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The Stochastic Vortex Simulation of an Unsteady Viscous Flow in a Multiconnected Domain

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Abstract

Vortex methods applied to flows in multiconnected domains are studied. The objective of the work is to construct a vortex method which eliminates the possibility of non-physical solutions in the pressure fields calculated a posteriori from the velocity and vorticity fields. Application is made in the case of an annular domain. It is shown that certain additional constrains concerning the production of vorticity on the boundary have to be imposed.

1 Introduction

Vortex methods are powerful tools for the numerical analysis of unsteady viscous flows of incompressible fluid. The key idea of these methods is to simulate directly the advection-diffusion process of vorticity transport and vorticity generation due to boundary conditions imposed on a velocity field. In principle, vortex methods are grid-free, which makes them particularly recommended for flows with higher Reynolds numbers.

In 1973 Chorin published his classical paper [4] where the idea of a stochastic approach to fluid dynamics was formulated. Since then many researchers have followed this idea and have contributed to further developments. Significant progress in both theoretical and computational aspects of vortex methods has been achieved. Rigorous mathematical results concerning convergence (under certain restrictions) have been obtained [1, 7]. The computational efficiency of the methods has been substantially improved by introducing fast algorithms for vortex interaction [3, 5, 6, 11]. Certain generalizations for three-dimensional flows have been recently proposed (see [2, 9, 10, 14]).

The mathematical basis for vortex methods is the equation of the vorticity transport (Helmholtz equation) accompanied by the continuity equation along with initial-boundary conditions for the velocity field. Two features of this formulation are of great importance. First, the pressure field is eliminated. Secondly, there is no explicit boundary conditions for the vorticity. In the random vortex method, one attempts to find a solution as an ensemble average of a number of realisations of a stochastic velocity field generated by a large assemblage of small vortex particles (or vortex blobs). In order to satisfy the boundary conditions, new particles are continuously created giving rise to the flux of vorticity through boundaries of the flow domain.

Although the pressure field does not appear explicitly, it can be calculated a posteriori from the velocity and vorticity fields. However, a serious difficulty appears when the domain of the flow is multiply connected: there exist solutions for the Helmholtz equation giving physically meaningless, multivalued pressure fields. Thus the question is how to construct a vortex method which eliminates the possibility of non-physical solutions.

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In this study we show how it can be done in the case of an annular domain. It turns out that certain additional constrains concerning the production of vorticity on the boundary have to be imposed.

2 Formulation of the problem

We consider a nonstationary motion of a viscous fluid in the annular domain between a pair of coaxial circular contours (Fig. 1). The contours rotate with angular velocities η_1 and η_2 which can depend on time. The dynamics is governed by the following set of equations (written in the polar coordinates):

- Helmholtz equation $\partial_t \omega + u_r \partial_r \omega + \frac{1}{r} u_\theta \partial_\theta \omega = \nu(\partial_{rr}\omega + \frac{1}{r}\partial_r \omega + \frac{1}{r^2}\partial_{\theta\theta}\omega)$
- Definition of vorticity $\omega = \frac{1}{r}(r\partial_r u_\theta + u_\theta - \partial_\theta u_r)$
- Continuity equation $\partial_r u_r + \frac{1}{r}u_r + \frac{1}{r}\partial_\theta u_\theta = 0$

The boundary conditions are:

$$u_\theta(t, R_1, \theta) = \eta_1(t) R_1 \equiv U_1, \quad u_r(t, R_1, \theta) = 0$$
$$u_\theta(t, R_2, \theta) = \eta_2(t) R_2 \equiv U_2, \quad u_r(t, R_2, \theta) = 0$$

The initial conditions are:

$$\mathbf{V}(t=0) \equiv 0 \text{ (no motion)}, \quad \eta_1(t=0) = 0, \quad \eta_2(t=0) = 0$$

The numerical method is a modification of the stochastic vortex method in the version proposed by Styczek [13].

3 The computational method

The velocity field $\mathbf{V}$ can be expressed as the sum of four components:

$$\mathbf{V} = V_W + V_O + V_A + V_G \tag{1}$$

where the meaning of the components is as follows:

V_W = the velocity induced by new vortex blobs created on the boundary
V_O = the velocity induced by previously created vortex blobs which are still in the flow
V_A = an additional potential velocity field, such that $\oint_S V_A \cdot ds = 0$ for any contour S
V_G = the velocity induced by a point vortex located in the center of the boundary circles; the role of this vortex and how to calculate its circulation Γ_G will be explained later.

The vorticity field is approximated by vortex blobs. If γ, ρ and (x_0, y_0) denote respectively the charge of vorticity (circulation), the radius of the core and the position of the center of a blob, then the velocity induced at any point (x, y) is given by

$$u = -\frac{\gamma}{2\pi} \cdot \frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}, \quad v = \frac{\gamma}{2\pi} \cdot \frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}$$

when $(x-x_0)^2 + (y-y_0)^2 > \rho^2$, and by

$$u = -\frac{\gamma}{2\pi\rho^2}(y - y_0), \quad v = \frac{\gamma}{2\pi\rho^2}(x - x_0)$$

otherwise.

