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MODELING AND OPTIMIZATION OF A NON-SYMMETRIC PLATE.

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Abstract. In the first part of this paper we present a limit model for non-symmetric elastic plates derived from three-dimensional elasticity by use of an asymptotic method and we prove existence and uniqueness of a solution of this model. In the second part we try the shape optimization of the plate, we obtain the existence of an optimal profile and we propose its numerical solution by means of penalty methods.

Key words: Non-symmetric plates, asymptotic analysis, shape optimization, penalty methods.
Mathematics subject classification: 73K10, 73K40, 73C02, 35C20.

1. Introduction

The use of asymptotic methods for obtention and mathematical justification of plate models in the framework of the theory of elasticity has been shown to be a successful technique during the last decades. First fundamental contributions in this direction were obtained by Ciarlet - Destuynder [7] with the justification of the classical bi-harmonic model of Kirchhoff - Love for the bending of symmetric plates. The application of this method to different situations (linear and nonlinear elasticity, composite and anisotropic materials, static, dynamic and thermoelastic cases, homogenization and so on) has provided important contributions. A complete analysis of plate models with exhaustive bibliographic references may be found in Ciarlet [6]. Asymptotic methods have also reached an important development in the case of shells (Destuynder [11], Ciarlet - Paumier [9], Ciarlet - Miara [8]) and rods (Trabucho - Viaño [16], Cimetière et al. [10]).

All of the above works have been exclusively devoted to the study of symmetric plates. The case of non-symmetric plates has so far remained, as far as we know, completely unreported. The first aim of this paper is the obtention of a mechanical model for non-symmetric plates in the framework of linear elasticity. In section 2 we describe the physical problem, we obtain the mathematical model by use of an asymptotic analysis of the three-dimensional problem and we prove existence and uniqueness of a solution for such a model.

In section 3, we study the optimal design problem of minimizing the weight of the non-symmetric plate under some geometric and technological constraints, with some imposed bounds on the deflection. In structural optimization, these problems can be formulated as state constrained optimal control problems governed by an elliptic differential equation, the control being a small parameter appearing in the coefficients of the differential operator (cf. Casas [3], Haug - Arora [13], Hlavacek - Bock - Lovisek [14], [15], Haslinger - Neittanmaki [12]). In this section we pose such a problem and obtain the existence of, at least, an optimal profile.

In order to carry out the numerical solution of the optimal shape problem, we propose in the last section the use of penalty techniques, obtaining necessary optimality conditions and some convergence results.

2. Modeling of a non-symmetric plate

Let ε be a positive real parameter and ω be a domain in R^2 with coordinates axis x_1x_2. Let $h \in W^{2,\infty}(\omega)$ be a "thickness" function verifying

$$h(x_1,x_2) \geq \delta > 0, \text{ for all } (x_1,x_2) \in \bar{\omega}. \tag{1}$$

We consider the non-symmetric elastic plate of thickness εh occupying the reference configuration $\overline{\Omega^\varepsilon}$ defined by

$$\Omega^\varepsilon = \{(x_1,x_2,x_3^\varepsilon) \ : \ (x_1,x_2) \in \omega, \ 0 < x_3^\varepsilon < \varepsilon h(x_1,x_2)\}. \tag{2}$$

We denote by $\partial_\alpha^\varepsilon v = \partial_\alpha v$ the derivative $\partial v/\partial x_\alpha$ and by $\partial_3^\varepsilon v$ the derivative $\partial v/\partial x_3^\varepsilon$. Here and along the whole work we use, as it is customary in elasticity theory, the summation convention on repeated indices, supposing that the Latin indices range over $\{1,2,3\}$ and Greek ones over $\{1,2\}$.

We study the physical problem corresponding to the mechanical behavior of a non-symmetric elastic plate, supposed to be clamped on the lateral surface and submitted to body and surface forces. We assume the constitutive material of the plate to be a homogeneous isotropic elastic material of Saint Venant-Kirchhoff's type with Young's modulus E and Poisson's ratio ν. Then, in the linearized elasticity framework, the displacement field u^ε and the Piola-Kirchhoff stress tensor σ^ε are the solution of the following mixed variational formulation (see Ciarlet [5]):

$$u^\varepsilon \in V(\Omega^\varepsilon), \ \sigma^\varepsilon \in \Sigma(\Omega^\varepsilon):$$

$$\int_{\Omega^\varepsilon} \left(\frac{1+\nu}{E}\sigma_{ij}^\varepsilon - \frac{\nu}{E}\sigma_{kk}^\varepsilon \delta_{ij}\right)\tau_{ij}^\varepsilon dx^\varepsilon - \int_{\Omega^\varepsilon} \gamma_{ij}^\varepsilon(u^\varepsilon)\tau_{ij}^\varepsilon dx^\varepsilon = 0, \forall \tau^\varepsilon \in \Sigma(\Omega^\varepsilon), \tag{3}$$

