ESAIM: Proceedings
Élasticité, Viscoélasticité et Contrôle Optimal
Huitièmes Entretiens du Centre Jacques Cartier
URL: http://www.emath.fr/proc/Vol.2/

ESAIM: Proc., Vol. 2, 1997, 99–111

INSTABILITIES IN SLIP DEPENDENT FRICTION

IOAN R. IONESCU AND JEAN-CLAUDE PAUMIER

Abstract. Quasi-static loading process and dynamic process modelling the interaction with slip displacement friction between an elastic body and a rigid body are considered. In these processes, stick-slip motions are related to the earthquake instabilities. On the contact interface we use the friction Coulomb law with a slip dependent friction coefficient in the case of a prescribed normal pressure. In quasi-static loading, the qualitative behaviour of the solution is decided by the competition between two parameters which involve the geometry, the normal stress, the elasticity properties and the dependence on the slip displacement of the friction coefficient. We explain how slow loads generate time discontinuities and space nonhomogeneities on the contact interface at a large time scale. In the dynamic process we provide a linear stability analysis: rapidly after the initial perturbation, the dominant exponential part governs the time and space evolution of the slip during the initiation phase. For large times, the slip velocity behaves qualitatively as it would be in the case of a propagating crack. Between these two phases a transition phase exists caracterised by an extremely high apparent velocity (supersonic) of the rupture front.

Key words: Antiplane Elasticity, Slip Dependent Friction, Static and Dynamic instabilities, Stick-Slip Motions, Catrastrophic Behavior, Bifurcations, Earthquake Source, Initiation Phase
Mathematics subject classification: 73C35, 35B35, 35B32

1. Introduction

The investigation of friction laws on geological faults emerges as a key issue for earthquake modelling. Generally speaking we deal with cycles of two different processes. The first one is a quasistatic long term (100-3000years) process of slow loading and the other one is a dynamic short term (10-50sec) process of rapid unloading. That is why both the dynamic and quasi-static analysis are necessary to complete decription of the phenomenon.

An important step in the understanding of the *stick-slip phenomenon* was done by Rabinowicz [1959] which pointed out that *the coefficient of friction μ varies, rather than be constant, with the tangential displacement, i.e. $\mu = \mu(|u_\tau|)$*. Stick-slip is then a result of the *slip weakening*, i.e. the fall of the friction force with slip. The importance of slip dependent friction was demonstrated from theoretical and experimental points of view (Brace and Byerlee [1966], Byerlee [1967],[1970], Byerlee and Brace [1968]). We do not intend to discuss the details of a particular friction law but rather we want to investigate the qualitative implications of a slip dependent friction.

A class of models consists of a totally concentrated mass sliding on a frictional surface and pulled by a spring. This single block slider has been used as an analogue of a fault for the discussion of the effect of the nonlinear friction laws deduced from laboratory experiments. The same problem of non linear friction can be adressed from the point of view of a model based on the elasticity equations of continuous media theory i.e. the contact problem at the boundary of an elastic body. As it follows from Ionescu & Paumier [1994] and Campillo et al [1996] the continuous model and the mass concentrated model have qualitative different behaviors with the same friction law. They pointed out the limitation of the use of such a simple analogy (i.e. the block slider model) to describe the properties of the relative motions of two media in contact with friction.

We consider here the contact problem between an elastic body and a foundation. On the contact interface we use the friction Coulomb law with a slip dependent friction coefficient in the special case of a prescribed normal pressure. The general problem (i.e. for an arbitrary geometry) was studied by Ionescu & Paumier [1996] in the (quasi-)static case. The anti-plane problem, studied here in both static and dynamic cases, was compiled as *the simplest example able to emphasise the main difficulties of the general problem.* It is sufficiently "simple" to permit some analytical computations and sufficiently "complicate" to point out some mathematical properties and some mechanical phenomena as non-uniqueness, bifurcation, catastrophic behaviour, dynamic instability, etc.

2. Problem statement

Let us consider the shearing of an infinite linear elastic slab bounded by the planes $x_1 = l$, $x_1 = 0$, $x_2 = h$ and $x_2 = 0$ as in Figure **??**. On $\Gamma_f = [0, l] \times \{h\} \times \mathbb{R}$ the slab is in contact with friction with a rigid body which pushes it with the constant normal force $\sigma_{22} = -S$ i.e.

$$\sigma(u)n \cdot n = -S \quad \text{on} \quad \Gamma_f, \tag{1}$$

where u is the displacement field, $\sigma = \sigma(u)$ is the stress tensor and n is the outward unit normal vector.

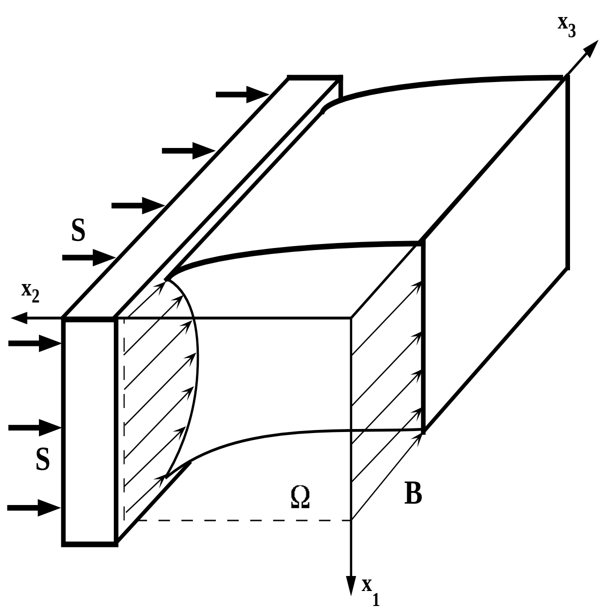

FIGURE 1. The geometry of the antiplane problem.