Since the parameters of all "old" blobs are known, the velocity V_O can be calculated. In order to gain reasonable efficiency with increasing number of interacting vortex blobs one has to resort to some fast summation algorithm.

To determine the components V_W and V_G, the circulation of new blobs $(\gamma_1, \ldots, \gamma_M)$ and the circulation of the central vortex Γ_G should be found. However this problem cannot be solved separately from the evaluation of the potential component V_A. An elegant method to overcome this difficulty was proposed by Styczek [13].

The decomposition of Eq. 1, written on the boundaries, yields for ∂D_1:

$$\begin{aligned} U_1 &= (V_W)_\theta + (V_O)_\theta + (V_A)_\theta + (V_G)_\theta \\ 0 &= (V_W)_r + (V_O)_r + (V_A)_r \end{aligned} \tag{2a}$$

and for ∂D_2:

$$\begin{aligned} U_2 &= (V_W)_\theta + (V_O)_\theta + (V_A)_\theta + (V_G)_\theta \\ 0 &= (V_W)_r + (V_O)_r + (V_A)_r \end{aligned} \tag{2b}$$

Since the velocity V_A has no circulation, its normal and tangential components on the boundary are related by a linear operator L. The definition of L is strictly connected with the Neumann boundary problem for a harmonic function in a considered flow domain—we will return to this later. As yet we can write formally

$$(V_A)_\theta = L(V_A)_r \tag{3}$$

From the boundary conditions we have that $(V_A)_r = -(V_O)_r - (V_W)_r$. Then the application of the operator L and substitution in Eq. 2 results in

$$U_i = (V_W)_\theta + (V_O)_\theta + (V_G)_\theta - L(V_W)_r - L(V_O)_r, \quad i = 1, 2 \tag{4}$$

If we assume that the velocity V_W is induced by a singular layer of vorticity, then Eq. 4 is an integral equation with a singular kernel of Cauchy type. The unknown is a linear density of circulation of the vortex layer. In numerical calculations this singular structure is approximated by M vortex blobs, each adjacent to a boundary segment $[s_j, s_{j+1}]$, $j = 1, \ldots, M$. Equation 4 is solved in the mean integral sense on each segment, i.e.

$$\int_{s_j}^{s_{j+1}} \left[(V_W)_\theta + (V_O)_\theta + (V_G)_\theta - L(V_W)_r - L(V_O)_r\right](s)\, ds = 0, \quad j = 1, \ldots, M \tag{5}$$

Here s denotes the arclenght parametrization of the boundary and $s_{M+1} = s_1$.

This method has two important features. First, the integration of terms like $L(V)_r$ gives

$$\int_{s_j}^{s_{j+1}} L(V)_r\, ds = \Phi(s_{j+1}) - \Phi(s_j) \equiv \Delta\Phi_j \tag{6}$$

i.e. the increment along the segment $[s_j, s_{j+1}]$ of a harmonic function Φ, satisfying the Neumann boundary condition $\partial_r \Phi = V_r$. Thus we do not have to calculate $L(V)_r$, we need only to know the boundary value of Φ. The second advantage is that Stokes' theorem is satisfied exactly by the velocity field in each time step (see Appendix).

The new vortex blobs are born always at fixed positions on the boundary and their cores have fixed radii. Thus we can introduce two functions, $T_i(s)$ and $N_i(s)$, describing a distribution of, respectively, the θ and r components of the velocity induced on the boundary by the vortex blob adjacent to the segment $[s_i, s_{i+1}]$. Then the velocity V_W on the boundary can be expressed as

$$(V_W)_r(s) = \sum_{i=1}^{M} \gamma_i N_i(s), \quad (V_W)_\theta(s) = \sum_{i=1}^{M} \gamma_i N_i(s) \tag{7}$$

After substitution of Eq. 7 into Eq. 5 we obtain

$$\sum_{j=1}^{M} \left[\int_{s_i}^{s_{i+1}} (T_j - LN_j)(s)\, ds \right] \gamma_j = \int_{s_i}^{s_{i+1}} \left[-(V_O)_\theta - (V_G)_\theta + L(V_O)_r + U_k \right](s)\, ds, \quad i = 1, \ldots, M \tag{8}$$

where subscript k is equal 1 or 2 when the segment $[s_i, s_{i+1}]$ belongs to ∂D_1 or ∂D_2, respectively.