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \gamma_{ij}^\varepsilon(v^\varepsilon) dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon da^\varepsilon, \ \forall v^\varepsilon \in V(\Omega^\varepsilon), \tag{4}$$

where

$$\Gamma_0^\varepsilon = \{(x_1,x_2,x_3^\varepsilon) \ : \ (x_1,x_2) \in \partial\omega, \ 0 < x_3^\varepsilon < \varepsilon h(x_1,x_2)\}, \tag{5}$$

$$\Gamma_+^\varepsilon = \{(x_1,x_2,x_3^\varepsilon) \ : \ (x_1,x_2) \in \bar{\omega}, \ x_3^\varepsilon = \varepsilon h(x_1,x_2)\}, \tag{6}$$

$$\Gamma_-^\varepsilon = \{(x_1,x_2,x_3^\varepsilon) \ : \ (x_1,x_2) \in \bar{\omega}, \ x_3^\varepsilon = 0\}, \tag{7}$$

$$V(\Omega^\varepsilon) = \{v^\varepsilon \equiv (v_i^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3 \ : \ v_i^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\}, \tag{8}$$

$$\Sigma(\Omega^\varepsilon) = [L^2(\Omega^\varepsilon)]_S^9 = \{\tau^\varepsilon \equiv (\tau_{ij}^\varepsilon) \in [L^2(\Omega^\varepsilon)]^9 \ : \ \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon\}, \tag{9}$$

and $\gamma^\varepsilon(u^\varepsilon) = (\gamma_{ij}^\varepsilon(u^\varepsilon))$ is the linearized strain tensor

$$\gamma_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon). \tag{10}$$

We consider the reference symmetric plate of constant thickness 1 occupying the volume $\overline{\Omega}$ where

$$\Omega = \{(x_1,x_2,x_3) \ : \ (x_1,x_2) \in \omega, \ 0 < x_3 < 1\} = \omega \times (0,1), \tag{11}$$

and we define

$$\Gamma_0 = \{(x_1,x_2,x_3) \ : \ (x_1,x_2) \in \partial\omega, \ 0 < x_3 < 1\} = \partial\omega \times (0,1), \tag{12}$$

$$\Gamma_+ = \{(x_1,x_2,x_3) \ : \ (x_1,x_2) \in \omega, \ x_3 = 1\} = \omega \times \{1\}, \tag{13}$$

$$\Gamma_- = \{(x_1,x_2,x_3) \ : \ (x_1,x_2) \in \omega, \ x_3 = 0\} = \omega \times \{0\}. \tag{14}$$

We can define the change of variable from Ω^ε to the fixed domain Ω:

$$\pi^\varepsilon : x \equiv (x_1, x_2, x_3) \in \overline{\Omega} \longrightarrow \pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3 h(x_1, x_2)) \equiv (x_1, x_2, x_3^\varepsilon) \equiv x^\varepsilon \in \overline{\Omega^\varepsilon}. \tag{15}$$

Now we scale the different fields appearing in the variational formulation (see Ciarlet - Destuynder [7]) and we define $u(\varepsilon)$ and $\sigma(\varepsilon)$ by

$$u_\alpha(\varepsilon)(x) = u_\alpha^\varepsilon(x^\varepsilon),\ \ u_3(\varepsilon)(x) = \varepsilon u_3^\varepsilon(x^\varepsilon), \tag{16}$$

$$\sigma_{\alpha\beta}(\varepsilon)(x) = \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon), \sigma_{\alpha 3}(\varepsilon)(x) = \varepsilon^{-1}\sigma_{\alpha 3}^\varepsilon(x^\varepsilon), \sigma_{33}(\varepsilon)(x) = \varepsilon^{-2}\sigma_{33}^\varepsilon(x^\varepsilon). \tag{17}$$

Also we assume that the applied forces are such that

$$f_\alpha^\varepsilon(x^\varepsilon) = f_\alpha(x),\ \ f_3^\varepsilon(x^\varepsilon) = \varepsilon f_3(x), \tag{18}$$

$$g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon g_\alpha(x),\ \ g_3^\varepsilon(x^\varepsilon) = \varepsilon^2 g_3(x), \tag{19}$$

where $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε. Then, we obtain that $(u(\varepsilon), \sigma(\varepsilon))$ is the only solution of the following scaled variational problem posed in Ω :

$$\begin{aligned}
& u(\varepsilon) \in V(\Omega), \tau \in \Sigma(\Omega) : \\
& -\int_\Omega h\gamma_{ij}^h(u(\varepsilon))\tau_{ij}dx + \int_\Omega h\{\frac{1+\nu}{E}\sigma_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\sigma_{\gamma\gamma}(\varepsilon)\delta_{\alpha\beta}\}\tau_{\alpha\beta}dx + \\
& \varepsilon^2\int_\Omega h\{2\frac{1+\nu}{E}\sigma_{\alpha 3}(\varepsilon)\tau_{\alpha 3} - \frac{\nu}{E}(\sigma_{33}(\varepsilon)\tau_{\alpha\alpha} + \sigma_{\alpha\alpha}(\varepsilon)\tau_{33})\}dx + \\
& \varepsilon^4\int_\Omega h\frac{1}{E}\sigma_{33}(\varepsilon)\tau_{33}dx = 0,\ \forall\tau \in \Sigma(\Omega),
\end{aligned} \tag{20}$$

