1/2}} + \frac{\gamma_{\max}^2}{\mu_{\min}} \left(\int_0^t \max_j \left| \frac{\Delta u_j}{\Delta x_j} \right|^2 \right) C_1 \\
&\leq C_1 \left(\frac{1}{\underline{\rho}} + \frac{\gamma_{\max}^2}{\mu_{\min}} C_4 \right)
\end{aligned}$$

and finally

$$\min(1,t) \sum_{j=0}^{J-1} \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} + \int_0^t \min(1,s) \sum_{j=0}^{J-1} \Delta x_j (\sigma_j')^2 \leq \max(\mu_{\max}, \overline{\rho}) C_1 \left(\frac{1}{\underline{\rho}} + \frac{\gamma_{\max}^2}{\mu_{\min}} C_4 \right).$$

□

1.4. **Stability and homogenization results**

The estimates obtained previously allow the proof of the theorem 1.6.

End of the proof of the theorem 1.6. It remains to show that $T_{\max} = +\infty$. By contradiction, we suppose $T_{\max} < +\infty$. We can then take $T_0 = T_{\max}$ in the previous estimates.

- For all $t \in]0, T_{\max}[$ and $j \in \{0, \cdots, J-1\}$, $|u_{j+1/2}(t)| \leq C_5(T_{\max}, \mu_\pm, \gamma_\pm, \mathfrak{c}_{\mathrm{v}\pm}, C_0, \overline{\mathcal{E}}, \overline{\rho_0}, \underline{\rho_0}, \overline{\theta_0}, L_0)$ thus $|U(t)| \underset{t\to T_{\max}}{\not\longrightarrow} +\infty$. Moreover, we get

$$|x_{j+1/2}(t)| \leq |x_{0,j+1/2}| + \int_0^t |u_{j+1/2}(s)| ds \leq 1 + T_{\max} C_5,$$

  thus $|X(t)| \underset{t\to T_{\max}}{\not\longrightarrow} +\infty$.
- Moreover, combining (10.4) and the remark 1.2, we get

$$\liminf_{t\to T_{\max}} \min_j \Delta x_j \geq \frac{1}{JL} > 0$$

- At last, by (10.3), $\liminf_{t\to T_{\max}} \min_j \theta_j \geq \underline{\theta} > 0$.

Hence an absurdity using the lemma 1.7. □

In the following we establish some links between (7) and (1).

Proposition 1.23. *One defines on $[0,T] \times \mathbb{T}$ the functions $\hat{c}_J, \hat{\rho}_J, \hat{m}_J, \hat{\theta}_J, \hat{u}_J, \hat{v}_J, \hat{\sigma}_J$, by for all $(t,x) \in [0,T] \times \mathbb{T}$,*

$$\begin{aligned}
\hat{c}_J(t,x) &= c_j(t) \text{ if } x \in [x_{j-1/2}(t), x_{j+1/2}(t)[, \\
\hat{\rho}_J(t,x) &= \rho_j(t) \text{ if } x \in [x_{j-1/2}(t), x_{j+1/2}(t)[, \\
\hat{m}_J(t,x) &= \rho_{j+1/2}(t) u_{j+1/2}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[,
\end{aligned}$$

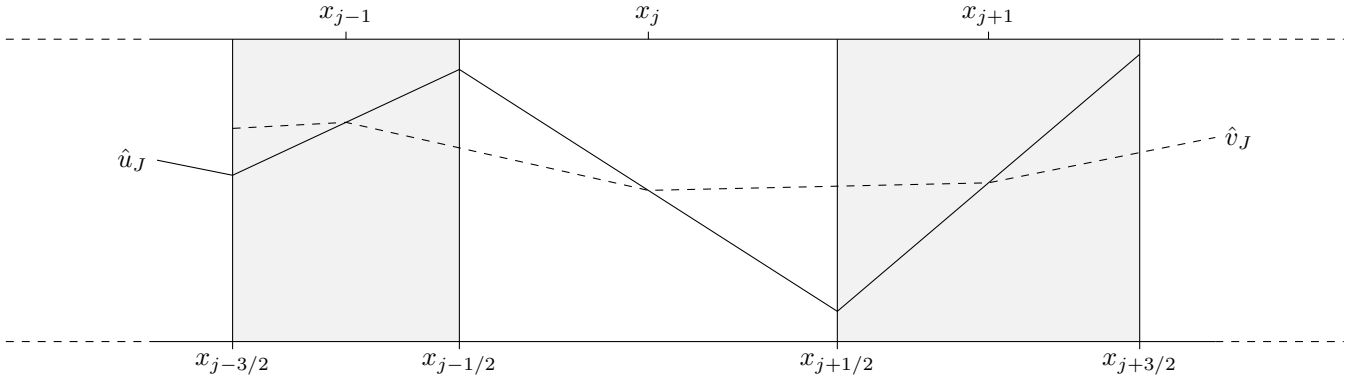
$$\hat{\theta}_J(t,x) = \theta_j(t) \text{ if } x \in [x_{j-1/2}(t), x_{j+1/2}(t)[,$$

$$\hat{u}_J(t,x) = u_{j-1/2}(t) + (x - x_{j-1/2}(t))\frac{\Delta u_j}{\Delta x_j}(t) \text{ if } x \in [x_{j-1/2}(t), x_{j+1/2}(t)[,$$

$$\hat{v}_J(t,x) = u_j(t) + (x - x_j(t))\frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[,$$

$$\hat{\sigma}_J(t,x) = \sigma_j(t) + (x - x_j(t))\frac{\Delta \sigma_{j+1/2}}{\Delta x_{j+1/2}}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[.$$

FIGURE 3. Representation of $\hat{u}_J$ and $\hat{v}_J$



We get, on $[0,T] \times \mathbb{T}$,

$$\begin{cases} \partial_t \hat{c}_J + \hat{u}_J \partial_x \hat{c}_J = 0 & (33.1) \\ \partial_t \hat{\rho}_J + \partial_x(\hat{\rho}_J \hat{u}_J) = 0 & (33.2) \\ \partial_t \hat{m}_J + \partial_x(\hat{m}_J \hat{v}_J) = \partial_x \hat{\sigma}_J & (33.3) \\ \partial_t(\hat{\rho}_J\, \mathfrak{c}_{\mathrm{v}}(\hat{c}_J)\hat{\theta}_J) + \partial_x(\hat{\rho}_J\, \mathfrak{c}_{\mathrm{v}}(\hat{c}_J)\hat{\theta}_J \hat{u}_J) = \hat{\sigma}_J \partial_x \hat{u}_J - \hat{r}^1_J & (33.4) \end{cases}$$

where

$$\hat{r}^1_J = (x - x_j(t))\frac{\Delta \sigma_{j+1/2}}{\Delta x_{j+1/2}}(t)\frac{\Delta u_j}{\Delta x_j}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[.$$

Proof. The proof is a direct computation using (7), and remarking in particular that using (7.5), we have on $]0,T[$, for all $j \in \{0, \cdots J-1\}$,

$$\partial_t \mathbb{1}_{[x_{j-1/2}, x_{j+1/2}[} + \hat{u}_J \partial_x \mathbb{1}_{[x_{j-1/2}, x_{j+1/2}[} = 0, \tag{34}$$

$$\partial_t \mathbb{1}_{[x_j, x_{j+1}[} + \hat{v}_J \partial_x \mathbb{1}_{[x_j, x_{j+1}[} = 0. \tag{35}$$

□

In the following, for $f \in \{\mu, \mathfrak{c}_{\mathrm{v}}, \gamma, R, p\}$, we will denote $\hat{f}_J := f(\hat{c}_J)$.

The Lemma 1.24 and the following Proposition 1.25 express respectively the closeness between (33) and (1), and uniform bounds on $(\hat{c}_J, \hat{\rho}_J, \hat{\theta}_J, \hat{u}_J, \hat{\sigma}_J)$, derived from the Section 1.3. Their proofs are purely computational and will be detailed in the appendix.

Lemma 1.24. *Let $T>0$. Denote $\hat r_J^2 := \hat\sigma_J - \mu(\hat c_J)\partial_x\hat u_J + R(\hat c_J)\hat\rho_J\hat\theta_J$. Then*

$$\sup_{0\le t\le T}\|\hat u_J-\hat v_J\|_{L^2(\mathbb{T})}\le\frac{L\sqrt{C_3}}{\sqrt{12}J}, \tag{36.1}$$

$$\|\hat r_J^2\|_{L^2([0,T]\times\mathbb{T})}\le\frac{L}{\sqrt{3}J}\sqrt{C_1}, \tag{36.2}$$

$$\|\hat r_J^1\|_{L^1([0,T]\times\mathbb{T})}\le\frac{L^{3/2}}{4J}(C_1+TC_3), \tag{36.3}$$

$$\sup_{0\le t\le T}\|\hat r_J^1\|_{L^1(\mathbb{T})}\le L\sqrt{C_1C_3}, \tag{36.4}$$

$$\text{for all } \varphi\in C_c^1(]0,1[),\quad \sup_{0\le t\le T}\left|\int_0^1\hat m_J\varphi-\int_0^1\hat\rho_J\hat u_J\varphi\right|\le\frac{\overline{\rho}C_5L}{4J}\|\varphi'\|_\infty+\frac{\overline{\rho}\sqrt{C_3}L}{4J}\|\varphi\|_\infty. \tag{36.5}$$

Proposition 1.25. *Let $T>0$. Then, on $[0,T]$,*

$$\hat c_J\in\{0,1\}, \tag{37.1}$$

$$\underline{\rho}\le\hat\rho_J\le\overline{\rho}, \tag{37.2}$$

$$\underline{\theta}\le\hat\theta_J\le\overline{\theta}, \tag{37.3}$$

$$|\hat u_J|\le C_5, \tag{37.4}$$

$$|\hat m_J|\le\overline{\rho}C_5, \tag{37.5}$$

and

$$\sup_{0\le t\le T}\int_0^1\hat\sigma_J^2+\int_0^T\int_0^1(\partial_x\hat\sigma_J)^2\le LC_1, \tag{37.6}$$

$$\sup_{0\le t\le T}\int_0^1(\partial_x\hat u_J)^2\le C_3, \tag{37.7}$$

$$\int_0^T\|\partial_x\hat u_J\|_\infty^2\le C_4, \tag{37.8}$$

$$\int_0^T\int_0^1(\partial_t\hat u_J+\hat u_J\partial_x\hat u_J)^2\le C_6, \tag{37.9}$$

$$\sup_{0\le t\le T}\min(1,t)\int_0^1(\partial_x\hat\sigma_J)^2+\int_0^T\min(1,s)\int_0^1(\partial_t\hat\sigma_J+\hat v_J\partial_x\hat\sigma_J)^2\le 2LC_7. \tag{37.10}$$

We can now show the convergence of (33) to (1) :

Theorem 1.26 (arrow (a) in Figure 1)**.** *For all $J\in\mathbb{N}^*$, let $(x^J_{0,j+1/2},c^J_{0,j},\rho^J_{0,j},\theta^J_{0,j},u^J_{0,j+1/2})_{j=0,\cdots J-1}$ satisfy* H1–H4 *with $\underline{\rho_0},\overline{\rho_0},\underline{\theta_0},\overline{\theta_0}$ and C_0,ε_0,L_0 independant of J. Suppose moreover that*

$$(\hat c_{0,J},\hat\rho_{0,J},\hat\theta_{0,J})\underset{J\to+\infty}{\longrightarrow}(c_0,\rho_0,\theta_0)\quad\text{in } L^2(\mathbb{T})^3,\qquad\qquad \hat u_{0,J}\underset{J\to+\infty}{\rightharpoonup}u_0\quad\text{in } H^1(\mathbb{T}).$$

Then, up to a subsequence, $(\hat c_J,\hat\rho_J,\hat\theta_J,\hat u_J)$ converges strongly in $L^2(0,T,L^2(\mathbb{T}))^3\times L^2(0,T,H^1(\mathbb{T}))$ to some "à la Hoff" solution of (1)*, with initial conditions $(c_0,\rho_0,\theta_0,u_0)$.* □

Sketch of the proof. The proof of this theorem is very similar to the stability result in [3] (Theorem 4.1). In the following, convergences are valid up to an extraction.