Equation 8 can be written in a slightly different form, i.e.

$$\sum_{j=1}^{M} \left[\int_{s_i}^{s_{i+1}} T_j(s)\, ds - \Delta\Phi_{i,j} \right] \gamma_j = - \int_{s_i}^{s_{i+1}} (V_O)_\theta(s)\, ds - \frac{s_{i+1} - s_i}{2\pi R_k} \Gamma_G + \Delta\Phi_{i,O} + U_k(s_{i+1} - s_i), \quad i = 1, \ldots, M \tag{9}$$

The increments of the boundary values of the harmonic potentials have appeared (see Eq. 6).

The matrix of the linear algebraic system (9) depends only on the geometric characteristics of the flow domain and the vortex blobs generated on the boundary. Hence it is fixed and can be calculated and factorized in advance. In opposition, the right-hand-side terms have to be evaluated in each time step. However the system can not be solved yet since the right-hand side of (9) contains an unknown circulation Γ_G. Thus, to make the problem solvable, we need an additional equation.

Now we consider the pressure field. In principle it can be reconstructed from the velocity and vorticity fields. Because of low regularity of the vorticity representation (a piecewise constant function of space variables) a weak formulation is used. The solution is then obtained by the finite element method (see for example [8]).

The solution procedure is usually more complicated when the flow domain is multiply connected. The general difficulty is related to the existence of velocity fields which satisfy the Helmholtz and continuity equations but produce multivalued pressure fields. Thus a numerical method should be designed, which filters out all non-physical "solutions".

If the pressure is a univalued function of space variables then the integral of the gradient ∇P along any closed contour is equal to zero. Introducing the following function,

$$F = P + \frac{1}{2} V^2 \tag{10}$$

where V denotes the absolute value of the velocity, one can derive the equation

$$\frac{d}{ds} F = -\frac{\partial}{\partial t} v_s + \omega v_n - \nu \frac{d}{dn} \omega \tag{11}$$

Here s and n denote, respectively, the tangential and normal directions. Integration of Eq. 11 along a closed contour C yields

$$-\frac{d}{dt} \oint_C v_s\, ds + \oint_C \omega v_n\, ds - \nu \oint_C \frac{d}{dn} \omega\, ds = 0 \tag{12}$$

since for a univalued pressure field

$$[F] \equiv \oint_C \frac{d}{ds} F \, ds = 0 \tag{13}$$

The condition (12) should be fulfilled for any contour located inside the domain. It is satisfied automatically for contours not surrounding the inner circle ∂D_1. Actually the necessary and sufficient conditions for the pressure correctness are (see [8] and [12])

$$-\frac{d}{dt} \oint_{\partial D_k} v_s \, ds + \oint_{\partial D_k} \omega v_n \, ds - \nu \oint_{\partial D_k} \frac{d}{dn} \omega \, ds = 0, \quad k = 1, 2 \tag{14}$$

or, taking into account the boundary conditions,

$$-\nu \oint_{\partial D_k} \frac{d}{dn} \omega \, ds = R_k \frac{d}{dt} \Omega_k, \quad k = 1, 2 \tag{15}$$

Equation 15 has obvious physical meaning—it says that the flux of vorticity through the contour ∂D_k is equal to the rate of change of the circulation of the velocity field. In particular, when Ω_k is fixed then this flux is zero.

The vortex blobs produce piecewise constant distribution of vorticity and the normal derivative in the left hand side of (15) does not exist. Instead of trying to apply (15) directly, we consider the balance of vorticity production during one time step. Since in this method the velocity and vorticity satisfy Stokes' theorem exactly, it is sufficient to impose the condition (15) only on one component of the boundary, say ∂D_2.

In the stochastic vortex method the boundary conditions for the velocity are fulfilled (in the mean integral sense defined earlier) at the beginning of each time step, i.e. before new blobs leave the boundary. From Eq. 9, one concludes that:

$$\oint_{\partial D_2} \left[(V_O)_\theta^{n-1} + (V_W)_\theta^n \right] \, ds = \Gamma_2^{n-1} - \Gamma_G^{n-1} \tag{16}$$

where $\Gamma_2^{n-1} = 2\pi R_2 U_2(t_{n-1})$. Subsequently all vortex blobs are moved to new positions according to the Ito differential equations (written here in Cartesian coordinates):

$$\begin{aligned} dx(t) &= v_x\big(t, x(t), y(t)\big) dt + \sqrt{2\nu} \, dW_x \\ dy(t) &= v_y\big(t, x(t), y(t)\big) dt + \sqrt{2\nu} \, dW_y \end{aligned} \tag{17}$$

where W_x and W_y are independent Wiener processes. As a result the boundary conditions become violated and

$$\oint_{\partial D_2} (V_O)_\theta^n \, ds \neq \Gamma_2^{n-1} - \Gamma_G^{n-1}$$

The change of circulation on ∂D_2 occurs for the following reasons:

A. vortex blobs which are in contact with the boundary ∂D_2 change their positions;

B. some blobs jump randomly outside ∂D_2 and are eliminated; and,

C. some blobs jump randomly inside ∂D_1 and are eliminated.

The vorticity flux through ∂D_2 is due to events A and B. This flux should be compensated by vorticity creation in the beginning of the next step so that total production of vorticity on ∂D_2 during one time step is zero. It results in the following equation:

$$\Gamma_G^{n-1} + \oint_{\partial D_2} (V_O)_\theta^n \, ds - \Gamma_2^{n-1} + \Gamma_{\mathrm{OUT},1}^{n-1} + \Omega_{W,2}^n = 0 \tag{18}$$

The symbol $\Gamma_{\mathrm{OUT},1}^{n-1}$ denotes the sum of circulations of the vortex blobs that penetrated the interior of ∂D_1 and were eliminated in the previous time step, while $\Omega_{W,2}^n$ is the contribution of new blobs to the vorticity of the flow.

It is important to point out that the contribution of the blob to the vorticity field is not identical to the blob's circulation. When a vortex blob is located sufficiently close to the boundary then a part of its vortex core is outside the flow domain and the contribution to the vorticity field differs from the blob's circulation. However, explicit use of $\Omega_{W,2}^n$ is not very convenient. The accurate calculation of this quantity is a strenuous geometric problem, particularly for more general shapes of the boundary. Fortunately, Eq. 18 can be reformulated.

If we add the equations of system (9) which concern the boundary ∂D_2, the result is as follows:

$$\sum_{(1)} \gamma_j^n + \Omega_{W,2}^n = - \oint_{\partial D_2} (V_O)_\theta^n \, ds - \Gamma_G^n + \Gamma_2^n \tag{19}$$

From Eqs. 18 and 19 we conclude that

$$\Gamma_G^n - \Gamma_G^{n-1} = \Gamma_2^n - \Gamma_2^{n-1} + \Gamma_{\mathrm{OUT},1}^{n-1} - \sum_{(1)} \gamma_j^n \tag{20}$$

Equation 20 supplements the system (9) and contains only easily computable components. Thus in each time step we solve (9) and (20) in order to determine the circulation of the new blobs and the new value of the circulation Γ_G. Next the components of the total velocity V_W and V_G are calculated in the centers of all vortex blobs.

The potential velocity field V_A can be determined from its boundary distribution using the Cauchy integral. However, in the case considered here, the method of Fourier expansion is more convenient. This means that we seek a complex potential function Φ_A in the form of the following series:

$$\Phi_A = \sum_{n=-\infty}^{n=\infty} w_n z^n = \sum_{n=-\infty}^{n=\infty} w_n r^n e^{in\theta} \tag{21}$$

where the coefficients w_n are to be determined. The components of V_A are obtained by differentiation, i.e.

$$\begin{aligned} (V_A)_r &= \frac{\partial \Phi_A}{\partial r} = \sum_{n=-\infty}^{n=\infty} \frac{n}{2} (r^{n-1} w_n - r^{-(n+1)} w_{-n}^*) e^{in\theta} \\ (V_A)_\theta &= \frac{1}{r} \frac{\partial \Phi_A}{\partial \theta} = i \sum_{n=-\infty}^{n=\infty} \frac{n}{2} (r^{n-1} w_n + r^{-(n+1)} w_{-n}^*) e^{in\theta} \end{aligned} \tag{22}$$

The known distributions of $(V_A)_r$ on ∂D_1 and ∂D_2 can be expanded in the Fourier series:

$$(V_A)_r(R_k, \theta) = \sum_{n=-\infty}^{n=\infty} C_n^k e^{in\theta}, \quad k = 1, 2 \tag{23}$$

Comparing (23) with the expansion (22) of $(V_A)_r$ written for R_k, $k = 1, 2$, one obtains the system of the following algebraic problems:

$$\begin{aligned} \frac{n}{2}(R_1^{n-1} w_n - R_1^{-(n+1)} w_{-n}^*) = C_n^1 \\ \frac{n}{2}(R_2^{n-1} w_n - R_2^{-(n+1)} w_{-n}^*) = C_n^2 \end{aligned} \tag{24}$$

Obviously, it is sufficient to solve this system for $n > 0$ only, since $C_n^k = (C_n^k)^*$ (the boundary data are real functions) and

$$\oint_{\partial D_k} (V_A)_r(\theta) R_k \, d\theta = 0, \quad k = 1, 2 \tag{25}$$

Once the coefficients are calculated the velocity can be determined from (22). To improve efficiency, an interpolating grid and a FFT algorithm are applied.

The most time consuming part of the calculations during one time step is the determination of the induced velocity. To make the simulation realisable in reasonable time, we applied the Greengard-Rokhlin method with self-adapting clustering.