$$\int_\Omega h\sigma_{ij}(\varepsilon)\gamma_{ij}^h(v)dx = \int_\Omega hf_iv_idx + \int_{\Gamma_\pm} h^*(\varepsilon)g_iv_ida,\ \forall v \in V(\Omega), \tag{21}$$

where

$$V(\Omega) = \{v \equiv (v_i) \in [H^1(\Omega)]^3\ :\ v_i = 0 \text{ on } \Gamma_0\}, \tag{22}$$

$$\Sigma(\Omega) = [L^2(\Omega)]_S^9, \tag{23}$$

$$h^*(\varepsilon) = [1 + \varepsilon^2(\partial_1 h)^2 + \varepsilon^2(\partial_2 h)^2]^{1/2}, \tag{24}$$

and $\gamma^h(v) \equiv (\gamma_{ij}^h(v))$ is the generalized strain tensor defined by

$$\gamma_{\alpha\beta}^h(v) = \frac{1}{2}[\partial_\alpha v_\beta + \partial_\beta v_\alpha - x_3h^{-1}\partial_\alpha h\partial_3 v_\beta - x_3h^{-1}\partial_\beta h\partial_3 v_\alpha], \tag{25}$$

$$\gamma_{\alpha 3}^h(v) = \frac{1}{2}[\partial_\alpha v_3 + h^{-1}\partial_3 v_\alpha - x_3h^{-1}\partial_\alpha h\partial_3 v_3], \tag{26}$$

$$\gamma_{33}^h(v) = h^{-1}\partial_3 v_3. \tag{27}$$

In order to obtain the convergence of the scaled three-dimensional problem as ε tends to zero we are going to show several technical results. In what follows, $|\cdot|_{0,D}$ and $||\cdot||_{m,D}$ denote the $L^2(D)$-norm and $H^m(D)$-norm, respectively.

Let $E^h(\Omega)$ be the space

$$E^h(\Omega) = \{v \equiv (v_i) \in [L^2(\Omega)]^3\ :\ \gamma_{ij}^h(v) \in L^2(\Omega)\},$$

endowed with the norm

$$||\, v\, ||_\Omega^h = \{|\, v\, |_{0,\Omega}^2 + \sum_{i,j=1}^3 |\, \gamma_{ij}^h(v)\, |_{0,\Omega}^2\}^{1/2}, \text{ for all } v \in E^h(\Omega). \tag{28}$$

We have the results whose complete proofs can be seen in [1].

THEOREM 1. *$[H^1(\Omega)]^3 = E^h(\Omega)$ and the norms $||\cdot||^h_\Omega$ and $||\cdot||_{1,\Omega}$ are equivalent.*

We introduce the space of displacements of Kirchhoff-Love

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) \in V(\Omega) \ : \ \gamma^h_{i3}(v) = 0\}. \tag{29}$$

We have the following characterization for this space.

THEOREM 2. *The space $V^h_{KL}(\Omega)$ is characterized by*

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) \ : \ v_3(x_1,x_2,x_3) = \zeta_3(x_1,x_2), \zeta_3 \in H^2_0(\omega),$$
$$v_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_1,x_2) - x_3 h(x_1,x_2)\partial_\alpha\zeta_3(x_1,x_2), \zeta_\alpha \in H^1_0(\omega) \tag{30}$$

REMARK 3. We write (30) in the following shortened form

$$V^h_{KL}(\Omega) = \{v \equiv (v_i) \ : \ v_3 = \zeta_3, \ v_\alpha = \zeta_\alpha - x_3 h \partial_\alpha \zeta_3,$$
$$\zeta_3 \in H^2_0(\omega), \ \zeta_\alpha \in H^1_0(\omega)\}$$

and we note that the following mapping is an isomorphism

$$j : \zeta \equiv (\zeta_1,\zeta_2,\zeta_3) \in [H^1_0(\omega)]^2 \times H^2_0(\omega) \longrightarrow$$
$$\longrightarrow j(\zeta) = (\zeta_1 - x_3 h\partial_1\zeta_3, \zeta_2 - x_3 h \partial_2 \zeta_3, \zeta_3) \in V^h_{KL}(\Omega.) \tag{31}$$

THEOREM 4. *The semi-norm $|\cdot|^h_\Omega$ defined by*

$$v \in V(\Omega) \longrightarrow | v |^h_\Omega = \{\sum_{i,j=1}^{3} | \gamma^h_{ij}(v) |^2_{0,\Omega}\}^{1/2}$$

is a norm over $V(\Omega)$ equivalent to $||\cdot||_{1,\Omega}$.