(1) Thanks to (37.7) and (37.9), we show that there exists some $u \in L^\infty(0,T,L^2(\mathbb{T})) \cap L^2(0,T,L^\infty(\mathbb{T}))$ such that

$$\int_0^T \|\hat{u}_J - u\|_\infty^2 \underset{J\to+\infty}{\longrightarrow} 0. \tag{38}$$

We then deduce by (37.8) that, defining the characteristic function X_J (resp. X) associated to $\hat{u}_J$ (resp. u) by, for all $x \in \mathbb{T}$, $t, s \in [0,T]$,

$$X_J(t,s,x) = x + \int_s^t \hat{u}_J(t, X_J(t,\tau,x))d\tau \qquad X(t,s,x) = x + \int_s^t u(t, X(t,\tau,x))d\tau$$

we get

$$X_J \underset{J\to+\infty}{\longrightarrow} X \quad \text{in } C([0,T]^2 \times \mathbb{T}).$$

As $\hat{c}_J$ is simply transported by $\hat{u}_J$, it is sufficiant to obtain the existence of a $c \in L^\infty(0,T,L^\infty(\mathbb{T}))$, $c \in \{0,1\}$ such that

$$\hat{c}_J \underset{J\to+\infty}{\longrightarrow} c \quad \text{in } L^\infty(0,T,L^2(\mathbb{T})). \tag{39}$$

(See [3], proof of Theorem 4.1, step 1 and 2).

(2) Equations (37.2), (37.3), (37.5) and (37.6) allows to show the existence of $\rho, \theta, m, g \in L^\infty(0,T,L^\infty(\mathbb{T}))$, $\sigma \in L^2(0,T,H^1(\mathbb{T})) \cap L^\infty(0,T,L^2(\mathbb{T}))$ such that

$$\underline{\rho} \le \rho \le \overline{\rho}, \qquad \underline{\theta} \le \theta \le \overline{\theta}, \qquad m \le \overline{\rho} C_5, \qquad \underline{\rho}\underline{\theta} \le g \le \overline{\rho}\overline{\theta} \qquad \text{a.e. on } [0,T]\times\mathbb{T},$$

$$\hat{\rho}_J \underset{J\to+\infty}{\longrightarrow} \rho, \qquad \hat{\theta}_J \underset{J\to+\infty}{\longrightarrow} \theta, \qquad \hat{m}_J \underset{J\to+\infty}{\longrightarrow} m, \qquad \hat{\rho}_J\hat{\theta}_J \underset{J\to+\infty}{\longrightarrow} g \qquad \text{in } L^\infty(0,T,L^\infty(\mathbb{T})) - \star,$$

$$\hat{\sigma}_J \underset{J\to+\infty}{\rightharpoonup} \sigma \text{ in } L^2(0,T,H^1(\mathbb{T})).$$

Moreover, because of the strong convergence (38) and thanks to (36.1), we get

$$\hat{u}_J, \hat{v}_J \underset{J\to+\infty}{\longrightarrow} u \quad \text{in } L^2(0,T,L^2(\mathbb{T})) \tag{40}$$

then

$$\hat{\rho}_J\hat{u}_J \underset{J\to+\infty}{\rightharpoonup} \rho u \quad \text{in } L^2(0,T,L^2(\mathbb{T})).$$

Thus using (36.5), we obtain $m = \rho u$. Remark that from (37.7) we get

$$\hat{u}_J \underset{J\to+\infty}{\rightharpoonup} u \quad \text{in } L^2(0,T,H^1(\mathbb{T})). \tag{41}$$

Finally, (41), (39) and (36.2) give

$$\sigma = \mu(c)\partial_x u - R(c)g.$$

(3) Equations (37.7) and (37.10) are suffisiant to prove moreover the strong convergence

$$\hat{\sigma}_J \underset{J\to+\infty}{\longrightarrow} \sigma \quad \text{in } L^2(0,T,L^2(\mathbb{T})). \tag{42}$$

(See [3], Section 5.1).

(4) Passing to the limit in (33) (using (36.3)), we get

$$\begin{cases} \partial_t c + u\partial_x c = 0 & (43.1)\\ \partial_t \rho + \partial_x(\rho u) = 0 & (43.2)\\ \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x(\mu(c)\partial_x u - R(c)g) & (43.3)\\ \partial_t(\mathfrak{c}_{\mathrm{v}}(c)g) + \partial_x(\mathfrak{c}_{\mathrm{v}}(c)gu) = (\mu(c)\partial_x u - R(c)g)\partial_x u. & (43.4) \end{cases}$$

Note that

$$\begin{aligned}
&\partial_t(\hat\rho_J\hat\theta_J\ln(\hat\rho_J\hat\theta_J))+\partial_x(\hat u_J\hat\rho_J\hat\theta_J\ln(\hat\rho_J\hat\theta_J))\\
&\qquad=(\partial_t(\hat\rho_J\hat\theta_J)+\partial_x(\hat u_J\hat\rho_J\hat\theta_J))(\ln(\hat\rho_J\hat\theta_J)+1)+(\partial_t\hat\rho_J+\hat u_J\partial_x\hat\rho_J)\hat\theta_J\\
&\qquad=\left(\frac{\hat\sigma_J\partial_x\hat u_J}{\hat{\mathfrak{c}}_{\mathrm{v}J}}-\frac{\hat r_J^1}{\hat{\mathfrak{c}}_{\mathrm{v}J}}\right)(\ln(\hat\rho_J\hat\theta_J)+1)-\hat\rho_J(\partial_x\hat u_J)\hat\theta_J\\
&\qquad=\frac{\hat\sigma_J^2}{\hat\mu_J\hat{\mathfrak{c}}_{\mathrm{v}J}}(\ln(\hat\rho_J\hat\theta_J)+1)+\frac{\hat R_J\hat\sigma_J}{\hat\mu_J\hat{\mathfrak{c}}_{\mathrm{v}J}}\hat\rho_J\hat\theta_J\ln(\hat\rho_J\hat\theta_J)+\frac{(\hat\gamma_J-2)\hat\sigma_J}{\hat\mu_J}\hat\rho_J\hat\theta_J\\
&\qquad-\frac{\hat R_J}{\hat\mu_J}(\hat\rho_J\hat\theta_J)^2-\frac{\hat r_J^1}{\hat{\mathfrak{c}}_{\mathrm{v}J}}(\ln(\hat\rho_J\hat\theta_J)+1).
\end{aligned}$$

From (36.3), (37.2) and (37.3), we get

$$\frac{\hat r_J^1}{\hat{\mathfrak{c}}_{\mathrm{v}J}}(\ln(\hat\rho_J\hat\theta_J)+1)\underset{J\to+\infty}{\longrightarrow}0\quad\text{ in }L^1(0,T,L^\infty(\mathbb{T})).$$

Thus [3], proof of Theorem 4.1, step 4, can be perfectly adapted in order to obtain

$$\hat\rho_J\hat\theta_J\underset{J\to+\infty}{\longrightarrow}g\quad\text{ in }L^2(0,T,L^2(\mathbb{T})).\tag{44}$$

Then (42), (44) and (36.2) give

$$\partial_x\hat u_J\underset{J\to+\infty}{\longrightarrow}\partial_x u\quad\text{in }L^2(0,T,L^2(\mathbb{T})).$$

Considering now $\hat\rho_J\ln\hat\rho_J$, some convexity arguments give

$$\hat\rho_J\underset{J\to+\infty}{\longrightarrow}\rho\quad\text{ in }L^2(0,T,L^2(\mathbb{T})).\tag{45}$$

(See [3], proof of Theorem 4.1, step 5). From (44) and (45) we get

$$\hat\theta_J\underset{J\to+\infty}{\longrightarrow}\theta\quad\text{ in }L^2(0,T,L^2(\mathbb{T})).$$

Moreover, $g=\rho\theta$, which concludes the proof.

□

It is possible to carry out a simultaneous convergence from the semi-discrete to the continuous and a transition from the mesoscopic to the macroscopic point of view. For instance, this may be achieved by considering a succession of pure fluid cells + and − with dimensions that vanish as the size decreases. We obtain the following result, combining arguments of [5] and [5]:

Theorem 1.27 (arrow (b) in Figure 1)**.** *For all $J\in\mathbb{N}^*$, let $(x^J_{0,j+1/2},c^J_{0,j},\rho^J_{0,j},\theta^J_{0,j},u^J_{0,j+1/2})_{j=0,\cdots J-1}$ satisfy* H1–H4 *with $\underline{\rho_0},\overline{\rho_0},\underline{\theta_0},\overline{\theta_0}$ and C_0,ε_0,L_0 independant of J. Suppose moreover that there exists some $\alpha_{0,\pm},\rho_{0,\pm},\theta_{0,\pm}\in L^\infty(\mathbb{T})$, $u_0\in H^1(\mathbb{T})$ such that*

$$\hat u_{0,J}\underset{J\to+\infty}{\rightharpoonup}u_0\ \text{ in }H^1(\mathbb{T})$$

and for all $\beta\in C(\mathbb{R}^3)$,

$$\beta(\hat c_{0,J},\hat\rho_{0,J},\hat\theta_{0,J})\underset{J\to+\infty}{\rightharpoonup}\alpha_{0,+}\beta(1,\rho_{0,+},\theta_{0,+})+\alpha_{0,-}\beta(0,\rho_{0,-},\theta_{0,-})\ \text{ in }L^\infty([0,T]\times\mathbb{T})-\star.\tag{46}$$

Then there exists some $\alpha_\pm$, $\rho_\pm$, $\theta_\pm \in L^\infty([0,T]\times\mathbb{T})$, $u \in L^2(0,T,H^1(\mathbb{T}))$ *such that, up to the extraction of a subsequence,*

$$\hat{u}_J \underset{J\to+\infty}{\rightharpoonup} u \text{ in } L^2(0,T,H^1(\mathbb{T})).$$

and for all $\beta \in C(\mathbb{R}^3)$,

$$\beta(\hat{c}_J,\hat{\rho}_J,\hat{\theta}_J) \underset{J\to+\infty}{\rightharpoonup} \alpha_+\beta(1,\rho_+,\theta_+) + \alpha_-\beta(0,\rho_-,\theta_-) \text{ in } L^\infty([0,T]\times\mathbb{T}) - \star. \tag{47}$$

Moreover $(\alpha_\pm,\rho_\pm,\theta_\pm,u)$ *verifies* (4) *with initial conditions* $(\alpha_{0,\pm},\rho_{0,\pm},\theta_{0,\pm},u_0)$.