Along $\Gamma_u = [0,l] \times \{0\} \times \mathbb{R}$ the displacement is imposed: $u_1 = 0$, $u_2 = 0$, $u_3 = B$ and on $\Gamma_c = \{0,l\} \times [0,h] \times \mathbb{R}$ the normal displacement and the tangential stress are vanishing: $u_1 = 0$, $\sigma_{12} = \sigma_{13} = 0$.

Let us assume that the displacement field is 0 with respect to Ox_1, u_2 depends on x_2 alone and u_3 does not depend on x_3. Since no perturbation of the equilibrum in the x_2 direction is considered we get $u_2(x_2) = -\frac{S}{\lambda+2G}x_2$ where $\lambda, G > 0$ are the Lamé coefficients. Let us denote by $\Omega :=]0,l[\times]0,h[$ and by $w := u_3 - B(1 - x_2/h)$.

2.1. The static case

The slip dependent friction law on Γ_f in the static case is described by:

$$\sigma_\tau(u) = -S\mu(|u_\tau|)u_\tau/|u_\tau| \quad \text{if} \quad u_\tau \neq 0 \ \text{ on } \ \Gamma_f, \tag{2}$$

$$|\sigma_\tau(u)| \leq \mu(0)S \quad \text{if} \quad u_\tau = 0 \ \text{ on } \ \Gamma_f, \tag{3}$$

where u_τ and σ_τ are the tangential displacement and tangential stress respectively.

Using the above assumptions (usually accepted for an anti-plane problem) the equilibrum equation $div\ \sigma = 0$ and the boundary conditions leads to the following static problem (SP).

Find $w : \Omega \longrightarrow \mathbb{R}$ such that:

$$\Delta w = 0 \quad \text{in} \quad \Omega, \tag{4}$$

$$\partial_{x_1} w(l,x_2) = \partial_{x_1} w(0,x_2) = w(x_1,0) = 0, \quad \forall\ x_1,x_2 \in]0,l[\times]0,h[, \tag{5}$$

$$G\partial_{x_2} w(x_1,h) + \mu(|w(x_1,h)|)S sign(w(x_1,h)) = q, \quad \text{if} \quad w(x_1,h) \neq 0, \tag{6}$$

$$|G\partial_{x_2} w(x_1,h) - q| \leq \mu(0)S \quad \text{if} \quad w(x_1,h) = 0, \tag{7}$$

where $-q$ the tangential stress, which corresponds to the "stuck case", i.e. $q := GB/h$.

2.2. The dynamic case

The slip dependent friction law on Γ_f in the dynamic case is described by:

$$\sigma_\tau(u) = -S\mu(|u_\tau|)\partial_t u_\tau/|\partial_t u_\tau| \quad \text{if} \quad \partial_t u_\tau \neq 0 \ \text{ on } \ \Gamma_f. \tag{8}$$

$$|\sigma_\tau(u)| \leq \mu(|u_\tau|)S \quad \text{if} \quad \partial_t u_\tau = 0 \ \text{ on } \ \Gamma_f, \tag{9}$$

The momentum balance law $div\ \sigma = \rho\partial_{tt}u$ and the boundary conditions leads to the following dynamic problem (DP).

Find $w : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ such that:

$$\partial_{tt} w(t) = c^2 \Delta w(t) \quad \text{in} \quad \Omega, \tag{10}$$

$$\partial_{x_1} w(t,l,x_2) = \partial_{x_1} w(t,0,x_2) = w(t,x_1,0) = 0, \tag{11}$$

$$\begin{aligned} G\partial_{x_2} w(t,x_1,h) + \mu(|w(t,x_1,h)|)S sign(\partial_t w(t,x_1,h)) &= q, \\ \text{if} \quad \partial_t w(t,x_1,h) &\neq 0, \end{aligned} \tag{12}$$

$$|G\partial_{x_2} w(t,x_1,h) - q| \leq \mu(|w(t,x_1,h)|)S \quad \text{if} \quad \partial_t w(t,x_1,h) = 0, \tag{13}$$

$$w(0) = w_0, \quad \partial_t w(0) = w_1 \quad \text{in} \quad \Omega, \tag{14}$$

where $c = \sqrt{G/\rho}$ is the shear velocity and w_0, w_1 are the initial conditions.

3. The static analysis

General results concerning the static analysis of the contact problem with slip displacement dependent friction in elasticity was obtained by Ionescu & Paumier [1996]. For the reader's convenience we recall here these results in the particular case of the antiplane problem.