COROLLARY 5. *The mapping*

$$v \in V^h_{KL}(\Omega) \longrightarrow | v |^h_\Omega = \{\sum_{\alpha,\beta=1}^{2} | \gamma^h_{\alpha\beta}(v) |^2_{0,\Omega}\}^{1/2}$$

is a norm over $V^h_{KL}(\Omega)$ equivalent to the norm $||\cdot||_{1,\Omega}$.

COROLLARY 6. *The mapping $||\cdot||_{KL}$ defined by (see (30))*

$$v \in V^h_{KL}(\Omega) \longrightarrow || v ||_{KL} = \{|| \zeta_1 ||^2_{1,\omega} + || \zeta_2 ||^2_{1,\omega} + || \zeta_3 ||^2_{2,\omega}\}^{1/2}$$

is a norm over $V^h_{KL}(\Omega)$ equivalent to the norm $|\cdot|^h_\Omega$ and, consequently, to the norm $||\cdot||_{1,\Omega}$.

The first step in order to obtain convergence of the sequence $(u(\varepsilon), \sigma(\varepsilon))$ consists of the following a priori estimates.

THEOREM 7. *There exist $\varepsilon_0 > 0$ and a constant $C = C(h,\omega)$ such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$|| u(\varepsilon) ||_{1,\Omega} \le C, \quad | \sigma_{\alpha\beta}(\varepsilon) |_{0,\Omega} \le C,$$

$$| \varepsilon\sigma_{\alpha 3}(\varepsilon) |_{0,\Omega} \le C, \quad | \varepsilon^2 \sigma_{33}(\varepsilon) |_{0,\Omega} \le C.$$

COROLLARY 8. *The sequences $\{u(\varepsilon)\}_{\varepsilon>0}$ and $\{\sigma(\varepsilon)\}_{\varepsilon>0}$ verify the following weak convergences*

$$u(\varepsilon) \rightharpoonup u \quad \textit{in } V(\Omega),$$

$$\sigma_{\alpha\beta}(\varepsilon) \rightharpoonup \sigma_{\alpha\beta} \quad \textit{in } L^2(\Omega),$$

$$\varepsilon\sigma_{\alpha 3}(\varepsilon) \rightharpoonup 0 \quad \textit{in } L^2(\Omega),$$

$$\varepsilon^2\sigma_{33}(\varepsilon) \rightharpoonup 0 \quad \textit{in } L^2(\Omega),$$

where u is the unique element in $V^h_{KL}(\Omega)$ solution of the problem

$$\int_\Omega \frac{E}{1-\nu^2} h\{(1-\nu)\gamma^h_{\alpha\beta}(u)\gamma^h_{\alpha\beta}(v) + \nu\gamma^h_{\alpha\alpha}(u)\gamma^h_{\beta\beta}(v)\}\, dx = \int_\Omega hf_i v_i\, dx + \int_{\Gamma_\pm} g_i v_i\, da, \text{ for all } v \in V^h_{KL}(\Omega), \tag{32}$$

and $\sigma_{\alpha\beta}$ is given by

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\{(1-\nu)\gamma^h_{\alpha\beta}(u) + \nu\gamma^h_{\mu\mu}(u)\delta_{\alpha\beta}\}. \tag{33}$$

COROLLARY 9. *The following strong convergences hold true:*

$$u(\varepsilon) \to u \quad in\ V(\Omega),$$
$$\sigma_{\alpha\beta}(\varepsilon) \to \sigma_{\alpha\beta} \quad in\ L^2(\Omega),$$
$$\varepsilon\sigma_{\alpha 3}(\varepsilon) \to 0 \quad in\ L^2(\Omega),$$
$$\varepsilon^2\sigma_{33}(\varepsilon) \to 0 \quad in\ L^2(\Omega).$$

REMARK 10. We can make the limit problem (32) more explicit by taking

$$u = (\xi_1 - x_3 h\partial_1\xi_3, \xi_2 - x_3 h\partial_2\xi_3, \xi_3), \tag{34}$$

$$v = (\zeta_1 - x_3 h\partial_1\zeta_3, \zeta_2 - x_3 h\partial_2\zeta_3, \zeta_3), \tag{35}$$

for $\xi_\alpha, \zeta_\alpha \in H^1_0(\omega)$ and $\xi_3, \zeta_3 \in H^2_0(\omega)$ and by letting $g^+_i(x_1,x_2) = g_i(x_1,x_2,1)$, $g^-_i(x_1,x_2) = g_i(x_1,x_2,0)$. Then the limit problem takes the following form:

$$\frac{E}{1-\nu^2}\{\int_\omega h[(1-\nu)\gamma_{\alpha\beta}(\xi)\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi)\gamma_{\beta\beta}(\zeta)] -$$
$$\int_\omega \frac{h^2}{2}[(1-\nu)(\gamma_{\alpha\beta}(\xi)\partial_{\alpha\beta}\zeta_3 + \gamma_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\xi_3) + \nu(\gamma_{\alpha\alpha}(\xi)\partial_{\beta\beta}\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\partial_{\beta\beta}\xi_3)]$$
$$+\int_\omega \frac{h^3}{3}[(1-\nu)\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\zeta_3 + \nu\partial_{\alpha\alpha}\xi_3\partial_{\beta\beta}\zeta_3]\ \} =$$
$$\int_\omega h[\int_0^1 f_i]\zeta_i - \int_\omega h^2[\int_0^1 x_3 f_\alpha]\partial_\alpha\zeta_3 + \int_\omega [g^-_i + g^+_i]\zeta_i - \int_\omega h g^+_\alpha \partial_\alpha\zeta_3,$$
$$\text{for all } \zeta_\alpha \in H^1_0(\omega),\ \zeta_3 \in H^2_0(\omega).$$

REMARK 11. If we assume an asymptotic expansion of the form

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon^2(u^2, \sigma^2) + \varepsilon^4(u^4, \sigma^4) + \dots$$

and if we substitute this expression into the scaled variational problem, we obtain that the first term (u^0, σ^0) verifies

$$u^0 = u, \quad \sigma^0_{\alpha\beta} = \sigma_{\alpha\beta},$$
$$\sigma^0_{i3} = x_3\sigma^0_{i\beta}\partial_\beta h - \int_0^{x_3} \partial_\beta(h\sigma^0_{i\beta}) - g^-_i - \int_0^{x_3} hf_i.$$

REMARK 12. We can return to the reference configuration Ω^ε taking into account that u is a limit approximation of $u(\varepsilon)$. Performing the inverse change of variable and the inverse scaling, we obtain that the limit displacement field $u^{0\varepsilon}$ in Ω^ε is given by

$$u^{0\varepsilon}_\alpha(x^\varepsilon) = u_\alpha(x) = \xi_\alpha(x_1,x_2) - x_3 h(x_1,x_2)\partial_\alpha\xi_3(x_1,x_2) =$$
$$= \xi_\alpha(x_1,x_2) - \varepsilon^{-1}x^\varepsilon_3\partial_\alpha\xi_3(x_1,x_2) = \xi^\varepsilon_\alpha(x_1,x_2) - x^\varepsilon_3\partial_\alpha\xi^\varepsilon_3(x_1,x_2),$$
$$u^{0\varepsilon}_3(x^\varepsilon) = \varepsilon^{-1}u_3(x) = \varepsilon^{-1}\xi_3(x_1,x_2) = \xi^\varepsilon_3(x_1,x_2),$$

where

$$\xi^\varepsilon_\alpha = \xi_\alpha, \quad \xi^\varepsilon_3 = \varepsilon^{-1}\xi_3.$$

Letting $g_i^{\varepsilon+}(.,.) = g_i^\varepsilon(.,.,\varepsilon h(.,.))$ and $g_i^{\varepsilon-}(.,.) = g_i^\varepsilon(.,.,0)$ we have $g_\alpha^{\varepsilon\pm} = \varepsilon g_\alpha^\pm$ and $g_3^{\varepsilon\pm} = \varepsilon^2 g_3^\pm$. Then, we obtain that $\xi^\varepsilon_\alpha \in H_0^1(\omega)$ and $\xi^\varepsilon_3 \in H_0^2(\omega)$ are the unique solution of the following problem which is the obtained model

$$\frac{E}{1-\nu^2}\{\int_\omega \varepsilon h[(1-\nu)\gamma_{\alpha\beta}(\xi^\varepsilon)\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi^\varepsilon)\gamma_{\beta\beta}(\zeta)] -$$

$$\int_\omega \frac{\varepsilon^2 h^2}{2}[(1-\nu)(\gamma_{\alpha\beta}(\xi^\varepsilon)\partial_{\alpha\beta}\zeta_3 + \gamma_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\xi_3^\varepsilon) + \nu(\gamma_{\alpha\alpha}(\xi^\varepsilon)\partial_{\beta\beta}\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\partial_{\beta\beta}\xi_3^\varepsilon)]$$

$$+ \int_\omega \frac{\varepsilon^3 h^3}{3}[(1-\nu)\partial_{\alpha\beta}\xi_3^\varepsilon\partial_{\alpha\beta}\zeta_3 + \nu\partial_{\alpha\alpha}\xi_3^\varepsilon\partial_{\beta\beta}\zeta_3]\ \} =$$

$$\int_\omega [\int_0^{\varepsilon h} f_i^\varepsilon + g_i^{\varepsilon-} + g_i^{\varepsilon+}]\zeta_i - \int_\omega [\int_0^{\varepsilon h} x_3^\varepsilon f_\alpha^\varepsilon + \varepsilon h g_\alpha^{\varepsilon+}]\partial_\alpha\zeta_3,$$