Sketch of the proof. (1) As in the proof of the Theorem 1.26, (37.1)–(37.9) and (36.1) allow to show the existence of $\rho,\theta,m \in L^\infty(0,T,L^\infty(\mathbb{T}))$, $u,\sigma \in L^2(0,T,H^1(\mathbb{T}))\cap L^\infty(0,T,L^2(\mathbb{T}))$ such that, up to an extraction,

$$\begin{array}{llll} \underline{\rho} \le \rho \le \overline{\rho}, & \underline{\theta} \le \theta \le \overline{\theta}, & m \le \overline{\rho}C_5 & \text{a.e. on } [0,T]\times\mathbb{T}, \\ \hat{\rho}_J \underset{J\to+\infty}{\longrightarrow} \rho, & \hat{\theta}_J \underset{J\to+\infty}{\longrightarrow} \theta, & \hat{m}_J \underset{J\to+\infty}{\longrightarrow} \rho u & \text{in } L^\infty(0,T,L^\infty(\mathbb{T})) - \star, \end{array}$$

$$\hat{u}_J \underset{J\to+\infty}{\longrightarrow} u \quad \text{in } L^2(0,T,L^2(\mathbb{T})), \tag{48}$$

$$\hat{\sigma}_J \underset{J\to+\infty}{\longrightarrow} \sigma \quad \text{in } L^2(0,T,L^2(\mathbb{T})). \tag{49}$$

We easily obtain from (33.3) the momentum equation

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x\sigma.$$

However, we cannot expect strong convergence on $(\hat{c}_J)$. In particular, the expression of σ is not immediately known.

Let $K := [0,1]\times[\underline{\rho},\overline{\rho}]\times[\underline{\theta},\overline{\theta}]$. Then, for $\beta \in C^1(K)$, the function $\beta_J : (t,x) \mapsto \beta(\hat{c}_J(t,x),\hat{\rho}_J(t,x),\hat{\theta}_J(t,x))$ is bounded in $L^\infty(0,T,L^\infty(\mathbb{T}))$, uniformly in J. Thus there exists some $\overline{\beta} \in L^\infty(0,T,L^\infty(\mathbb{T}))$ such that, up to a subsequence,

$$\beta_J \underset{J\to+\infty}{\longrightarrow} \overline{\beta} \quad \text{in } L^\infty(0,T,L^\infty(\mathbb{T})) - \star.$$

Moreover, because $(C^1(K),\|\cdot\|_\infty)$ is separable, we can choose a subsequence valid for each function $\beta \in C^1(K)$. Starting from

$$\hat{\sigma}_J = \mu(\hat{c}_J)\partial_x\hat{u}_J - R(\hat{c}_J)\hat{\rho}_J\hat{\theta}_J + \hat{r}_J^2$$

then dividing by $\mu(\hat{c}_J)$ and passing to the limit, we get

$$\overline{\left(\frac{1}{\mu(c)}\right)}\sigma = \partial_x u - \overline{\left(\frac{R(c)\rho\theta}{\mu(c)}\right)}$$

hence

$$\sigma = \frac{1}{\overline{1/\mu(c)}}\partial_x u - \frac{1}{\overline{1/\mu(c)}}\overline{\left(\frac{R(c)\rho\theta}{\mu(c)}\right)}.$$

It therefore appears that the study of the weak limit of non-linear functions of $\hat{c}_J,\hat{\rho}_J,\hat{\theta}_J$ is sufficient to pass to the limit in equation (33).

(2) Let $\beta \in C^1(K)$. Then,

$$\begin{aligned}\partial_t\beta_J + \partial_x(\hat{u}_J\beta_J) &= (\partial_t\hat{c}_J)\partial_c\beta_J + (\partial_t\hat{\rho}_J)\partial_\rho\beta_J + (\partial_t\hat{\theta}_J)\partial_\theta\beta_J + (\partial_x\hat{u}_J)\beta_J \\ &\quad + (\hat{u}_J\partial_x\hat{c}_J)\partial_c\beta_J + (\hat{u}_J\partial_x\hat{\rho}_J)\partial_\rho\beta_j + (\hat{u}_J\partial_x\hat{\theta}_J)\partial_\theta\beta_J \\ &= -(\hat{\rho}_J\partial_x\hat{u}_J)\partial_\rho\beta_J + \frac{\hat{\sigma}_J\partial_x\hat{u}_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\rho}_J}\partial_\theta\beta_J + (\partial_x\hat{u}_J)\beta_J - \frac{\hat{r}^1_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\rho}_J}\partial_\theta\beta_J. \end{aligned} \tag{50}$$

Thus

$$\begin{aligned}\partial_t\beta_J + \partial_x(\hat{u}_J\beta_J) &= -\hat{\sigma}_J\frac{\hat{\rho}_J\partial_\rho\beta_J}{\hat{\mu}_J} - \frac{\hat{R}_J\hat{\rho}_J^2\hat{\theta}_J\partial_\rho\beta_J}{\hat{\mu}_J} + \hat{\sigma}_J^2\frac{\partial_\theta\beta_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\rho}_J\hat{\mu}_J} + \hat{\sigma}_J\frac{\hat{R}_J\hat{\theta}_J\partial_\theta\beta_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\mu}_J} + \hat{\sigma}_J\frac{\beta_J}{\hat{\mu}_J} \\ &\quad + \frac{\hat{R}_J\hat{\rho}_J\hat{\theta}_J\beta_J}{\hat{\mu}_J} + \hat{r}_J^2\frac{\hat{\rho}_J\partial_\rho\beta_J}{\hat{\mu}_J} - \hat{r}_J^2\hat{\sigma}_J\frac{\partial_\theta\beta_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\rho}_J\hat{\mu}_J} - \hat{r}_J^2\frac{\beta_J}{\hat{\mu}_J} - \frac{\hat{r}^1_J}{\hat{\mathfrak{c}_{\mathrm{v}}}_J\hat{\rho}_J}\partial_\theta\beta_J.\end{aligned}$$

Passing to the limit, using in particular (48), (49), (36.2), (36.3), we get

$$\partial_t\overline{\beta} + \partial_x(u\overline{\beta}) = -\sigma\overline{\frac{\rho\partial_\rho\beta}{\mu(c)}} - \overline{\frac{R(c)\rho^2\theta\partial_\rho\beta}{\mu(c)}} + \sigma^2\overline{\frac{\partial_\theta\beta}{\mathfrak{c}_{\mathrm{v}}(c)\rho\mu(c)}} + \sigma\overline{\frac{R(c)\theta\partial_\theta\beta}{\mathfrak{c}_{\mathrm{v}}(c)\mu(c)}} + \sigma\overline{\frac{\beta}{\mu(c)}} + \overline{\frac{R(c)\rho\theta\beta}{\mu(c)}}. \tag{51}$$

Defining $\nu : [0,T]\times\mathbb{T}\to\mathbb{P}(K), (t,x)\mapsto\nu_{t,x}$ by

$$\text{for all } t\in[0,T],\ \text{for all } x\in\mathbb{T},\ \text{for all } \beta\in C^1(K),\quad \langle\nu_{t,x},\beta\rangle = \overline{\beta(c,\rho,\theta)},$$

we deduce from (51) that ν verifies the kinetic equation

$$\partial_t\nu + \partial_x(u\nu) - \frac{\sigma+R(c)\rho\theta}{\mu(c)}\nu - \partial_\rho\left(\rho\frac{\sigma+R(c)\rho\theta}{\mu(c)}\nu\right) + \partial_\theta\left(\frac{\sigma(\sigma+R(c)\rho\theta)}{\mathfrak{c}_{\mathrm{v}}\, c\mu(c)\rho}\nu\right) = 0. \tag{52}$$

(3) For $J\in\mathbb{N}^*$, we consider $\nu^J : [0,T]\times\mathbb{T}\to\mathbb{P}(K), (t,x)\mapsto\nu^J_{t,x}$ where

$$\text{for all } t\in[0,T],\ \text{for all } x\in\mathbb{T},\ \text{for all } \beta\in C^1(K),\quad \langle\nu^J_{t,x},\beta\rangle = \beta_J = \beta(\hat{c}_J,\hat{\rho}_J,\hat{\theta}_J).$$

Note that (50) and (36.4) give that $\partial_t\beta_J + \hat{u}_J\partial_x\beta_J$ is bounded in $L^\infty(0,T,L^1(\mathbb{T}))$ uniformly in J. By repeating the proof of the Lemma 5.1 of [3], we get

$$\nu^J \underset{J\to+\infty}{\longrightarrow} \nu \text{ in } C(0,T,\mathbb{P}(\mathbb{T}\times K)). \tag{53}$$

But (46) can be rewritten as

$$\text{for all } x\in\mathbb{T},\quad \nu^J_{0,x} \underset{J\to+\infty}{\longrightarrow} \alpha_{0,+}(x)\delta_{(1,\rho_{0,+}(x),\theta_{0,+}(x))} + \alpha_{0,-}(x)\delta_{(0,\rho_{0,-}(x),\theta_{0,-}(x))} \quad \text{in } \mathbb{P}(K). \tag{54}$$

Thus (53) and (54) give

$$\text{for all } x\in\mathbb{T},\quad \nu_{0,x} = \alpha_{0,+}(x)\delta_{(1,\rho_{0,+}(x),\theta_{0,+}(x))} + \alpha_{0,-}(x)\delta_{(0,\rho_{0,-}(x),\theta_{0,-}(x))}. \tag{55}$$

(4) In [3] the authors prove that (52) with initial condition (55) has a unique solution (see Section 5.3), which is (4) with initial conditions $(\alpha_{0,\pm},\rho_{0,\pm},\theta_{0,\pm},u_0)$ (see Section 5.4). This concludes the proof.

□

2. Mesoscopic and macroscopic schemes

2.1. **Mescoscopic discretization**

Discretizing in time the semi-discrete system (7), we obtain a numerical scheme associate with the mesoscopic system of Navier-Stokes with temperature and without conductivity (1). The aim of this study is to numerically achieve both the convergence from the discrete to the continuous and the homogenization simultaneously when $J \to +\infty$.