3.1. The variational approach

We consider the space

$$V = \{v \in H^1(\Omega)/v(x_1, 0) = 0 \text{ a.e. } x_1 \in (0, l)\}$$

and let $\mathcal{W} : V \longrightarrow \mathbb{R}$ be the energy function

$$\mathcal{W}(v) = \frac{G}{2}\int_\Omega |\nabla v|^2 + \int_0^l H(|v(x_1, h)|)dx_1 - q\int_0^l v(x_1, h)dx_1 \tag{15}$$

where H is the primitive $H(u) = \int_0^u \mu(s)ds$.

Let us denote by $j : V \times V \longrightarrow \mathbb{R}_+$ and $f : V \longrightarrow \mathbb{R}$ the functions:

$$j(u, v) = \int_{\Gamma_f} S\mu(|u|)|v|dx, \qquad f(v) = q\int_{\Gamma_f} v dx, \qquad \forall u, v \in V.$$

Then the following quasi-variational inequality represents the *variational approach* of (SP):

Find $w \in V$, such that

$$\int_\Omega \nabla w \cdot \nabla(w - v)dx + j(w, w) - j(w, v) \leq f(w - v), \quad \forall v \in V. \tag{16}$$

Then we have the following result:

Theorem 3.1. (**i**) If $w \in V$ is a local extremum for $\mathcal{W}$ then w is a solution of (16).
(**ii**) There exists at least a global minimum for $\mathcal{W}$.

3.2. The eigenvalue problem

Let us consider now the following eigenvalue problem: Find $\varphi \in V, \varphi \neq 0$ and $b \in \mathbb{R}$ such that

$$\Delta\varphi(x_1, x_2) = 0 \quad \text{in} \quad \Omega, \tag{17}$$

$$\varphi(x_1, 0) = \partial_{x_1}\varphi(l, x_2) = \partial_{x_1}\varphi(0, x_2) = 0, \;\; G\partial_{x_2}\varphi(x_1, h) = b\,\varphi(x_1, h). \tag{18}$$

After some algebra, we get the eigenvalues:

$$b_0 \;=\; G/h, \quad b_n = \frac{Gn\pi}{l}\text{cth}\left(\frac{n\pi h}{l}\right) \quad \forall n \geq 1, \tag{19}$$

and the corresponding eigenfunctions:

$$\varphi_0(x_1, x_2) = x_2/h, \varphi_n(x_1, x_2) = C_n\text{sh}\left(\frac{n\pi}{l}x_2\right)\cos\left(\frac{n\pi}{l}x_1\right) \quad \forall n \geq 1,$$

where C_n are choosen such that $\int_0^l \varphi_n^2(x_1, h)dx_1 = 1$.

3.3. Homogeneous sliding solutions

Now we will focus our attention on the homogeneous sliding solutions of (SP), i.e. we look for w^h solution of (4)-(7) such that $\partial_{x_1} w^h(x_1, h) = 0$ for all $x_1 \in]0, l[$. If we denote by $U := w^h(x_1, h)$ and introduce the function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ given by

$$g(u) := b_0 u + \mu(u) S, \qquad \forall u \in \mathbb{R}_+$$

then we get:

$$\begin{aligned} g(U) = q \quad &\text{if} \quad U > 0, \\ q \in [-g(0), g(0)] \quad &\text{if} \quad U = 0, \\ -g(-U) = q \quad &\text{if} \quad U < 0. \end{aligned} \tag{20}$$

In order to fix the ideas we will suppose for simplicity that g is increasing on $[0, \alpha_1]$ and on $[\alpha_2, +\infty)$ and it is deacreasing on $[\alpha_1, \alpha_2]$. If $U \geq 0$ from (20) we obtain the branch of homogeneous slip $U = g^{-1}(q)$.

Let us analyse in the next the dependence of the homogeneous slip U on the load q when q is increasing, i.e. we imagine a *quasi-static loading process.* As we can see in Figure 2 there exists *no (global) continuous solutions* $U : I \longrightarrow \mathbb{R}$ for I an open interval with $q_1, q_2 \in I$ where $q_1 = g(\alpha_1), q_2 = g(\alpha_2)$. One can consider the *(perfect) delay convention*, which comes from the catastrophe theory (see Poston and Stewart [1978]) to select as physical solution the solution which jumps (has a discontinuity) as late as possible (i.e. when it has no other choice). Hence in a quasi-static process, when the load q is increasing, there exist equilibrum positions where a *typical catastrophic event* is present (for instace the point C in Figure 2).

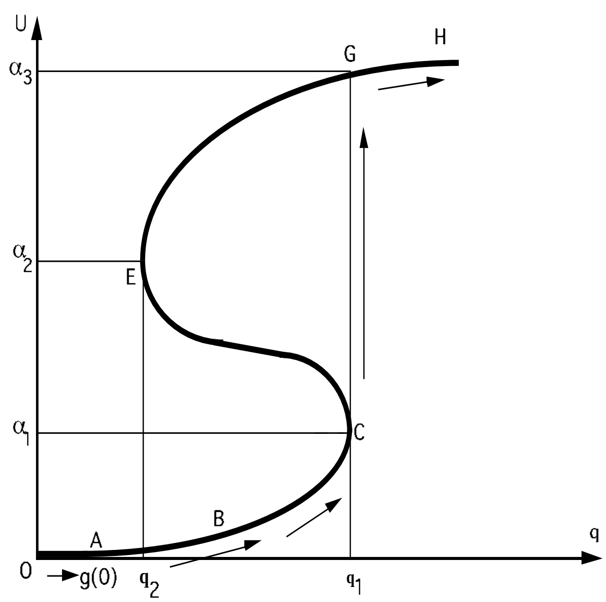

FIGURE 2. The homogeneous slip $U = g^{-1}(q)$ versus the tangential load q. Solutions of the quasistatic loading process selected by the (perfect) delay criterion (solid arrows).