$$\text{for all } \zeta_\alpha \in H_0^1(\omega),\ \zeta_3 \in H_0^2(\omega).$$

3. Optimization of a non-symmetric plate

We consider an homogeneous, isotropic, elastic non-symmetric plate occupying the reference configuration

$$\Omega^h = \{(x_1,x_2,x_3) \in I\!R^3\ :\ (x_1,x_2) \in \omega,\ 0 < x_3 < h(x_1,x_2)\}$$

clamped on its lateral surface and submitted to body forces of density $f^h \equiv (f_i^h)$ on Ω^h and surface forces of density $g^{h+} \equiv (g_i^{h+})$ on Γ_+^h and $g^{h-} \equiv (g_i^{h-})$ on Γ_-^h where

$$\Gamma_+^h = \{(x_1,x_2,x_3) \in I\!R^3\ :\ (x_1,x_2) \in \omega,\ x_3 = h(x_1,x_2)\},$$

$$\Gamma_-^h = \{(x_1,x_2,x_3) \in I\!R^3\ :\ (x_1,x_2) \in \omega,\ x_3 = 0\}.$$

We denote by $u^h \equiv (u_i^h) : \bar\Omega^h \longrightarrow I\!R^3$ the displacement field corresponding to the above conditions. In the sequel we identify the functions u^h, f^h and $g^{h\pm}$ with the functions $u(h) : \bar\Omega \longrightarrow R^3$, $f(h) : \Omega \longrightarrow R^3$ and $g^\pm(h) : \Gamma_\pm \longrightarrow R^3$ such that

$$u(h)(x_1,x_2,x_3) = u^h(x_1,x_2,x_3 h(x_1,x_2)),$$

$$f(h)(x_1,x_2,x_3) = f^h(x_1,x_2,x_3 h(x_1,x_2)),$$

$$g^+(h)(x_1,x_2,1) = g^{h+}(x_1,x_2,h(x_1,x_2)),$$

$$g^-(h)(x_1,x_2,0) = g^{h-}(x_1,x_2,0).$$

For simplicity, we assume from now on that the applied forces are such that $f(h) = f$ and $g^\pm(h) = g^\pm$, where f and $g^\pm$ are independent of h.

As it was shown in the previous section the displacement field $u(h)$ can be approximated by the only solution of the following variational problem:

$$u(h) \in V_{KL}^h(\Omega), \quad a_h(u(h),v) = l_h(v), \text{ for all } v \in V_{KL}^h(\Omega),$$

where for all $u, v \in V_{KL}^h(\Omega)$,

$$a_h(u,v) = \int_\Omega \frac{E}{1-\nu^2} h\{(1-\nu)\gamma^h_{\alpha\beta}(u)\gamma^h_{\alpha\beta}(v) + \nu\gamma^h_{\alpha\alpha}(u)\gamma^h_{\beta\beta}(v)\}\ dx, \quad (36)$$

$$l_h(v) = \int_\Omega h f_i v_i\ dx + \int_{\Gamma_+} g_i^+ v_i\ da + \int_{\Gamma_-} g_i^- v_i\ da. \quad (37)$$

The optimal design problem deals with minimizing the weight of the non-symmetric plate Ω^h, or equivalently its volume (since the material is homogeneous), that we denote by $J(h) = \int_\omega h$.

The plate must be designed in such a way that allowed deflections be less than a given bound e. As $u_3(h) \in H_0^2(\omega)$ we impose:

$$\| u_3(h) \|_{\infty,\omega} \leq e.$$

In addition, we impose to the thickness some constraints of technological nature:

i) The plate is designed in order to be constructed later, and this construction would not be possible if the plate is too thick or too thin. This fact leads us to impose the constraint:

$$0 < a \leq h(x) \leq b < \infty, \text{ for all } x \in \omega.$$

ii) We also need that the variation of the thickness be slow and progressive. Then, we require

$$\| \partial_\alpha h \|_{\infty,\omega} \leq c, \quad \| \partial_{\alpha\beta} h \|_{\infty,\omega} \leq d.$$

These technological constraints lead us to work in the set of feasible thickness:

$$U_{ad} = \{h \in W^{2,\infty}(\omega) \ : \ a \leq h(x) \leq b, \| \partial_\alpha h \|_{\infty,\omega} \leq c, \| \partial_{\alpha\beta} h \|_{\infty,\omega} \leq d\}$$

which is a convex, closed and bounded subset of $W^{2,\infty}(\omega)$.