Let $J \in \mathbb{N}^*$. Consider a mesh with J moving cells whose positions depend on the number of the iterate n. The extremities of the cells are denoted by $x^n_{1/2}, \cdots, x^n_{J-1/2}$. Initially, starting from a volume fraction α_0 (the ratio of fluid + in the macroscopic mixture), with a finite number of discontinuities, we construct (c^0_j) and $(x^0_{j+1/2})$ such that

- H'1 $0 \leq x^0_{1/2} < x^0_{3/2} < \cdots < x^0_{J-1/2} < 1$,
- H'2 there exists some $L_0 > 0$ independant of J such that $\max_{0\leq j\leq J-1} \Delta x^0_{j+1/2} \leq L_0 \min_{0\leq j\leq J-1} \Delta x^0_{j+1/2}$,
- H'3 for each point of discontinuity $\tilde{x} \in \mathbb{T}$ of α_0, there exists some $\tilde{j} \in \{0, \cdots J-1\}$ such that $\tilde{x} = x_{\tilde{j}+1/2}$,
- H'4 for each $j \in \{0, \cdots J-1\}$, $c^0_j \in \{0, 1\}$.
- H'5 for all $j \in \{0, \cdots J-1\}$, $c^0_{2j} + c^0_{2j+1} = \displaystyle\int_{x^0_{2j-1/2}}^{x^0_{2j+3/2}} \alpha_0$.

For $j \in \{0, \cdots J-1\}$, we will discretize initial density ρ^0_j, initial temperature θ^0_j, and initial velocity u^0_j by

$$\rho^0_j = \rho_{0,+}(x^0_j)c^0_j + \rho_{0,-}(x^0_j)(1-c^0_j), \qquad \theta^0_j = \theta_{0,+}(x^0_j)c^0_j + \theta_{0,-}(x^0_j)(1-c^0_j), \qquad u^0_{j+1/2} = u_0(x^0_{j+1/2}),$$

where

$$x^0_j = \frac{x^0_{j-1/2} + x^0_{j+1/2}}{2}.$$

and $\rho_{0,\pm}, \theta_{0,\pm}$ are supposed to be continuous at the points x^0_j, and $u_0 \in H^1(\mathbb{T})$.

Once $(u^n_{j+1/2})$, $(x^n_{j+1/2})$, (ρ^n_j) and (θ^n_j) are known at a time step n, we can define, as for the semi-discrete system, (Δu^n_j), (Δx^n_j), $(\Delta x^n_{j+1/2})$ and $(\rho^n_{j+1/2})$.

We choose the same splitting as in [5] (in the barotropic case). At each step, we begin by computing the velocity, then the new positions of cells, the density and the temperature.

More precisely, the scheme can be written, for all $j \in \{0, \cdots J-1\}$ and $n \in \mathbb{N}$,

$$\begin{cases} c^{n+1}_j = c^n_j, & (56.1)\\ \rho^n_{j+1/2}\Delta x^n_{j+1/2}u^{n+1}_{j+1/2} = \rho^n_{j+1/2}\Delta x^n_{j+1/2}u^n_{j+1/2} - \Delta t^n(p(c^n_{j+1},\rho^n_{j+1},\theta^n_{j+1}) - p(c^n_j,\rho^n_j,\theta^n_j)) & \\ \qquad + \Delta t^n\left(\mu(c^n_{j+1})\dfrac{u^{n+1}_{j+3/2} - u^{n+1}_{j+1/2}}{\Delta x^n_{j+1}} - \mu(c^n_j)\dfrac{u^{n+1}_{j+1/2} - u^{n+1}_{j-1/2}}{\Delta x^n_j}\right), & (56.2)\\ x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t^n u^{n+1}_{j+1/2}, & (56.3)\\ \rho^{n+1}_j \Delta x^{n+1}_j = \rho^n_j \Delta x^n_j, & (56.4)\\ \rho^n_j \Delta x^n_j \theta^{n+1}_j = \rho^n_j \Delta x^n_j \theta^n_j + \Delta t^n\left(\mu(c^n_j)\dfrac{\Delta u^{n+1}_j}{\Delta x^n_j} - p(c^n_j,\rho^n_j,\theta^n_j)\right)\dfrac{\Delta u^{n+1}_j + \Delta u^n_j}{2\,\mathfrak{c}_{\mathrm{v}}(c^n_j)} & (56.5) \end{cases}$$

with $\Delta t^n > 0$ a time step.

Remark 2.1. Proving conservation of the order of positions, i.e.

$$\text{for all } n, \quad 0 \leq x^n_{1/2} < x^n_{3/2} < \cdots < x^n_{J-1/2} < 1 \quad \text{(within one index rotation)}$$

seems difficult. However this condition is essential in order to avoid $\Delta x^n_j \leq 0$. But this does not happen in our numerical simulations, if the Courant-Friedrich-Levy condition

$$\|u^n\|_\infty \Delta t^n \leq \min_j \Delta x^n_{j+1/2}$$

is enforced for each n. It is important to note that, in the event that $x^n_{1/2}$ becomes negative during the computations, the value is to be increased by 1 in order to remain on the torus.

Remark 2.2. It is not clear at first glance that $(u^{n+1}_{j+1/2})$ is well-defined by (56.2) (the definition being implicit). However, this equation cas be rewritten in a matrix form as

$$(\mathbb{M} + \Delta t^n (\mathbb{I} - \mathbb{S})^T \mathbb{D}_n^{-1} (\mathbb{I} - \mathbb{S})) U^{n+1} = \mathbb{M} U^n + \Delta t^n (\mathbb{I} - \mathbb{S})^T P^n \tag{57}$$

where $\mathbb{M} = \text{diag}(\rho^n_{j+1/2} \Delta x^n_{j+1/2})$, $\mathbb{D}_n = \text{diag}(\Delta x^n_j / \mu(c^n_j))$, $U^n = (u^n_j)$, $P^n = (p(c^n_j, \rho^n_j, \theta^n_j) =: p^n_j)$ and $\mathbb{S}$ is the permutation matrix associated with the cycle $(1234 \cdots J)$.

Then, the left matrix $\mathbb{A}_n = \mathbb{M} + \mu \Delta t^n (\mathbb{I} - \mathbb{S})^T \mathbb{D}_n^{-1} (\mathbb{I} - \mathbb{S})$ is indeed invertible as a sum of a positive definite symmetric matrix $\mathbb{M}$ and a positive semi definite symmetric matrix.

Remark 2.3. As the scheme is only linearly implicit, it is very practical to use. Just like the barotropic scheme ([5]), tested in many cases, its stability has never been called into question. However, to obtain a priori estimates as in the semi-discrete system seems out of reach for the time being. The main issue comes from the explicit definition of the pressure: note that in [15], a proof of the convergence of a scheme (with static cells) for the barotropic case in Navier Stokes is given, but this scheme is totally implicit.

Remark 2.4. The scheme (56) is conservative in the sense that, defining E^n_j by

$$\rho^n_{j+1/2} \Delta x^n_{j+1/2} E^n_{j+1/2} = \frac{\rho^n_{j+1/2} \Delta x^n_{j+1/2} (u^n_{j+1/2})^2}{2} + \frac{\rho^n_j \Delta x^n_j \theta^n_j}{2} + \frac{\rho^n_{j+1} \Delta x^n_{j+1} \theta^n_{j+1}}{2},$$

we get

$$\rho^n_{j+1/2} \Delta x^n_{j+1/2} E^{n+1}_{j+1/2} = \rho^n_{j+1/2} \Delta x^n_{j+1/2} E^n_{j+1/2} + \Delta t^n (\sigma^{n+1/2}_{j+1} u^{n+1/2}_{j+1} - \sigma^{n+1/2}_j u^{n+1/2}_j)$$

where

$$\sigma^{n+1/2}_j = \mu(c^n_j) \frac{\Delta u^{n+1}_j}{\Delta x^n_j} - p(c^n_j, \rho^n_j, \theta^n_j).$$

2.2. **Macroscopic discretization**

Imitating the mesoscopic discretization, we obtain a discretization of (4). We choose initially $(x^0_{j+1/2})$ verifying H'1-3, then for all $j \in \{0, \cdots J-1\}$,

$$\alpha^0_j = \alpha_0(x^0_j), \qquad \rho^0_{\pm,j} = \rho_\pm(x^0_j), \qquad \theta^0_{\pm,j} = \theta_\pm(x^0_j), \qquad u^0_{j+1/2} = u_0(x^0_{j+1/2}),$$

where

$$x^0_j = \frac{x^0_{j-1/2} + x^0_{j+1/2}}{2}.$$

Once $(u^n_{j+1/2})$, $(x^n_{j+1/2})$, (α^n_j), $(\rho^n_{\pm,j})$, $(\theta^n_{\pm,j})$ are known at a time step n, we define for $j \in \{0, \cdots J-1\}$,

$$\rho^n_j = \alpha^n_j \rho^n_{+,j} + (1 - \alpha^n_j) \rho^n_{-,j}, \qquad \theta^n_j = \alpha^n_j \theta^n_{+,j} + (1 - \alpha^n_j) \theta^n_{-,j},$$

$$\mu^n_j = \frac{\mu_+ \mu_-}{\alpha^n_j \mu_- + (1 - \alpha^n_j) \mu_+}, \qquad p^n_j = \frac{\alpha_+ p_+(\rho^n_{+,j}, \theta^n_{+,j}) \mu_- + \alpha_- p_-(\rho^n_{-,j}, \theta^n_{-,j})}{\alpha^n_j \mu_- + (1 - \alpha^n_j) \mu_+}.$$

Moreover, once $(u^{n+1}_{j+1/2})$ is known, we define for $j \in \{0, \cdots J-1\}$

$$\sigma^{n+1/2}_{\pm,j} = \mu_\pm \frac{\Delta u^{n+1}_j}{\Delta x^n_j} - p_\pm(\rho^n_{\pm,j}, \theta^n_{\pm,j}), \qquad \sigma^{n+1/2}_j = \mu^n_j \frac{\Delta u^{n+1}_j}{\Delta x^n_j} - p^n_j.$$