We can now give a result on stability and uniqueness of the homogeneous sliding. Let U be a solution of (20) corresponding to $w^h = U\varphi_0$, the solution of (4)-(7). Then we have:

THEOREM 3.2. (**i**) if $-S\inf_{u\in R_+}\mu'(u) < b_0$ holds then w^h is the unique solution of (4)-(7) in V,
(**ii**) if $b_0 + S\mu'(|U|) > 0$ holds then w^h corresponds to an (isolated) local minimum for $\mathcal{W}$ in V.

3.4. NONHOMOGENEOUS BIFURCATION OF THE HOMOGENEOUS SLIDING

Using the Liapounov-Schmidt method we get the following results on the branches of solutions and bifurcation. Let us suppose in the next that $\mu \in C^2$, $\mu(0) = \mu''(0) = 0$. Let (q_0, U_0) be a solution of (20). In a neighborhood $\mathcal{V}$ of $(q_0, U_0\varphi_0) \in \mathbb{R}\times V$ we have:

THEOREM 3.3. (**i**) There exists a branch $(q, w^h) = (q, U\varphi_0)$ of solutions for (20) with $U(q_0) = U_0$ which is the unique branch of homogeneous sliding solutions,
(**ii**) If $b_n + S\mu'(|U_0|) \neq 0$ for all $n \geq 1$ then (q, w^h) is the unique branch of solutions for (4)-(7) in $\mathcal{V}$,
(**iii**) If $b_j + S\mu'(|U_0|) = 0$ with $j \geq 1$ and $\mu''(|U_0|) \neq 0$ then $(t_0, U_0\varphi_0)$ is a cusp bifurcation point, where the branch of homogeneous sliding solutions (q, w^h) intersects tranversally a branch of nonhomogeneous sliding solutions (q, w^j) (see Figure 3).

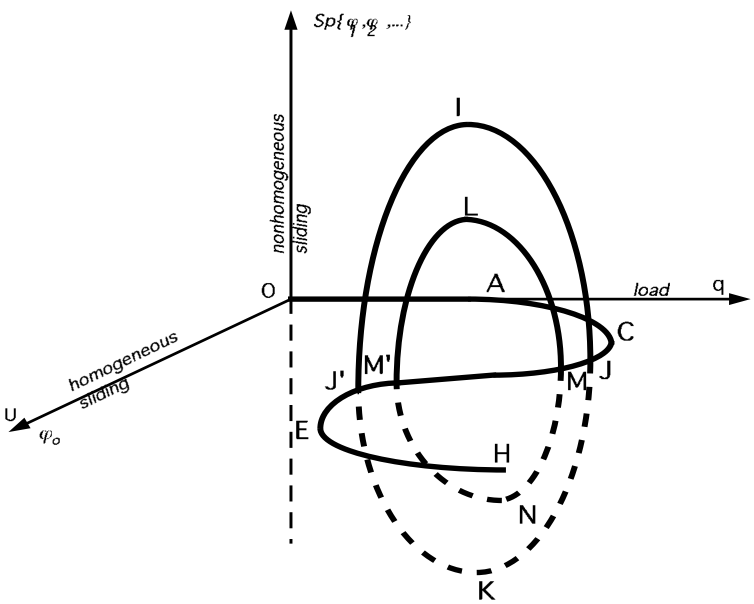

FIGURE 3. Four cusp bifurcation points (J , J', M and M'), where the branch of homogeneous sliding (OACJMM'J'EH) intersects transversally two nonhomogeneous sliding branches (IJKJ' and LMNM').

Having in mind the above results we must take into account in our analysis not only the homogeneous solution but also nonhomogeneous sliding solutions. In this case we remark that *the perfect delay criterion does not merely work.*Hence, with a quasistatic analysis it is difficult to predict the new equilibrum position. Only the *stability analysis of the homogeneous sliding in the associated dynamical problem* can give some answers to this question.

4. The dynamic analysis

As far as we know, the existence and unicity of a solution to (10)-(14) is still an open problem. But, since the one-dimensional problem (called here homogeneous solutions of the antiplane problem) has an unique solution, one can conjecture the same result the two-dimensional case.

Since our intention is to study the evolution of the elastic system near an (unstable) equilibrum position w^{eq} (one of the solution of the static case) we shall denote by $\bar{w}(t) = w(t) - w^{eq}$ the perturbation and by $\bar{w}_0 = w_0 - w^{eq}$ and $\bar{w}_1 = w_1$ the initial perturbations.