If we define the set $K_h = \{v \in V_{KL}^h(\Omega) \ : \ \| v_3 \|_{\infty,\omega} \leq e\}$ which is a convex, closed subset of $V_{KL}^h(\Omega)$, we can formulate the following optimal design problem:

$$(P) \quad Find\ \overline{h} \in U_{ad}\ such\ that: \ u(\overline{h}) \in K_{\overline{h}},$$

$$J(\overline{h}) = inf\{J(h) : h \in U_{ad}, u(h) \in K_h\}.$$

Our first aim is to study the existence of a solution for problem (P). In order to do so, we consider J as a function defined in the space $W^{1,\infty}(\omega)$

$$J : h \in W^{1,\infty}(\omega) \longrightarrow J(h) = \int_\omega h \in I\!R$$

and we define the set $B = \{h \in U_{ad} \ : \ u(h) \in K_h\}$ that we assume to be non empty. Then, problem (P) can be written in the following form:

$$(P) \quad Find\ \overline{h} \in B\ such\ that\ J(\overline{h}) = inf\{J(h) : h \in B\}.$$

Since J is continuous, to obtain the existence of a solution it should be enough to prove that B is compact in $W^{1,\infty}(\omega)$.

In order to prove this fact, we write problem (P) in a more suitable form. We recall that the definition of Ω^h implicitly assumes that h is in the subset $A(\omega)$ of the space $W^{1,\infty}(\omega)$ given by

$$A(\omega) = \{h \in W^{2,\infty}(\omega) \ : \ h(x) \geq \delta(h) > 0, \text{ for all } x \in \omega\}.$$

We denote by $\mathbf{L}_2(V_{KL}^h(\Omega))$ the space of continuous bilinear forms on $V_{KL}^h(\Omega)$ and by $\mathbf{L}(V_{KL}^h(\Omega), V_{KL}^h(\Omega)')$ the space of continuous linear forms from $V_{KL}^h(\Omega)$ on its dual. We also introduce the functions

$$T : h \in A(\omega) \longrightarrow T(h) = l_h \in V_{KL}^h(\Omega)',$$

$$F : h \in A(\omega) \longrightarrow F(h) = a_h \in \mathbf{L}_2(V_{KL}^h(\Omega)),$$

$$G : a \in \mathbf{L}_2(V_{KL}^h(\Omega)) \longrightarrow G(a) = A \in \mathbf{L}(V_{KL}^h(\Omega), V_{KL}^h(\Omega)'),$$

where

$$A(y)(z) = a(y,z), \text{ for all } y,z \in V^h_{KL}(\Omega).$$

Of course, G is an isometric isomorphism.

With each bilinear form $a_h = F(h)$ we associate the operator $A_h = G(a_h) = G(F(h))$. Since a_h is $V^h_{KL}(\Omega)-$elliptic for all $h \in A(\omega)$, it follows from Lax-Milgram lemma that for each $l_h \in V^h_{KL}(\Omega)'$ there exists a unique element $u(h) \in V^h_{KL}(\Omega)$ such that $A_h(u(h)) = l_h$. Then, A_h is an isomorphism from $V^h_{KL}(\Omega)$ onto $V^h_{KL}(\Omega)'$ and the following function is well defined and of class C^∞ :

$$u : h \in A(\omega) \subset W^{1,\infty}(\omega) \longrightarrow u(h) = A_h^{-1}(l_h) \in V^h_{KL}(\Omega) \subset [H^1(\Omega)]^3.$$

This is the key to prove that U_{ad} and B are compact in $W^{1,\infty}(\omega)$. Consequently, if l_h is of the form (37) then problem (P) has at least one solution.

REMARK 13. For $h, k \in A(\omega)$ we define

$$m_{h,k} = DT(h)(k) \in V^h_{KL}(\Omega)', \quad a_{h,k} = DF(h)(k) \in \mathbf{L}_2(V^h_{KL}(\Omega)).$$

Then we have from (37)

$$m_{h,k} = \int_\Omega k f_i v_i.$$

In the same way we obtain from (36) that for all $u, v \in V^h_{KL}(\Omega)$ given by (34),(35)

$$a_{h,k}(u,v) = \frac{E}{1-\nu^2}\{\int_\omega k[(1-\nu)\gamma_{\alpha\beta}(\xi)\gamma_{\alpha\beta}(\zeta) + \nu\gamma_{\alpha\alpha}(\xi)\gamma_{\beta\beta}(\zeta)] -$$

$$\int_\omega hk[(1-\nu)(\gamma_{\alpha\beta}(\xi)\partial_{\alpha\beta}\zeta_3 + \gamma_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\xi_3) + \nu(\gamma_{\alpha\alpha}(\xi)\partial_{\beta\beta}\zeta_3 + \gamma_{\alpha\alpha}(\zeta)\partial_{\beta\beta}\xi_3)] +$$

$$\int_\omega h^2 k[(1-\nu)\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\zeta_3 + \nu\partial_{\alpha\alpha}\xi_3\partial_{\beta\beta}\zeta_3]\ \}.$$

Let $A_{h,k} = G(a_{h,k}) \in \mathbf{L}(V^h_{KL}(\Omega), V^h_{KL}(\Omega)')$. Then we have

$$p = Du(h)(k) = A_h^{-1}[m_{h,k} - A_{h,k}(u(h))].$$

This equation will be helpful for the numerical computation of $Du(h)(k)$ in the next section.