The scheme writes

$$\begin{cases}
\rho^n_{j+1/2}\Delta x^n_{j+1/2} u^{n+1}_{j+1/2} = \rho^n_{j+1/2}\Delta x^n_{j+1/2} u^n_{j+1/2} - \Delta t^n (p^n_{j+1} - p^n_j) \\
\qquad\qquad + \Delta t^n \left(\mu^n_{j+1} \dfrac{u^{n+1}_{j+3/2} - u^{n+1}_{j+1/2}}{\Delta x^n_{j+1}} - \mu^n_j \dfrac{u^{n+1}_{j+1/2} - u^{n+1}_{j-1/2}}{\Delta x^n_j} \right), & (58.1) \\
x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t^n u^{n+1}_{j+1/2}, & (58.2) \\
\alpha^{n+1}_j = \alpha^n_j + \Delta t^n \dfrac{\alpha^n_j(1-\alpha^n_j)}{\alpha^n_j \mu_- + (1-\alpha^n_j)\mu_+} (\sigma^{n+1/2}_{-,j} - \sigma^{n+1/2}_{+,j}), & (58.3) \\
\rho^{n+1}_{+,j}\Delta x^{n+1}_j = \rho^n_{+,j}\Delta x^n_j + \Delta t^n \rho^n_{+,j} \dfrac{1-\alpha^n_j}{\alpha^n_j \mu_- + (1-\alpha^n_j)\mu_+} (\sigma^{n+1/2}_{+,j} - \sigma^{n+1/2}_{-,j}) \Delta x^n_j, & (58.4) \\
\rho^{n+1}_{-,j}\Delta x^{n+1}_j = \rho^n_{-,j}\Delta x^n_j + \Delta t^n \rho^n_{-,j} \dfrac{\alpha^n_j}{\alpha^n_j \mu_- + (1-\alpha^n_j)\mu_+} (\sigma^{n+1/2}_{-,j} - \sigma^{n+1/2}_{+,j}) \Delta x^n_j, & (58.5) \\
\rho^n_{+,j}\Delta x^n_j \theta^{n+1}_{+,j} = \rho^n_{+,j}\Delta x^n_j \theta^n_{+,j} + \Delta t^n \dfrac{(1-\alpha^n_j)/\mathfrak{c}_{v+}}{\alpha^n_j \mu_- + (1-\alpha^n_j)\mu_+} \sigma^{n+1/2}_j (\sigma^{n+1/2}_{-,j} - \sigma^{n+1/2}_{+,j}) \Delta x^n_j \\
\qquad\qquad + \Delta t^n \sigma^{n+1/2}_j \dfrac{\Delta u^{n+1}_j + \Delta u^n_j}{2\,\mathfrak{c}_{v+}}, & (58.6) \\
\rho^n_{-,j}\Delta x^n_j \theta^{n+1}_{-,j} = \rho^n_{-,j}\Delta x^n_j \theta^n_{-,j} + \Delta t^n \dfrac{\alpha^n_j/\mathfrak{c}_{v-}}{\alpha^n_j \mu_- + (1-\alpha^n_j)\mu_+} \sigma^{n+1/2}_j (\sigma^{n+1/2}_{+,j} - \sigma^{n+1/2}_{-,j}) \Delta x^n_j \\
\qquad\qquad + \Delta t^n \sigma^{n+1/2}_j \dfrac{\Delta u^{n+1}_j + \Delta u^n_j}{2\,\mathfrak{c}_{v-}}. & (58.7)
\end{cases}$$

Remark 2.5. The discretization formulation may seem a little complicated and detached from the continuous equations. Nevertheless, it allows to consider zones of pure fluid (i.e. α can take the value 0 or 1). In the case $0 < \alpha < 1$, the equation (58.4) for example can be rewritten in the form

$$\alpha^{n+1}_j \rho^{n+1}_{+,j} \Delta x^{n+1}_j = \alpha^n_j \rho^n_{+,j} \Delta x^n_j.$$

which is the natural way of discretizing (4.2).

The schemes (56) and (58) are used in [3] in order to obtain some numerical illustrations, which we reproduce here (Figure 4). We choose as initial condition, for $x \in \mathbb{T}$:

$$\rho_{0,+}(x), \theta_{0,+}(x) = \begin{cases} 2 & \text{if } 0.25 \le x \le 0.75 \\ 0.2 & \text{else} \end{cases}, \qquad \rho_{0,-}(x), \theta_{0,-}(x) = \begin{cases} 1 & \text{if } 0.25 \le x \le 0.75 \\ 0.2 & \text{else} \end{cases},$$

$$c_0(x) = \begin{cases} 1 & \text{if } \lfloor Jx \rfloor \text{ is even or } 0.25 \le x \le 0.75 \\ 0 & \text{else} \end{cases}, \qquad \alpha_{0,+}(x) = \begin{cases} 1/2 & \text{if } 0.25 \le x \le 0.75 \\ 1 & \text{else} \end{cases},$$

$$u_0(x) = 0, \qquad \rho_0 = c_0\rho_{0,+} + (1-c_0)\rho_{0,-}, \qquad \theta_0 = c_0\theta_{0,+} + (1-c_0)\theta_{0,-}, \qquad \alpha_{0,-} = 1 - \alpha_{0,+}.$$

We consider moreover $\mu_+ = 0.1$, $\mu_- = 0.2$, $\gamma_+ = 2$, $\gamma_- = 3$, $\mathfrak{c}_{v\pm} = 1$, $t = 0.1$. We take $J = 100$ for the visualization of ρ, θ, p in order to clearly observe the oscillations. We take $J = 1000$ for the vizualisation of

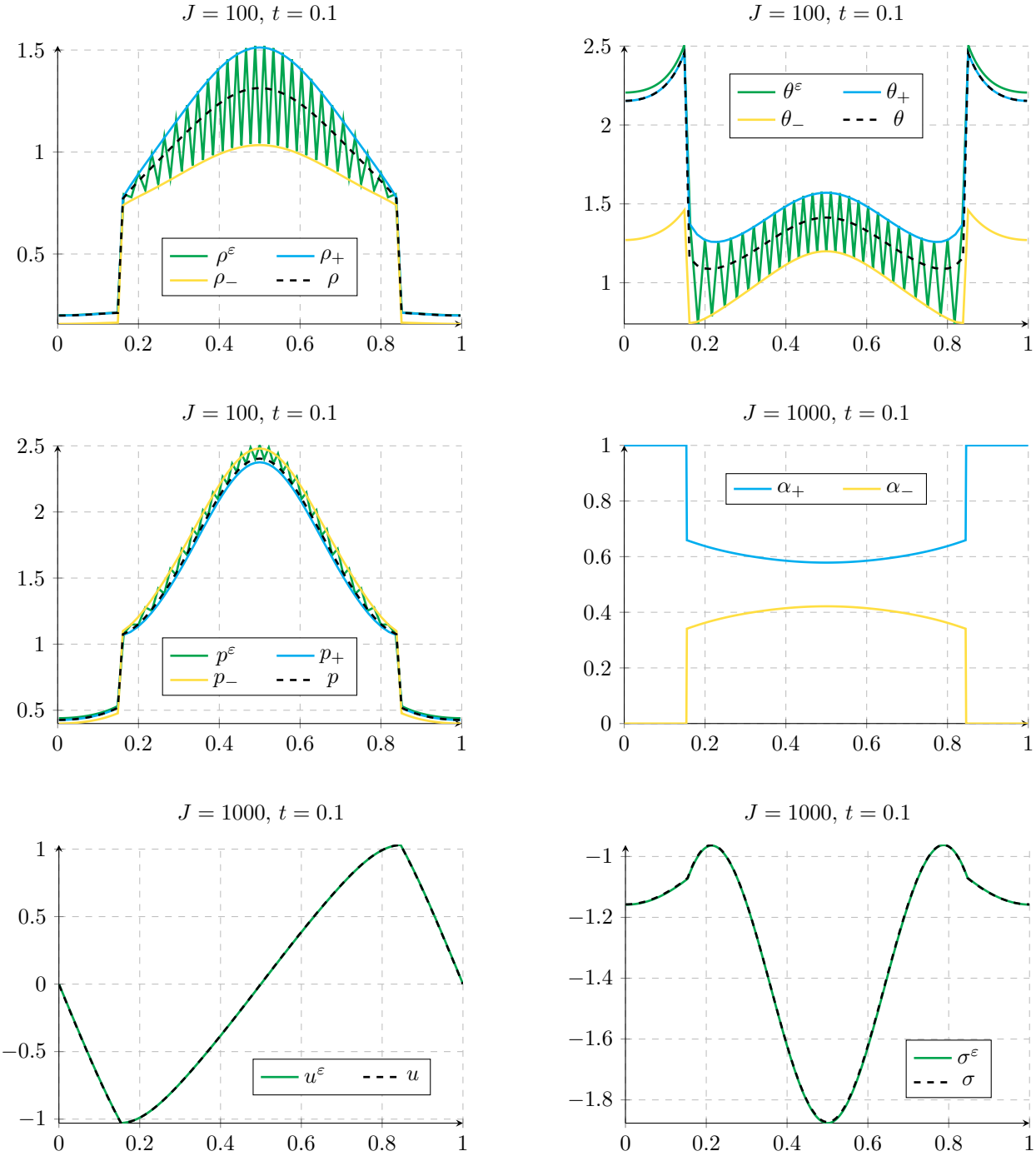
α, u, σ in order to see the discontinuities. Solutions marked with an epsilon are mesoscopic solutions, the others are macroscopic solutions.

Remark 2.6. We consider here a mixture with a pure zone initially (note that $c_0 = 0$ on $[0, 0.25[\cup]0.75, 1[$). This pure zone is transported but conserved through time (see $\alpha_\pm$).

Remark 2.7. The quantity ρ^ε (resp. θ^ε, p^ε) oscillates between ρ_- (resp. θ_-, p_-) and ρ_+ (resp. θ_+, p_+). Indeed no strong convergence is expected, but a binary behaviour (described by (47)). The weak limit ρ (resp. θ, p) can be interpreted as a weighted average.

Remark 2.8. The strong convergence of both u^ε and σ^ε is observed. It is noteworthy that these functions are in $H^1(\mathbb{T})$ in space, but not beyond this (refer to the singular points surrounding $x \simeq 0.15$, $x \simeq 0.85$).

FIGURE 4. Convergence of $\rho^\varepsilon, \theta^\varepsilon, p^\varepsilon, u^\varepsilon, \sigma^\varepsilon$



3. Discretization with conductivity

In all our discussions, we have dealt with the homogeneization of the Navier-Stokes equations for ideal gas, without heat conductivity. A priori estimates for Navier Stokes with conductivity are very different (see [16] in the continuous case, and [8] for a semi-discrete approach). Furthermore, the authors exclusively address a substantial conductivity constant. It appears that homogenisation with conductivity is more challenging than in the absence of conductivity. The only known result [12] deals with small time and data with few oscillation. However, the scheme (56) can easily be modified in order to consider heat-conductivity. In this subsection, we give some numerical illustration of stability when the heat conductivity tends to 0, and homogenization with low conductivity.