4 Results of the computations

The numerical calculations were carried out for the flow induced by the steady rotation of the inner circle while the outer circle was at rest. The radii of the circles were $R_1 = 1$ and $R_2 = 2$. The velocity of the inner boundary was $U_1 = \pi/4$. The number of vortices generated on each circle was 96. The Euler scheme was used to integrate the stochastic differential equations (17). Written in polar coordinates, the scheme is given by the following formulas:

$$\begin{aligned} r(t_{n+1}) &= r(t_n) + [V_r\big(t_n, r(t_n), \theta(t_n)\big) + \nu/r(t_n)]\Delta t + \sqrt{2\nu\Delta t}\, N_r \\ \theta(t_{n+1}) &= \theta(t_n) + [V_\theta\big(t_n, r(t_n), \theta(t_n)\big)\Delta t + \sqrt{2\nu\Delta t}\, N_\theta]/r(t_n) \end{aligned} \tag{26}$$

where

$$N_r = N_x \cos\theta(t_n) + N_y \sin\theta(t_n), \quad N_\theta = N_y \cos\theta(t_n) - N_x \sin\theta(t_n)$$

The symbols N_x and N_y denote a pair of statistically independent random variables with normalised Gaussian distribution. The time step Δt was equal to 0.05. The kinematic viscosity was such that the Reynolds number $Re = U_1(R_2 - R_1)/\nu$ was equal to 10 000.

The number of time steps was sufficient to obtain quasi-stationary pattern of the flow. This number was typically about one thousand. Then, the subsequent two hundred steps were accomplished to calculate the average velocity.

In Fig. 2, an instantaneous velocity field for the last time step is shown. The average velocity pattern is presented in Fig. 3. The flux of the flow is plotted versus time in Fig. 4. A principally oscillating behaviour with small randomly fluctuating component is clearly visible. We found that the time-averaged flux obtained in the calculations is smaller than the flux corresponding to the laminar, stationary solution. A typical instantaneous vorticity distribution is shown in Fig. 5. The pattern is a very irregular mixture of regions filled with intensive negative and positive vorticity. We found that peak, instantaneous values of vorticity were even 30 times greater that the averaged intensity.

An instantaneous pressure distribution was also calculated in the flow domain using the algorithm developed by Nowakowski and Rokicki [12]. The result is shown in Fig. 6. The resolution of fine details is poor because of the rather coarse finite element grid used in these calculations. However, the pressure field is physically meaningful—only negligibly small jump appears in the pressure distribution (not seen in the figure). Since the pressure correctness condition is based on the balance of the vorticity production during a time step, one can expect that the pressure jump will diminish for smaller time steps.

We found an additional effect of the pressure correctness condition—it stabilises the numerical process. When this condition is switched off then, after relatively small number of steps, the velocity attains locally very large values and most of the vortex blobs are rapidly swept from the flow domain. The pressure condition controls the generation of the vorticity near the boundaries and eliminates the phenomenon of accumulation of vorticity in separated clusters inducing the velocity of unrealistic magnitude.

5 Concluding remarks

The main goal of this study was the construction of the random vortex method applicable for unsteady flows in an annular domain. The treatment of boundary conditions is based on the relation between tangential and normal components of the potential part of the velocity field. The no-slip condition is satisfied in a mean integral sense on the system of boundary segments.

Since the domain is multiply connected, special attention was given to the pressure problem. A mechanism for controlling the production of vorticity has been incorporated. It consists in precise calculations of the vorticity flux through the boundary during each time step. This flux is exactly balanced by the vorticity production in the next step. As a result we obtain the velocity field which produces physically sensible pressures.

The results presented are of preliminary character. The spatial resolution of the flow patterns could be much improved by introducing smaller vortex blobs and a finer grid for the finite element solution of the pressure calculations. Nevertheless, these results show that the implementation of the pressure correctness condition works.

The method presented in this paper can be generalised to arbitrary multiply connected domains. The case of an external flow past a system of airfoils is a subject of current research work of the authors.

Appendix

We prove that Stokes formula is strictly satisfied in the presented method. We denote by D_1 the interior of the inner circle ∂D_1 and by D_2 the exterior of the outer circle ∂D_2.

Summation of Eqs. 18 concerning the inner circle results in

$$\sum_{i=1}^{M_1} \left[\oint_{\partial D_1} T_i(s)\, ds \right] \gamma_i = - \oint_{\partial D_1} (V_O)_\theta \, ds - \Gamma_C + \Gamma_1$$

which implies that

$$\sum_{i=1}^{M_1} \gamma_i - \int_D \omega_W^{(1)} \, d\sigma = - \int_{D_1} \omega_O \, d\sigma - \Gamma_C + \Gamma_1 \tag{A1}$$

where M_1 denotes the number of new vortex blobs generated on ∂D_1, $\omega_W^{(1)}$ means the vorticity distribution due to these blobs and $\Gamma_1 = 2\pi R_1 U_1$.