4.1. The linearized problem

Let us assume in the following that the slip $w(x_1, h)$ and the slip rate $\partial_t w(x_1, h)$ are positive (usualy accepted in seismology). In ordrer to give a linear analyse of (DP) we shall suppose that the initial perturbation is small and the nonlinear function μ may be approached in a neibourhood of w^{eq} by its linear approximation i.e.

$$\mu(w(t, x_1, h)) \approx \mu(w^{eq}(x_1, h)) + \mu'(w^{eq}(x_1, h))\bar{w}(t, x_1, h), \tag{21}$$

for all $0 \leq t \leq T_c$, where $[0, T_c]$ will be called in the following the "initiation phase". If we denote by

$$m(x_1) = -S\mu'(w^{eq}(x_1, h)) \tag{22}$$

then from (DP) we deduce the the following linear problem (LP)

$$\partial_{tt}\bar{w}(t) = c^2\Delta\bar{w}(t) \quad \text{in} \quad \Omega, \tag{23}$$

$$\partial_{x_1}\bar{w}(t, l, x_2) = \partial_{x_1}\bar{w}(t, 0, x_2) = \bar{w}(t, x_1, 0) = 0, \tag{24}$$

$$G\partial_{x_2}\bar{w}(t, x_1, h) = m(x_1)\bar{w}(t, x_1, h), \tag{25}$$

$$\bar{w}(0) = \bar{w}_0, \quad \partial_t\bar{w}(0) = \bar{w}_1 \quad \text{in} \quad \Omega. \tag{26}$$

4.2. The eigenvalue problem

Let us consider the following eingenvalue problem connected to (PL): Find $\phi \in V, \phi \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$c^2\Delta\phi(x_1, x_2) = -\lambda\phi(x_1, x_2) \quad \text{in} \quad \Omega, \tag{27}$$

$$\phi(x_1, 0) = \partial_{x_1}\phi(l, x_2) = \partial_{x_1}\phi(0, x_2) = 0, \tag{28}$$

$$G\partial_{x_2}\phi(x_1, h) = m(x_1)\phi(x_1, h). \tag{29}$$

Theorem 4.1. Let $m \in L^\infty(\Gamma_f)$. Then there exists an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues and $\lambda_n \longrightarrow +\infty$. To each λ_n it corresponds a finite dimensional subspace Φ_n of eigenfunctions.

Note that, generaly, λ_0 is not positive, hence there exist a finite number of negative eigenvalues.

Proof. Let us firstly deduce the following inequality :

$$||v||^2_{L^2(\Gamma_f)} \leq \bar{C}||v||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}, \quad \forall v \in V. \tag{30}$$

Indeed, if $v \in V$ then $v^2 \in W^{1,1}(\Omega)$ and using the trace continuity and the Friedrichs-Poincaré inequality in $W^{1,1}(\Omega)$ we get

$$||v^2||_{L^1(\Gamma_f)} \leq C_\gamma ||v^2||_{W^{1,1}(\Omega)} \leq C_\gamma C_F||\nabla(v^2)||_{L^1(\Omega)} =$$

$$2C_\gamma C_F||v\nabla v||_{L^1(\Omega)} \leq \bar{C}||v||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}.$$

Let $\alpha > 0$ be a constant and let us remark that (27) may be written as :

$$-\Delta\phi + \alpha\phi = \lambda'\phi \quad \text{in} \quad \Omega \tag{31}$$

where $\lambda' = \frac{\lambda}{c^2} + \alpha$. For all $f \in L^2(\Omega)$ let us denote by $u = T(f)$ the solution of the following problem:

$$-\Delta u + \alpha u = f \quad \text{in} \quad \Omega, \tag{32}$$

$$u(x_1, 0) = \partial_{x_1} u(l, x_2) = \partial_{x_1} u(0, x_2) = 0, \tag{33}$$

$$G\partial_{x_2} u(x_1, h) = m(x_1)u(x_1, h) \quad . \tag{34}$$

Indeed, $u \in V$ is the solution of :

$$a(u, v) \;=\; \int_\Omega fv dx \quad \forall v \in V, \tag{35}$$

with $a(u, v) := \int_\Omega \nabla u \cdot \nabla v dx + \alpha \int_\Omega uv dx - \frac{1}{G}\int_{\Gamma_f} muv dx$. If we compute $a(v, v)$ from (30) we get

$$\begin{aligned} a(v,v) \;&\geq\; ||\nabla v||^2_{L^2(\Omega)} + \alpha||v||^2_{L^2(\Omega)} - \frac{\bar{m}}{G}||v||^2_{L^2(\Gamma_f)} \\ &\geq\; ||\nabla v||^2_{L^2(\Omega)} + \alpha||v||^2_{L^2(\Omega)} - \frac{\bar{m}\bar{C}}{G}||v||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}, \end{aligned}$$

where $\bar{m} =: \text{essup}_{x\in\Gamma_f} m(x)$. For $\alpha =: \frac{1}{2}(1 + \frac{\bar{m}^2\bar{C}^2}{G^2})$ we obtain $a(v, v) \geq \frac{1}{2}||v||^2_{H^1(\Omega)}$ for all $v \in V$, hence there exists $u \in V$ the unique solution of (35). We have just proved that $T : L^2(\Omega) \longrightarrow V$ is a well defined, symmetric, continous and positive operator. Bearing in mind the compact imbending of V in $L^2(\Omega)$ we deduce that $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is also a compact operator, hence there exists $(\beta_n)_{n\in\mathbb{N}}$ a positive decreasing sequence of eigenvalues of T. Moreover $\beta_n \longrightarrow 0$ and to each β_n it corresponds a finite dimensional subspace $\Phi_n \subset V$ of eigenfunctions, i. e. $T(\phi) = \beta_n\phi$ for all $\phi \in \Phi_n$. Finally we get the statement of the theorem with $\lambda'_n = \frac{1}{\beta_n}$ and $\lambda_n = c^2(\frac{1}{\beta_n} - \alpha)$. □