If it seems natural to define conductivity inside the meshes as the conductivity of the present fluid, the definition of $\kappa_{j+1/2}$ is not that clear. As $\kappa\partial_x\theta$ is expected to be continuous, we are looking for some $\theta_{j+1/2}$, $\kappa_{j+1/2}$ verifying

$$\kappa(c_j^n)\frac{\theta_{j+1/2}^n-\theta_j^n}{\Delta x_j^n/2}=\kappa(c_{j+1}^n)\frac{\theta_{j+1}^n-\theta_{j+1/2}^n}{\Delta x_{j+1}^n/2}=\kappa_{j+1/2}^n\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x_{j+1/2}^n}$$

which gives

$$\kappa_{j+1/2}^n:=\frac{2\Delta x_{j+1/2}^n}{\Delta x_{j+1}^n/\kappa(c_j^n)+\Delta x_j^n/\kappa(c_{j+1}^n)}.$$

Thus, in order to obtain some discretization of the homogenization of Navier-Stokes with conductivity, we just have to replace (56.5) with

$$\begin{aligned}\rho_j^n\Delta x_j^n\theta_j^{n+1}=\rho_j^n\Delta x_j^n\theta_j^n+\Delta t^n\left(\mu(c_j^n)\frac{\Delta u_j^{n+1}}{\Delta x_j^n}-p(c_j^n,\rho_j^n,\theta_j^n)\right)\frac{\Delta u_j^{n+1}+\Delta u_j^n}{2\,\mathfrak{c}_{\mathrm{v}}(c_j^n)}\\+\Delta t^n\frac{2\Delta x_j^n/\,\mathfrak{c}_{\mathrm{v}}(c_j^n)}{\Delta x_{j-1/2}^n+\Delta x_{j+1/2}^n}\left(\kappa_{j+1/2}^{n+1}\frac{\theta_{j+1}^{n+1}-\theta_j^{n+1}}{\Delta x_{j+1/2}^{n+1}}-\kappa_{j-1/2}^{n+1}\frac{\theta_j^{n+1}-\theta_{j-1}^{n+1}}{\Delta x_{j-1/2}^{n+1}}\right),\end{aligned}$$

which is once again only linearly implicit.

In the following we denote by $\theta(t)$ an interpolation of $(\theta_j^n)_j$ for n such that $\sum_{k=0}^{n} \Delta t^k = t$. Choosing a pure fluid with for initial conditions

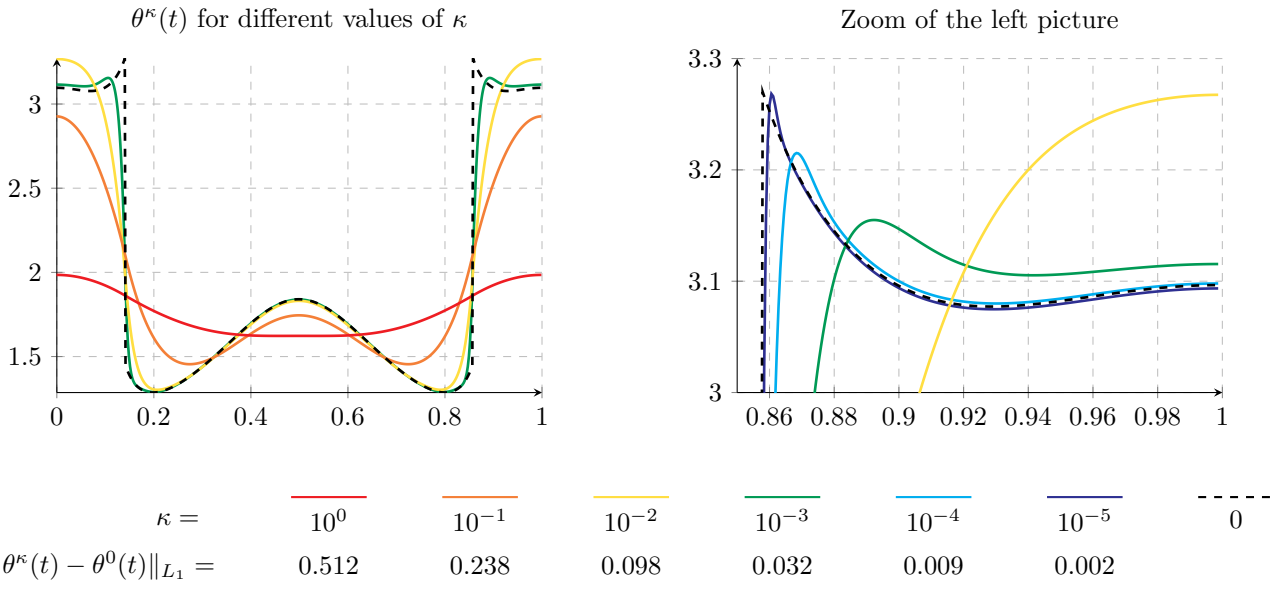
$$\text{for all } x \in \mathbb{T}, \quad \rho_0(x), \theta_0(x) = \begin{cases} 2 & \text{if } 0.25 \leq x \leq 0.75 \\ 0.125 & \text{else} \end{cases},$$

$$\text{for all } x \in \mathbb{T}, \qquad u_0(x) = 0$$

and $\mu = 0.1$, $\gamma = 2$, $\mathfrak{c}_{\mathrm{v}} = 1$, we get for $J = 1000$ cells, at time $t = 0.1$, the Figure 5.

We can then conjecture the convergence in the $L^\infty(0, T, L^1(\mathbb{T}))$-space, which will be the subject of a future paper.

FIGURE 5. Behaviour of θ^κ when $\kappa \to 0$ (case of a pure fluid)



Considering now a mesoscopic mixture of two fluids $\pm$, with initial conditions

$$\text{for all } x \in \mathbb{T}, \quad \rho_{0,\pm}(x), \theta_{0,+}(x) = \begin{cases} 2 & \text{if } 0.25 \leq x \leq 0.75, \\ 0.125 & \text{else} \end{cases}, \quad \theta_{0,-}(x) = \begin{cases} 1 & \text{if } 0.25 \leq x \leq 0.75 \\ 0.125 & \text{else} \end{cases},$$

$$\text{for all } x \in \mathbb{T}, \quad u_0(x) = 0, \quad c_0(x) = \begin{cases} 1 & \text{if } \lfloor Jx \rfloor \text{ is even} \\ 0 & \text{else} \end{cases}$$

and $\mu_\pm = 0.1$, $\gamma_+ = 2$, $\gamma_- = 3$, $\mathfrak{c}_{\mathrm{v}\pm} = 1$, we get for $J = 100$ cells, at time $t = 0.1$, the Figure 6. Once again, it seems that for each $t > 0$, $\theta^\kappa(t) \underset{\kappa \to 0}{\longrightarrow} \theta(t)$ in $L^1(\mathbb{T})$.

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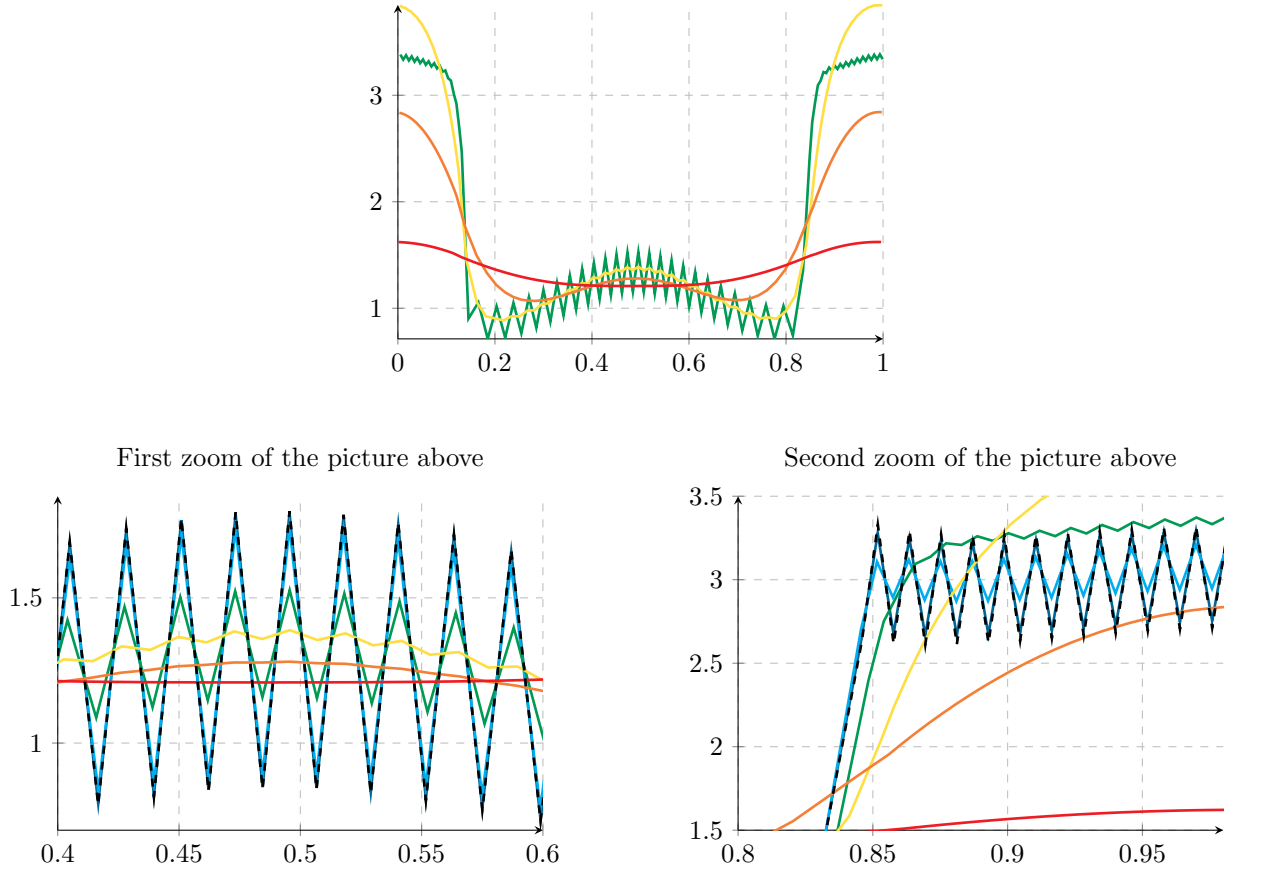
Figure 6. Behaviour of θ^κ when $\kappa \to 0$ (case of a mixture)
$\theta^\kappa(t)$ for different values of κ



$\kappa_+ =$	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	0
$\kappa_- =$	2×10^0	2×10^{-1}	2×10^{-2}	2×10^{-3}	2×10^{-4}	2×10^{-5}	0
$\|\theta^\kappa(t) - \theta^0(t)\|_{L^1} =$	0.631	0.339	0.333	0.200	0.049	0.008	

A. Proof of the Lemma 1.24.

Proof of (36.1). Let's fix $t \in [0,T]$ and $j \in \{0, J-1\}$. Remark that

$$\text{for all } x \in [x_j(t), x_{j+1/2}(t)[, \quad \hat{u}_J(t,x) = u_j(t) + \frac{\Delta u_j}{\Delta x_j}(t)(x - x_j(t))$$

$$\text{for all } x \in [x_{j+1/2}(t), x_{j+1}(t)[, \quad \hat{u}_J(t,x) = u_{j+1}(t) + \frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t)(x - x_{j+1}(t))$$

$$\begin{aligned}\text{for all } x \in [x_j(t), x_{j+1}(t)[, \quad \hat{v}_J(t,x) &= u_j(t) + \frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t)(x - x_j(t)) \\ &= u_{j+1}(t) + \frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t)(x_{j+1}(t) - x)\end{aligned}$$

thus

$$\begin{aligned}\int_{x_j(t)}^{x_{j+1}(t)} |\hat{u}_J - \hat{v}_J|^2(t,x)dx = &\left(\frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t) - \frac{\Delta u_j}{\Delta x_j}(t)\right)^2 \int_{x_j(t)}^{x_{j+1/2}(t)} (x - x_j(s))^2 ds \\ &+ \left(\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t)\right)^2 \int_{x_{j+1/2}(t)}^{x_{j+1}(t)} (x_{j+1}(s) - x)^2 ds.\end{aligned} \tag{59}$$