Summation of Eqs. 18 concerning the outer circle ∂D_2 yields

$$\sum_{i=1}^{M} \gamma_i \left[\oint_{\partial D_2} T_i(s)\, ds \right] \gamma_i = - \oint_{\partial D_2} (V_O)_\theta \, ds - \Gamma_C + \Gamma_2$$

which implies that

$$\sum_{i=1}^{M_1} \gamma_i - \int_D \omega_W^{(2)} \, d\sigma = -\Gamma_O + \int_{D_2} \omega_O \, d\sigma - \Gamma_C + \Gamma_2 \tag{A2}$$

where Γ_O denotes total charge of vorticity of old vortex blobs, $\omega_W^{(2)}$ means the vorticity distribution due to the blobs created on ∂D_2 and $\Gamma_2 = 2\pi R_2 U_2$. From (A1) and (A2) one concludes that

$$\int_D \omega_W^{(1)} \, d\sigma + \int_D \omega_W^{(2)} \, d\sigma + \Gamma_O - \int_{D_1 \cup D_2} \omega_O \, d\sigma = \Gamma_2 - \Gamma_1$$

or simply

$$\int_D [\omega_W^{(1)} + \omega_W^{(2)} + \omega_O] \, d\sigma = \Gamma_2 - \Gamma_1$$

as mentioned.

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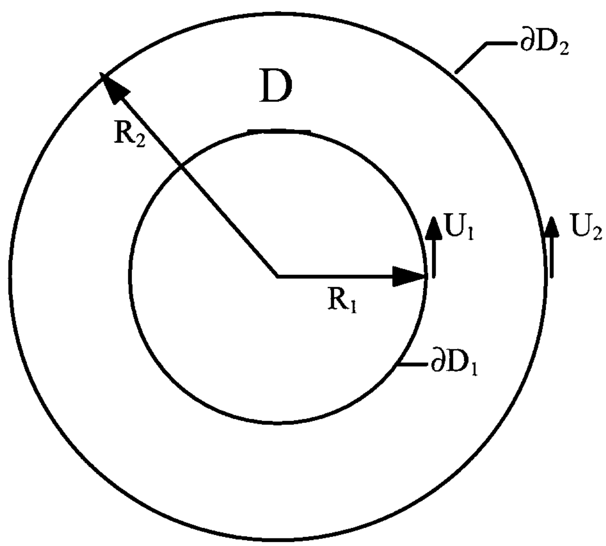

Figure 1: The computational domain.

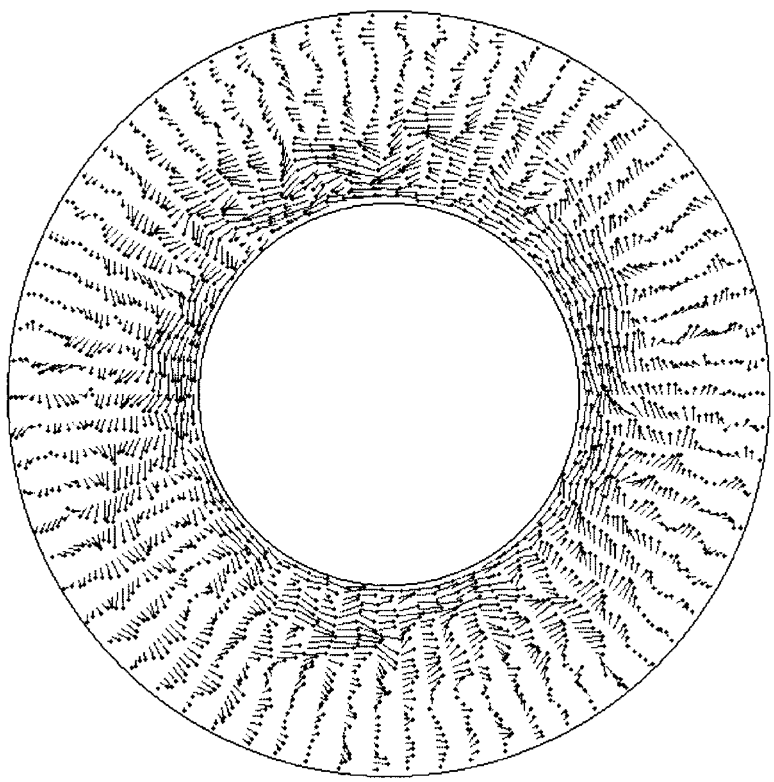

Figure 2: Instantaneous velocity field.