For $m(x_1) = m$ constant on Γ_f one can obtain, after some algebra, the analytical expression of the eigenvalues and eigenfunctions.

$$\lambda_{n0} = \frac{G}{\rho}[\frac{n^2\pi^2}{l^2} - \frac{r_0^2}{h^2}], \quad \phi_{n0} = C_{n0}\text{sh}(\frac{r_0}{h}x_2)\cos(\frac{n\pi}{l}x_1) \;\text{ if }\; m > b_0$$

$$\lambda_{n0} = \frac{G}{\rho}[\frac{n^2\pi^2}{l^2} + \frac{p_0^2}{h^2}], \quad \phi_{n0} = C_{n0}\sin(\frac{p_0}{h}x_2)\cos(\frac{n\pi}{l}x_1) \;\text{ if }\; m < b_0$$

$$\lambda_{n0} = \frac{G}{\rho}\frac{n^2\pi^2}{l^2}, \quad \phi_{n0} = C_{n0}x_2\cos(\frac{n\pi}{l}x_1) \text{ if } m = b_0$$

$$\lambda_{nk} = \frac{G}{\rho}[\frac{n^2\pi^2}{l^2} + \frac{p_k^2}{h^2}], \quad \phi_{nk} = C_{nk}\sin(\frac{p_k}{h}x_2)\cos(\frac{n\pi}{l}x_1) \text{ for all } m \in \mathbb{R}, k \geq 1,$$

where $r_0 > 1$ is the solution of the equation $xcth(x) = \frac{m}{b_0}$, $p_k \in]k\pi, (k+1)\pi[$ is the solution of the equation $x\mathrm{ctg}(x) = \frac{m}{b_0}$ for all $k \geq 0$ and C_{nk} are normalistion constants.

Note that $\lambda_{00} < \lambda_{01} < .... < \lambda_{0j} < 0 < \lambda_{0j+1}$ iff $b_{j+1} > m > b_j$ and $\lambda_{nk} > 0$ for all $n \geq 0, k \geq 1$.

4.3. The dynamic stability analysis

In order to fix the ideas let us denote by (λ_n, ϕ_n) the eigenvalues and the eigenfunctions of (27)-(29) and let N be such that $\lambda_0 < \lambda_1 < .... < \lambda_N < 0 < \lambda_{N+1} < ....$

Bearing in mind the above spectral results, one can give a generic form of the solution for (23)-(26) as follows:

$$\bar{w}(t) = \sum_{n=0}^{\infty}[\mathrm{ch}(\sqrt{-\lambda_n}t)W_n^0 + \frac{\mathrm{sh}(\sqrt{-\lambda_n}t)}{\sqrt{-\lambda_n}}W_n^1]\phi_n \quad \text{in } \Omega,$$

where $W_n^0 = \int_\Omega \phi_n(x)\bar{w}_0(x)dx, W_n^1 = \int_\Omega \phi_n(x)\bar{w}_1(x)dx$.

We remark that the part of the solution associated with negative eigenvalues will have an exponential growth with time. Hence, after a while this part will completely dominate the other part which has a wave-type evolution. This behavior is the expression of the instability caused by the slip weakening friction law. This is why we put $\bar{w} = \bar{w}^d + \bar{w}^w$, where $\bar{w}^d$ is the "dominant part" given by

$$\bar{w}^d(t) = \sum_{n=0}^{N}[\mathrm{ch}(\sqrt{-\lambda_n}t)W_n^0 + \frac{\mathrm{sh}(\sqrt{-\lambda_n}t)}{\sqrt{-\lambda_n}}W_n^1]\phi_n \quad \text{in } \Omega,$$

and $\bar{w}^w$ is the "wave part":

$$\bar{w}^w(t) = \sum_{n=N+1}^{\infty}[\cos(\sqrt{\lambda_n}t)W_n^0 + \frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}W_n^1]\phi_n \quad \text{in } \Omega.$$

This expression indicates that the coefficient of the exponential growth is larger for low values of the wavenumber n. In other words, the larger is the characteristic length, the larger is the Liapounov exponent. The use of the expression of the dominant part leads to a solution in which the perturbation has been severely smoothed by the finite wavenumber integration. The propagative terms are rapidly negligible and the shape of the slip distribution is almost perfectly described by the dominant part.

4.4. Numerical tests

The aim of this paragraph is to compare the theoretical results with some numerical tests. These tests were obtained by using a numerical approach of the nonlinear problem (10)-(14). Since the details of the numerical method is beyond the scope of the present paper, let us give here only a brief description of the numerical scheme. The second order partial differential equation (10) is written as a first order system involving the velocity and the two shear stress components. After splitting, an alternating direction method is used to reduce the problem to two hyperbolic systems in one space-dimension, for each time step. A classical finite difference scheme (Lax-Wendroff) is used in the discretisation of these systems. Concerning the nonlinear boundary condition (12)-(13) we used the integration along the characteristic lines (in the system following the x_2 direction) to deduce an instability capturing scheme.