From (7.9), we get

$$\frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t) = \frac{\Delta x_j(t)}{2\Delta x_{j+1/2}(t)}\frac{\Delta u_j}{\Delta x_j}(t) + \frac{\Delta x_{j+1}(t)}{2\Delta x_{j+1/2}(t)}\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t)$$

thus

$$\frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t) - \frac{\Delta u_j}{\Delta x_j}(t) = \frac{\Delta x_{j+1}(t)}{2\Delta x_{j+1/2}(t)}\left(\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_j}{\Delta x_j}(t)\right), \tag{60}$$

$$\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_{j+1/2}}{\Delta x_{j+1/2}}(t) = \frac{\Delta x_j(t)}{2\Delta x_{j+1/2}(t)}\left(\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_j}{\Delta x_j}(t)\right). \tag{61}$$

Moreover by Young's inequality

$$\sqrt{\Delta x_j(t)\Delta x_{j+1}(t)} \leq \Delta x_{j+1/2}(t). \tag{62}$$

Thus (60), (61), (62) and (10.4) in (59) gives

$$\begin{aligned}&\int_{x_j(t)}^{x_{j+1}(t)} |\hat{u}_J - \hat{v}_J|^2(t,x)dx \\ &= \left(\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_j}{\Delta x_j}(t)\right)^2 \left(\frac{|\Delta x_{j+1}(t)|^2}{4|\Delta x_{j+1/2}(t)|^2}\frac{(x_{j+1/2}(t) - x_j(t))^3}{3} + \frac{|\Delta x_j(t)|^2}{4|\Delta x_{j+1/2}(t)|^2}\frac{(x_{j+1}(t) - x_{j+1/2}(t))^3}{3}\right) \\ &= \frac{1}{48}\left(\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t) - \frac{\Delta u_j}{\Delta x_j}(t)\right)^2 \frac{|\Delta x_j(t)|^2|\Delta x_{j+1}(t)|^2}{\Delta x_{j+1/2}(t)} \\ &\leq \frac{1}{24}\left(\left|\frac{\Delta u_{j+1}}{\Delta x_{j+1}}(t)\right|^2 + \left|\frac{\Delta u_j}{\Delta x_j}(t)\right|^2\right)\frac{|\Delta x_j(t)|^2|\Delta x_{j+1}(t)|^2}{\Delta x_{j+1/2}(t)} \leq \frac{L^2}{24J^2}\left(\frac{(\Delta u_{j+1})^2}{\Delta x_{j+1}}(t) + \frac{(\Delta u_j)^2}{\Delta x_j}(t)\right).\end{aligned}$$

Hence, summing in j, we get by periodicity and using (10.7)

$$\int_0^1 |\hat{u}_J - \hat{v}_J|^2(t,x)dx = \sum_{j=0}^{J-1}\int_{x_j(t)}^{x_{j+1}(t)} |\hat{u}_J - \hat{v}_J|^2(t,x)dx \leq \frac{L^2}{12J^2}\sum_{j=0}^{J-1}\frac{(\Delta u_j)^2}{\Delta x_j}(t) \leq \frac{L^2}{12J^2}C_3.$$

□

Proof of (36.2). Remark that, for all $(t,x) \in [0,T] \times \mathbb{T}$,

$$\hat{\sigma}_J(t,x) = \sigma_j(t) + (x - x_j(t)) \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[,$$

$$\hat{\sigma}_J(t,x) = \sigma_{j+1}(t) - (x_{j+1}(t) - x) \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}(t) \text{ if } x \in [x_j(t), x_{j+1}(t)[.$$

thus, for all $(t,x) \in [0,T] \times \mathbb{T}$,

$$\begin{aligned} \hat{r}_J^2(t,x) &= \hat{\sigma}_J(t,x) - \mu(\hat{c}_J)\partial_x \hat{u}_J(t,x) + R(\hat{c}_J)\hat{\rho}_J\hat{\theta}_J(t,x) \\ &= \begin{cases} (x - x_j(t) \dfrac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}(t) & \text{if } x \in [x_j(t), x_{j+1/2}(t)[\\ -(x_{j+1}(t) - x) \dfrac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}(t) & \text{if } x \in [x_{j+1/2}(t), x_{j+1}(t)[. \end{cases} \end{aligned}$$

Then we get from (10.7) and the inequality

$$\text{for all } a, b \geq 0, \quad a^3 + b^3 \leq (a+b)^3,$$

$$\begin{aligned} &\int_0^T \int_0^1 |\hat{r}_J^2|^2 \\ &= \int_0^T \sum_{j=0}^{J-1} \int_{x_j}^{x_{j+1}} \left| \frac{x_{j+1} - x}{\Delta x_{j+1/2}} \sigma_j + \frac{x - x_j}{\Delta x_{j+1/2}} \sigma_{j+1} - \sigma_j \mathbb{1}_{[x_j, x_{j+1/2}[} - \sigma_{j+1} \mathbb{1}_{[x_{j+1/2}, x_{j+1}[} \right|^2 \\ &= \int_0^T \sum_{j=0}^{J-1} \int_{x_j}^{x_{j+1/2}} \left| \frac{x - x_j}{\Delta x_{j+1/2}} \right|^2 |\Delta\sigma_{j+1/2}|^2 + \int_{x_{j+1/2}}^{x_{j+1}} \left| \frac{x_{j+1} - x}{\Delta x_{j+1/2}} \right|^2 |\Delta\sigma_{j+1/2}|^2 \\ &= \int_0^T \sum_{j=0}^{J-1} \frac{(x_{j+1/2} - x_j)^3 + (x_{j+1} - x_{j+1/2})^3}{3} \left| \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}} \right|^2 \\ &\leq \int_0^T \sum_{j=0}^{J-1} \frac{(\Delta x_{j+1/2})^3}{3} \left| \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}} \right|^2 \leq \frac{L^2}{3J^2} C_1. \end{aligned}$$

□

Proof of (36.3). From (10.4), Young's inequality then (10.7), (10.5), we get

$$\begin{aligned} \int_0^T \int_0^1 |\hat{r}_J^1| &\leq \sum_{j=0}^{J-1} \int_0^T \int_{x_j}^{x_{j+1}} (x - x_j) \left| \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}} \right| \left| \frac{\Delta u_j}{\Delta x_j} \right| = \frac{1}{2} \sum_{j=0}^{J-1} \int_0^T |\Delta x_{j+1/2}|^2 \left| \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}} \right| \left| \frac{\Delta u_j}{\Delta x_j} \right| \\ &\leq \frac{L^{3/2}}{2J} \sum_{j=0}^{J-1} \int_0^T \sqrt{\Delta x_{j+1/2}} \left| \frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}} \right| \sqrt{\Delta x_j} \left| \frac{\Delta u_j}{\Delta x_j} \right| \\ &\leq \frac{L^{3/2}}{4J} \left(\sum_{j=0}^{J-1} \int_0^T \frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}} + \sum_{j=0}^{J-1} \int_0^T \frac{(\Delta u_j)^2}{\Delta x_j} \right) \leq \frac{L^{3/2}}{4J} (C_1 + TC_3). \end{aligned}$$

□

Proof of (36.4). Similarly, from (10.4), Hölder's inequality then (10.7), (10.5), we get on $[0,T]$

$$\begin{aligned}\int_0^1 |\hat{r}_J^1| &\le \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}} (x-x_j)\left|\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}\right|\left|\frac{\Delta u_j}{\Delta x_j}\right| \\ &= \frac{1}{2}\sum_{j=0}^{J-1}\Delta x_{j+1/2}|\Delta\sigma_{j+1/2}|\left|\frac{\Delta u_j}{\Delta x_j}\right| \le \frac{L}{2}\sum_{j=0}^{J-1}\left(\sqrt{\Delta x_j}|\sigma_j| + \sqrt{\Delta x_{j+1}}|\sigma_{j+1}|\right)\Delta x_j\left|\frac{\Delta u_j}{\Delta x_j}\right| \\ &\le \frac{L}{2}\left(2\sum_{j=0}^{J-1}\Delta x_j|\sigma_j|^2 + \Delta x_{j+1}|\sigma_{j+1}|^2\right)^{1/2}\left(\sum_{j=0}^{J-1}\Delta x_j\left|\frac{\Delta u_j}{\Delta x_j}\right|^2\right)^{1/2} \le LC_1{}^{1/2}C_3{}^{1/2}.\end{aligned}$$

□

Proof of (36.5). Let $\varphi \in C_c^1(]0,1[)$. Then, on $]0,T[$,

$$\begin{aligned}&\left|\int_0^1 \hat{m}_J\varphi - \int_0^1 \hat{\rho}_J\hat{u}_J\varphi\right| \\ &\le \sum_{j=0}^{J-1}\left|\int_{x_j}^{x_{j+1/2}}(\rho_{j+1/2}u_{j+1/2} - \rho_j\hat{u}_J)\varphi + \int_{x_{j+1/2}}^{x_{j+1}}(\rho_{j+1/2}u_{j+1/2} - \rho_{j+1}\hat{u}_J)\varphi\right| \\ &\le \sum_{j=0}^{J-1}\left|\int_{x_j}^{x_{j+1/2}}(\rho_{j+1/2}-\rho_j)u_{j+1/2}\varphi + \rho_j(u_{j+1/2}-\hat{u}_J)\varphi\right. \\ &\left.+\int_{x_{j+1/2}}^{x_{j+1}}(\rho_{j+1/2}-\rho_{j+1})u_{j+1/2}\varphi + \rho_{j+1}(u_{j+1/2}-\hat{u}_J)\varphi\right| \\ &\le \sum_{j=0}^{J-1}\left|(\rho_{j+1/2}-\rho_j)u_{j+1/2}\int_{x_j}^{x_{j+1/2}}\varphi - (\rho_{j+1}-\rho_{j+1/2})u_{j+1/2}\int_{x_{j+1/2}}^{x_{j+1}}\varphi\right| \\ &+ \|\varphi\|_\infty\sum_{j=0}^{J-1}\left(\rho_j\int_{x_j}^{x_{j+1/2}}|u_{j+1/2}-\hat{u}_J| + \rho_{j+1}\int_{x_{j+1/2}}^{x_{j+1}}|u_{j+1/2}-\hat{u}_J|\right) =: I_1 + I_2.\end{aligned}$$