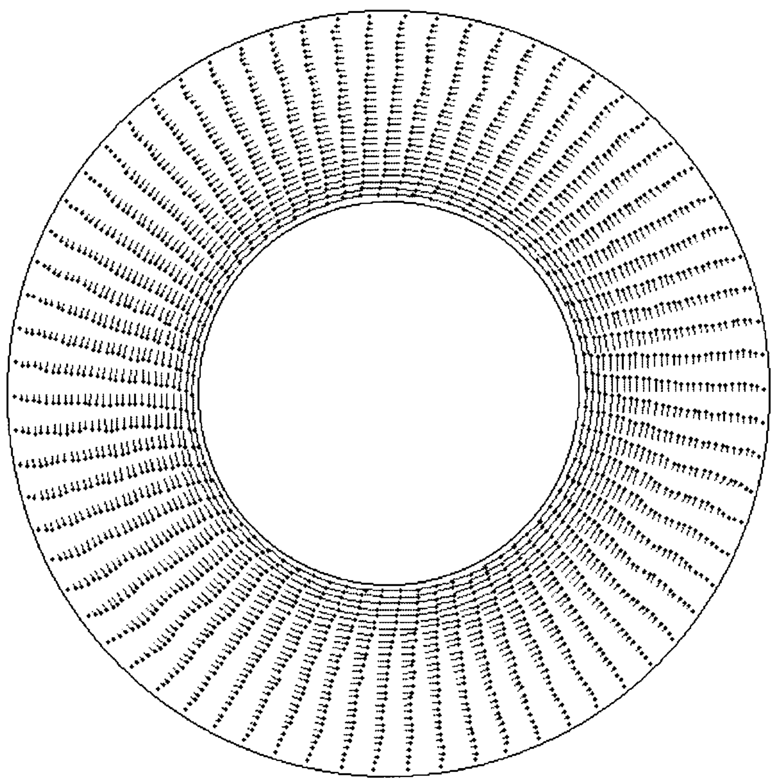

Figure 3: Time-averaged velocity field.

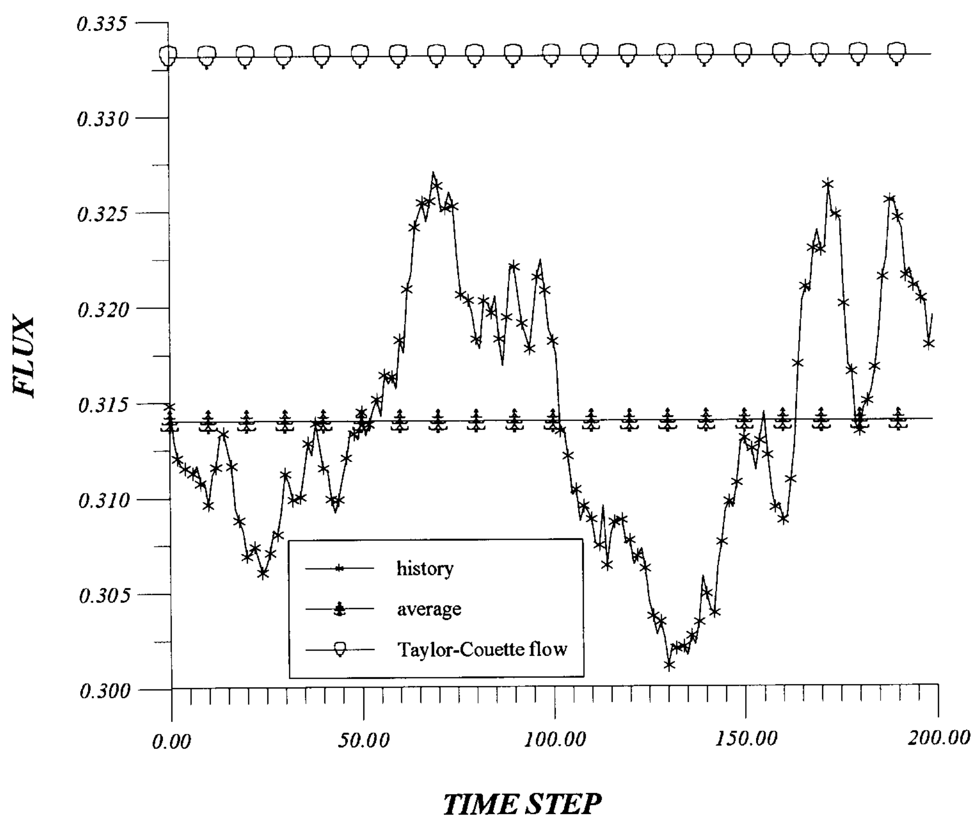

Figure 4: Flux of the flow.