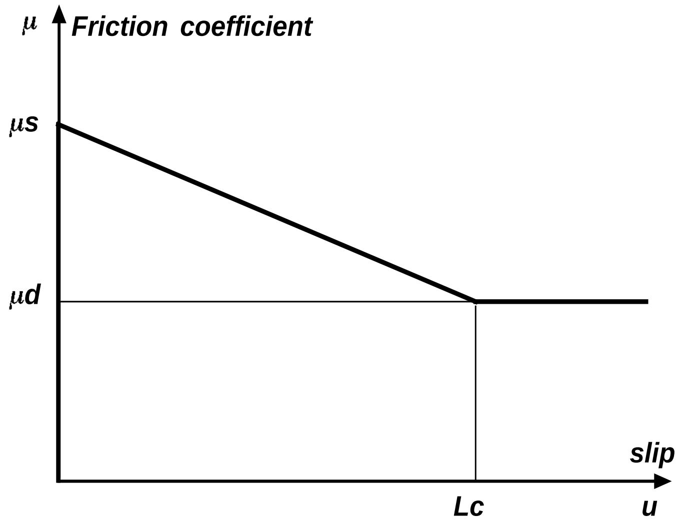

FIGURE 4. The friction coefficient as a function of the relative slip, used in the numerical tests.

For simplicity, let us assume in the following that the friction law has the form of a piecewise linear function:

$$\mu(u) = \mu_s - \frac{\mu_s - \mu_d}{L_c} u \text{ if } u \leq L_c, \quad \mu(x,u) = \mu_d \text{ if } u > L_c, \tag{36}$$

where u is the slip, μ_s and μ_d ($\mu_s > \mu_d$) are the static and dynamic friction coefficients, and L_c is the critical slip (see Figure 4). This piecewise linear function is a reasonable approximation of the experimental observations obtained by Ohnaka et al. [1987], Ohnaka [1995]. The equilibrum will be assumed to be $w^{eq}(x_1, x_2) = 0$ and $q = S\mu_s$. The initial condition corresponds to a slip velocity perturbation $\bar{w}_1$ while the initial slip perturbation $\bar{w}_0$ is 0. Since μ is linear on $[0, L_c]$ the above linear stability analysis is exact for $0 \leq w(t, x_1, h) \leq L_c$. Let T_c is a critical time for which the slip on the friction plane reaches the critical value L_c at least at one point and let us call in the following $[0, T_c]$ the "initiation phase".

For a better visualisation of the numerical solution we have choosen in our tests some generic values of the physical constants ($S = 1., G = 1., \rho = 1., L_c = 0.1, \mu_s = 1., \mu_d = 0.5, l = 4., h = 2.$). No qualitative differences are remarked when real geophysical values of the contants are used(see Campillo & Ionescu [1996]).

As we can see in Figure 5 rapidly after the initial perturbation, the dominant exponential part governs the time and space evolution of the slip until the critical slip L_c has been reached at some point of the surface for $t = T_c (T_c \approx 0.9$ in this numerical test). The "wave-type" part of the solution becames after a while negligible and has a small inflence on the shape

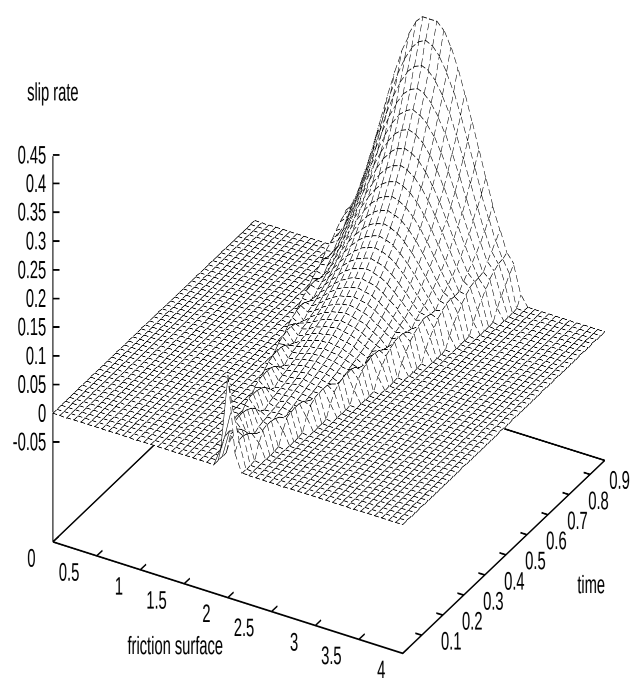

FIGURE 5. Slip velocity $\partial_t w$ on the frictional surface ($x_2 = h$) as a function of space (x_1) and time (t) computed using a finite difference method during the "initiation" stage ($t \in [0, T_c], T_c = 0.9$).

of the perturbation at the end of the initiation phase. The positive results of the confrontation of the analytical results with the numerical solution shows the possibility to use the expression of the dominant part to describe the properties of the initiation phase in our model.