4. Penalty method

In order to obtain the numerical solution of problem (P) we will use penalty techniques, defining a family of problems (P_δ) which approximate problem (P) and which have a suitable numerical solution. Besides, we will obtain necessary optimality conditions for the penalty problem (P_δ).

For $\delta > 0$ we define the function

$$J_\delta : h \in A(\omega) \longrightarrow J_\delta(h) = J(h) + \frac{1}{2\delta} \mid\mid u(h) - P_{K_h}(u(h)) \mid\mid^2_{KL} \in R$$

where $P_{K_h} : V^h_{KL}(\Omega) \longrightarrow K_h$ denotes the projection in the Hilbert space $V^h_{KL}(\Omega)$ over the closed and convex subset K_h, which is a continuous function (Barbu-Precupanu [3], Cea [11]). Thus, J_δ is also a continuous function.

We consider the following problem:

$$(P_\delta) \quad \textit{Find } h_\delta \in U_{ad} \textit{ such that } J_\delta(h_\delta) = inf\{J_\delta(h) : h \in U_{ad}\}.$$

Due to the compacity of U_{ad} and the continuity of J_δ, the problem (P_δ) has, at least, one solution.

If for each $\delta > 0$ we note h_δ a solution of problem (P_δ) then we have the following result.

THEOREM 14. *i) If* $\delta_1 < \delta_2$, *then*

$$J_{\delta_2}(h_{\delta_2}) \leq J_{\delta_1}(h_{\delta_1}) \leq \inf_{h \in B} J(h), \tag{38}$$

$$\| u(h_{\delta_1}) - P_{K_{h_{\delta_1}}}(u(h_{\delta_1})) \|_{KL} \leq \| u(h_{\delta_2}) - P_{K_{h_{\delta_2}}}(u(h_{\delta_2})) \|_{KL}, \tag{39}$$

$$J(h_{\delta_2}) \leq J(h_{\delta_1}) \leq \inf_{h \in B} J(h). \tag{40}$$

ii) The family $\{h_\delta \ : \ \delta > 0\}$ *is bounded in* $W^{2,\infty}(\omega)$ *and there exist sequences* $\{\delta_n\}$, $\{h_{\delta_n}\}$ *such that if* $n \to \infty$, *then*

$$\delta_n \to 0, \qquad h_{\delta_n} \to \overline{h} \text{ in } W^{1,\infty}(\omega).$$

Each of these limits $\overline{h}$ *is a solution of problem* (P).

iii) It is verified that

$$\lim_{\delta \to 0} J(h_\delta) = \lim_{\delta \to 0} J_\delta(h_\delta) = \inf_{h \in B} J(h).$$

As an immediate consequence, we obtain the following convergence result.

COROLLARY 15. *If problem* (P) *has a unique solution* $\overline{h}$ *and, for each* $\delta > 0$, h_δ *denotes a solution of the problem* (P_δ), *then as* $\delta \to 0$ *we have*

$$h_\delta \to \overline{h} \text{ in } W^{1,\infty}(\omega).$$

In order to obtain optimality conditions for the problem (P_δ) we first prove the following result.

THEOREM 16. *The function* $J_\delta : A(\omega) \longrightarrow R$ *is of class* C^1 *and*

$$DJ_\delta(h)(k) = J(k) + a_{h,k}(u(h), z_\delta) - m_{h,k}(z_\delta),$$

where $u(h)$ *and* z_δ *verify*

$$A_h(u(h)) = l_h, \tag{41}$$

$$p_\delta = \frac{I - P_{K_h}}{\delta}(u(h)), \tag{42}$$

$$A_h^*(z_\delta) + p_\delta = 0, \tag{43}$$

and where $A_h^* : V_{KL}^h(\Omega) \longrightarrow (V_{KL}^h(\Omega))'$ *is the adjoint operator of* A_h.

A point $h_\delta \in U_{ad}$ is said to be a stationary point for problem (P_δ) if and only if it verifies $DJ_\delta(h_\delta)(h - h_\delta) \geq 0, \ \forall h \in U_{ad}$. Since U_{ad} is a convex set, we have that if $h_\delta \in U_{ad}$ is a solution of (P_δ) then h_δ is a stationary point for (P_δ) (cf. Cea [4]). This fact supplies us the following necessary optimality condition.

THEOREM 17. $h_\delta \in U_{ad}$ *is a stationary point for* (P_δ) *if and only if there exist* $p_\delta, \ u_\delta, \ z_\delta \in V_{KL}^h(\Omega)$ *verifying*

$$A_{h_\delta}(u_\delta) = l_{h_\delta}, \quad p_\delta = \frac{I - P_{K_{h_\delta}}}{\delta}(u_\delta), \quad A_{h_\delta}^*(z_\delta) + p_\delta = 0,$$

$$J(h) - J(h_\delta) + a_{h_\delta, h - h_\delta}(u_\delta, z_\delta) - m_{h_\delta, h - h_\delta}(z_\delta) \geq 0, \ \forall h \in U_{ad}.$$

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