First, remark that for $j \in \{0,\cdots,J-1\}$,

$$\begin{aligned}&\rho_j\int_{x_j}^{x_{j+1/2}}|u_{j+1/2}-\hat{u}_J| + \rho_{j+1}\int_{x_{j+1/2}}^{x_{j+1}}|u_{j+1/2}-\hat{u}_J| \\ &= \rho_j\int_{x_j}^{x_{j+1/2}}(x_{j+1/2}-x)\left|\frac{\Delta u_j}{\Delta x_j}\right| + \rho_{j+1}\int_{x_{j+1/2}}^{x_{j+1}}(x-x_{j+1/2})\left|\frac{\Delta u_{j+1}}{\Delta x_{j+1}}\right| \\ &= \frac{\rho_j\Delta x_j}{8}|\Delta u_j| + \frac{\rho_{j+1}\Delta x_{j+1}}{8}|\Delta u_{j+1}|,\end{aligned}$$

thus from periodicity then (10.4)

$$I_2 \le \frac{\|\varphi\|_\infty}{4}\sum_{j=0}^{J-1}\rho_j\Delta x_j|\Delta u_j| \le \frac{\overline{\rho}LC_3{}^{1/2}}{4J}\|\varphi\|_\infty.$$

Moreover, for $j \in \{0, \cdots, J-1\}$, we obtain doing some integration by parts

$$\int_{x_j}^{x_{j+1/2}} \varphi = \frac{\Delta x_j}{2}\varphi(x_{j+1/2}) - \int_{x_j}^{x_{j+1/2}} (x - x_j)\varphi'(x)dx,$$

$$\int_{x_{j+1/2}}^{x_{j+1}} \varphi = \frac{\Delta x_{j+1}}{2}\varphi(x_{j+1/2}) + \int_{x_{j+1/2}}^{x_{j+1}} (x_{j+1} - x)\varphi'(x)dx$$

thus, from (10.9) and the inequalities

$$|\rho_{j+1/2} - \rho_j| \le \overline{\rho}, \quad |\rho_{j+1} - \rho_{j+1/2}| \le \overline{\rho},$$

$$\begin{aligned} I_1 \le \frac{1}{2}\sum_{j=0}^{J-1} \left|(\rho_{j+1/2} - \rho_j)\Delta x_j - (\rho_{j+1} - \rho_{j+1/2})\Delta x_{j+1}\right| |\varphi(x_{j+1/2})u_{j+1/2}| \\ + \overline{\rho}C_5 \sum_{j=0}^{J-1} \left(\left| \int_{x_j}^{x_{j+1/2}} (x - x_j)\varphi'(x)dx \right| + \left| \int_{x_{j+1/2}}^{x_{j+1}} (x_{j+1} - x)\varphi'(x)dx \right| \right). \end{aligned}$$

Remark that the first term in this last equation is zero because $2\rho_{j+1/2}\Delta x_{j+1/2} = \rho_j \Delta x_j + \rho_{j+1}\Delta x_{j+1}$. Finally, we get using (10.4)

$$\begin{aligned} I_1 &\le \overline{\rho}C_5\|\varphi'\|_\infty \sum_{j=0}^{J-1} \frac{|\Delta x_j|^2}{8} + \frac{|\Delta x_{j+1}|^2}{8} \\ &= \frac{\overline{\rho}C_5}{4}\|\varphi'\|_\infty \sum_{j=0}^{J-1} \Delta x_j^2 \le \frac{\overline{\rho}C_5 L}{4J}\|\varphi'\|_\infty. \end{aligned}$$

□

B. Proof of the Proposition 1.25.

Proof of (37.1)–(37.5). The result is immediate by the definition of $\hat{c}_J, \hat{\rho}_J, \hat{\theta}_J, \hat{u}_J, \hat{m}_J$ and using the bounds (10.1)–(10.3) and (10.9). Note that, for $j \in \{0, \cdots J-1\}$ and $t \in [0,T]$, $x \in [x_{j-1/2}(t), x_{j+1/2}(t)[$, we get

$$\hat{u}_J(t,x) = \frac{x_{j+1/2}(t) - x}{\Delta x_j(t)} u_{j-1/2}(t) + \frac{x - x_{j-1/2}(t)}{\Delta x_j(t)} u_{j+1/2}(t)$$

thus from (10.9),

$$\begin{aligned} |\hat{u}_J(t,x)| &\le \frac{x_{j+1/2}(t) - x}{\Delta x_j(t)} |u_{j-1/2}(t)| + \frac{x - x_{j-1/2}(t)}{\Delta x_j(t)} |u_{j+1/2}(t)| \\ &\le \frac{x_{j+1/2}(t) - x}{\Delta x_j(t)} C_5 + \frac{x - x_{j-1/2}(t)}{\Delta x_j(t)} C_5 = C_5. \end{aligned}$$

□

Proof of (37.6). Remark that if some function f is linear on each $[x_j, x_{j+1}[$, $j \in \{0, \cdots J-1\}$, then denoting $f_j := f(x_j)$ for each $j \in \{0, \cdots J-1\}$ we have

$$\int_0^1 f = \frac{1}{3}\sum_{j=0}^{J-1}(f_j^2 + f_{j+1}^2 + f_j f_{j+1})\Delta x_{j+1/2}. \tag{63}$$

Thus we get, on $[0,T]$, from Young's inequality, periodicity and (10.4),

$$\begin{aligned}\int_0^1 \hat\sigma_J^2 &= \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\left(\sigma_j + (x-x_j)\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}\right)^2 \mathbb{1}_{[x_j,x_{j+1}[} = \frac{1}{3}\sum_{j=0}^{J-1}(\sigma_j^2+\sigma_{j+1}^2+\sigma_j\sigma_{j+1})\Delta x_{j+1/2} \qquad (64)\\ &\le \frac{1}{2}\sum_{j=0}^{J-1}(\sigma_j^2+\sigma_{j+1}^2)\Delta x_{j+1/2} = \frac{1}{2}\sum_{j=0}^{J-1}\sigma_j^2(\Delta x_{j-1/2}+\Delta x_{j+1/2}) \le L\sum_{j=0}^{J-1}\sigma_j^2\Delta x_j.\end{aligned}$$

Moreover,

$$\int_0^T\int_0^1(\partial_x\hat\sigma_J)^2 = \sum_{j=0}^{J-1}\int_0^T\int_{x_j}^{x_{j+1}}\left|\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}\right|^2 = \sum_{j=0}^{J-1}\int_0^T\frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}}$$

thus (as $L \ge 1$), we obtain using (10.5)

$$\int_0^1\hat\sigma_J^2 + \int_0^T\int_0^1(\partial_x\hat\sigma_J)^2 \le LC_1.$$

□

Proof of (37.7)–(37.8). The result is obvious, noting that, for all $j \in \{0,\cdots J-1\}$, $t \in [0,T]$ and $x \in]x_{j-1/2}(t), x_{j+1/2}(t)[$,

$$\partial_x\hat u_J(t,x) = \frac{\Delta u_j(t)}{\Delta x_j(t)},$$

then using (10.7) and (10.8). □

Proof of (37.9). Straightforward computations using (34) and (7.5) give, on $[0,T]\times\mathbb{T}$,

$$\partial_t\hat u_J + \hat u_J\partial_x\hat u_J = \sum_{j=0}^{J-1}\left(u'_{j-1/2} + (x-x_{j-1/2})\frac{\Delta u'_j}{\Delta x_j}\right)\mathbb{1}_{[x_{j-1/2},x_{j+1/2}[}.$$

Hence, taking the square and integrating in time and in space, using (63), periodicity, Young's inequality then (10.10)

$$\begin{aligned}&\int_0^T\int_0^1(\partial_t\hat u_J + \hat u_J\partial_x\hat u_J)^2 = \int_0^T\sum_{j=0}^{J-1}\int_{x_{j-1/2}}^{x_{j+1/2}}\left(u'_{j-1/2}+(x-x_{j-1/2})\frac{\Delta u'_j}{\Delta x_j}\right)^2\\ &= \frac{1}{3}\int_0^T\sum_{j=0}^{J-1}\Delta x_j\left(|u'_{j-1/2}|^2+|u'_{j+1/2}|^2+u_{j-1/2}u_{j+1/2}\right) \le \frac{1}{2}\int_0^T\sum_{j=0}^{J-1}\Delta x_j(|u'_{j-1/2}|^2+|u'_{j+1/2}|^2)\\ &\le \frac{1}{2}\int_0^T\sum_{j=0}^{J-1}(\Delta x_{j+1}+\Delta x_j)|u'_{j+1/2}|^2 \le \int_0^T\sum_{j=0}^{J-1}\Delta x_{j+1/2}|u'_{j+1/2}|^2 \le C_6.\end{aligned}$$

□

Proof of (37.10). As in previous proof, from (35) and (7.5), we get, on $[0,T]\times\mathbb{T}$,

$$\begin{aligned}\partial_t\hat\sigma_J+\hat v_J\partial_x\hat\sigma_J&=\sum_{j=0}^{J-1}\partial_t\left(\sigma_j+(x-x_j)\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}\right)\mathbb{1}_{[x_j,x_{j+1}[}+\hat v_J\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}\mathbb{1}_{[x_j,x_{j+1}[}\\&=\sum_{j=0}^{J-1}\left(\sigma_j'-u_j\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}+(x-x_j)\frac{\Delta\sigma'_{j+1/2}}{\Delta x_{j+1/2}}-(x-x_j)\frac{\Delta\sigma_{j+1/2}\Delta u_{j+1/2}}{(\Delta x_{j+1/2})^2}\right)\mathbb{1}_{[x_j,x_{j+1}[}\\&+\left(u_j\frac{\Delta\sigma_{j+1/2}}{\Delta x_{j+1/2}}+(x-x_j)\frac{\Delta\sigma_{j+1/2}\Delta u_{j+1/2}}{(\Delta x_{j+1/2})^2}\right)\mathbb{1}_{[x_j,x_{j+1}[}\\&=\sum_{j=0}^{J-1}\left(\sigma_j'+(x-x_j)\frac{\Delta\sigma'_{j+1/2}}{\Delta x_{j+1/2}}\right)\mathbb{1}_{[x_j,x_{j+1}[}.\end{aligned}$$

The same computations as for (64) give

$$\int_0^1(\partial_t\hat\sigma_J+\hat v_J\partial_x\hat\sigma_J)^2\le L\sum_{j=0}^{J-1}(\sigma_j')^2\Delta x_j.$$

Multiplying by $t\mapsto\min(1,t)$, integrating on $[0,T]$ then adding for $t\in[0,T]$

$$\min(1,t)\int_0^1(\partial_x\hat\sigma_J)^2=\min(1,t)\sum_{j=0}^{J-1}\frac{(\Delta\sigma_{j+1/2})^2}{\Delta x_{j+1/2}},$$

we get (37.10) from (10.11) and $L\ge1$. □

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