The evolution of the system in the initiation phase is fundamental in understanding how the dynamic crack propagation develops after that. Since a part of the slipping surface already reached the critical slip while other parts are still at rest or in the exponential growth domain, the state of the system becomes complex. This stage cannot be simply described analytically but the finite difference solution, ploted in Figure 6, will help in showing the main characteristics of the beginning of the dynamic crack propagation (i.e. for $t > T_c \approx 0.9$). For large times, i.e. for $t > 1.25$ in Figure 6, the slip velocity behaves qualitatively as it would be in the case of a propagating crack. There is no singularity but a concentration of the slip velocity.

Also for large times, the peak of the slip velocity (the seismic rupture front) tends to propagate at the shear wave velocity but it is delayed with respect to the wave emitted at the initial perturbation . Between the steady state propagation of the rupture discussed just above and the initiation phase, a transition phase exists ($0.9 < t < 1.25$ in this numerical test). The state of the system at this stage is completly determined by the properties of the initiation phase. A striking feature is the extremely high apparent velocity (supersonic) of the rupture front when the rupture propagation begins. This phenomenon occurs because the critical slip is reached almost simultaneously on a patch of finite length. This length is indeed related to the shape of the slip distribution at the end of the initiation phase.

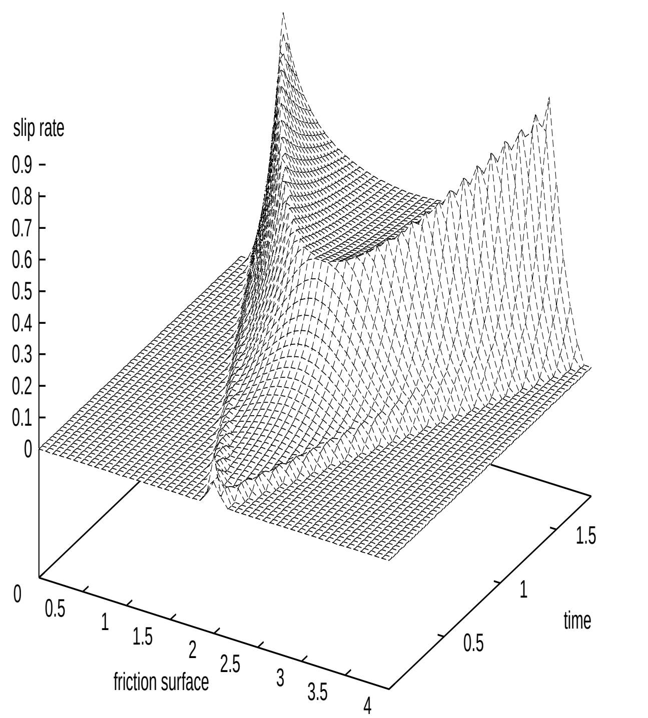

FIGURE 6. Slip velocity $\partial_t w$ on the frictional surface ($x_2 = h$) as a function of space (x_1) and time (t) computed using a finite difference method during the initiation stage ($t \in [0, T_c], T_c = 0.9$) and the transition to the steady state propagation($t > 1.25$). Note the supersonic velocity of the slip velocity concentration immediatly after the initiation phase. Then the rupture front velocity tends to the shear wave velocity.

REFERENCES

Brace W.F. and Byerlee J.D. [1966] Stick-Slip as a Mechanism for Earthquakes, Science, Vol. 153, 990-992

Byerlee J.D. [1967] Frictional Characteristics of Granit under High Confining Presure, J. Geophysical Res., vol. 72, No. 14, 3693-3648

Byerlee J.D. [1970] The Mechanism of Stick-Slip, Tectonophysics, 9, 475-468

Byerlee J.D. and Brace W.F. [1968] Stick-Slip, Stable Sliding and Earthquakes - Effect of Rock Type, Pressure, Strain Rate and Stiffness, J. Geophysical Res., Vol. 73, No. 18, 6031-6037

Campillo M. and Ionescu I.R. [1996] Initiation of antiplane shear instability under slip dependent friction, submitted to J. of Geophysical Research

Campillo M., Ionescu I.R., Paumier J.C. and Renard Y. [1996] On the dynamic sliding with friction of a rigid block and of an infinite elastic slab, Physics of the Earth and Planetary interiors, 96, 15-23

Ionescu I.R. and Paumier J.-C. [1994] On the Contact Problem with slip Rate Dependent Friction in Elastodynamics, European J. of Mech., A/Solids, **13**, n^o 4, 555-568

Ionescu I.R. and Paumier J.-C. [1996] On the Contact Problem with slip Dependent Friction in Elastostatics, Int. J. Engng. Scie., vol.34, 4, 471-491

Ohnaka M., Y. Kuwahara and K. Yamamoto [1987] - Constitutive relations between dynamic physical parameters near a tip of the propagation slip during stick-slip shear failure Tectonophysics 144 109-125

Ohnaka M. [1995] - Nonuniformity of the constitutive law parameters for shear rupture and quasistatic nucleation to dynamic rupture: a physical model of earthquake generation model Proceedings of US Academy of Sciences Colloquium on "Earthquake prediction: the scientific challenge" Irvine, California.

Poston T. and Stewart I. [1978] Catastrophe Theory and its Applications, Pitman

Rabinowicz E. [1959] A Study of Stick-Slip Process, in "Friction and wear "(R. Davies editor), Elviser, London, 149-161

LAMA,Université de Savoie, 73376-Le Bourget du Lac and LMC-IMAG, BP 53, 38041-Grenoble Cedex 9, France