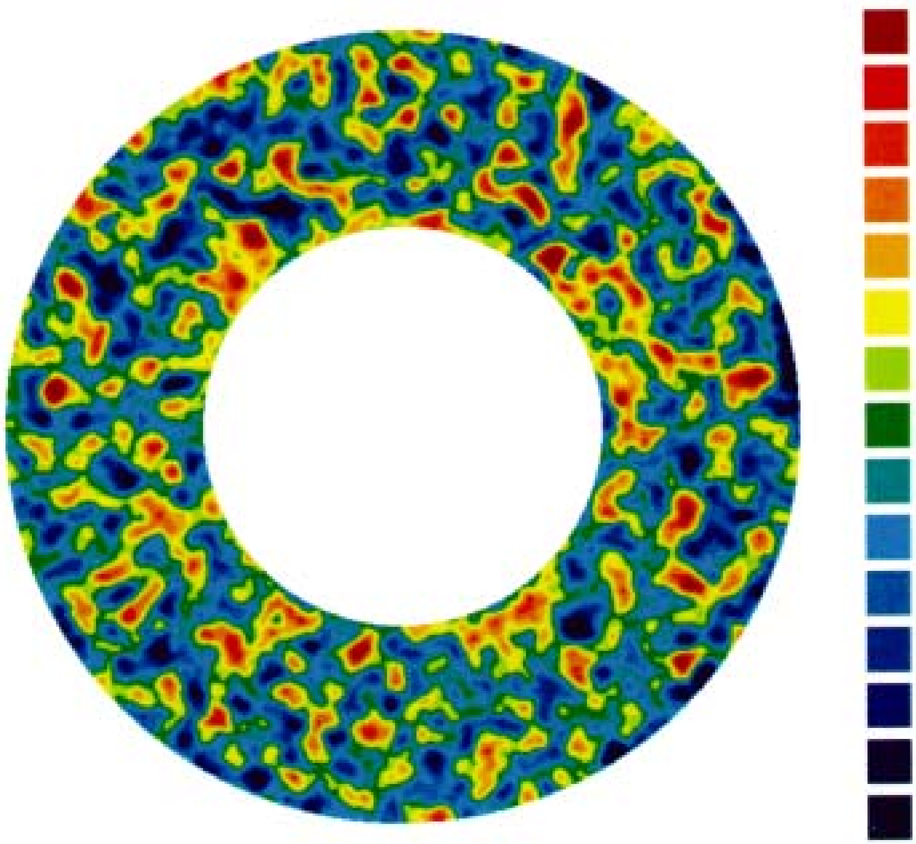

Figure 5: Instantaneous vorticity field.

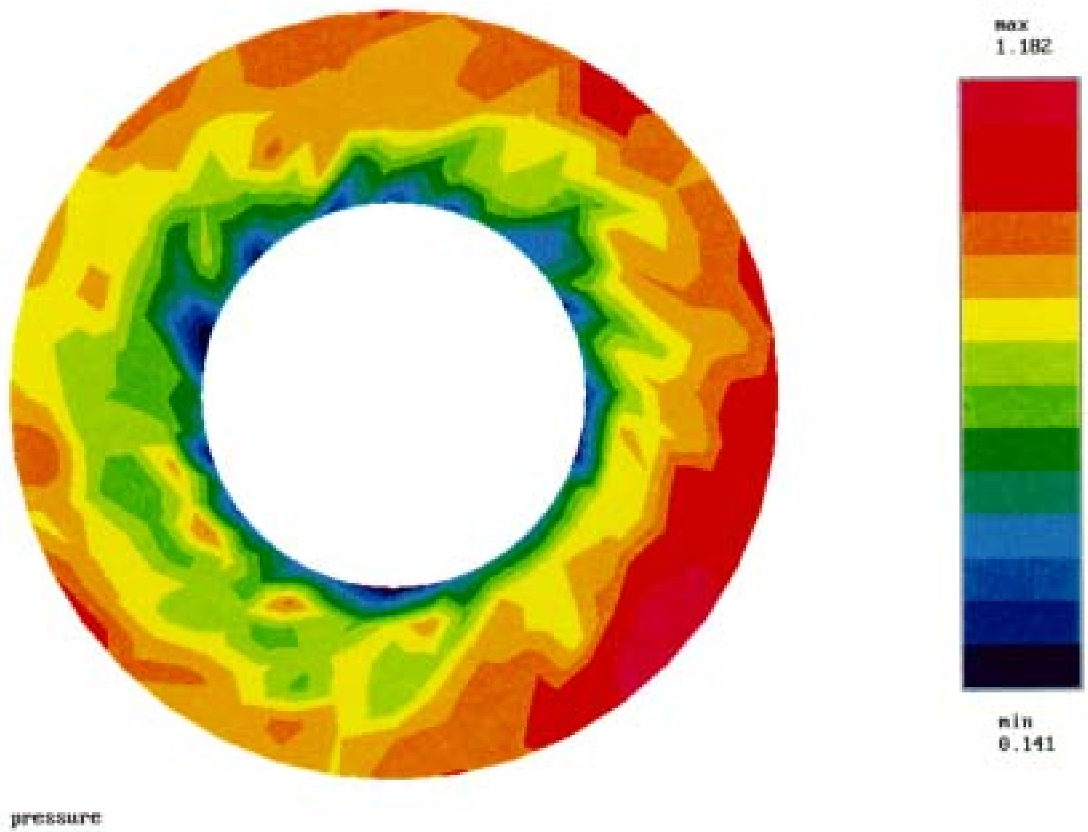

Figure 6: Instantaneous pressure field.