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ASYMPTOTIC CONSISTENCY OF THE POLYNOMIAL APPROXIMATION IN THE LINEARIZED PLATE THEORY. APPLICATION TO THE REISSNER-MINDLIN MODEL

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Abstract. We establish a partial link between two standard methods for deriving plate models from linearized three-dimensional elasticity: the asymptotic method, known to justify the Kirchhoff-Love model, and the polynomial reduction method. In the polynomial method, the reduced model is obtained by projecting the three-dimensional displacement on a closed subspace of admissible displacements, namely displacements that are polynomial with respect to the thickness variable. Our procedure characterizes minimal polynomial subspaces that are consistent with the Kirchhoff-Love model. In the same time, if a singular perturbation term is dropped in the equations of the lower degree model, we recover a Reissner-Mindlin model.

Key words: Plates, asymptotic method, projection method, polynomial basis, Reissner, Mindlin
Mathematics subject classification: 73K10, 73V25, 35B25

1. Introduction

Slender structures such as plates, shells, rods are of high technological interest. Numerous works have been devoted for more than a century to the construction of models that take into account the small thickness, or the small diameter, of the structure. These models replace the three-dimensional system of equations that describe the static or dynamical behavior of the stucture thought of as a body in $\mathbb{R}^3$ by a two-dimensional, or one-dimensional system. Such a dimensional reduction obviously makes the model easier to understand and easier to deal with.

In this work, we provide a partial link between two standard methods for deriving plate models from linearized three-dimensional elasticity: the asymptotic method and the polynomial reduction method. In the polynomial method, the reduced model is obtained by projecting the three-dimensional displacement on a closed subspace of admissible displacements, namely displacements that are polynomial with respect to the thickness variable. Generalizing a previous work ([16]) on the locking phenomenon, our procedure characterizes, in a sense made precise below, the minimal polynomial subspaces that are consistent with the Kirchhoff-Love model. As the asymptotic method made it clear ([6], [5]), this model must necessarily be recovered when the thickness goes to zero. In the same time, our results allow to recover, at least formally, Reissner-Mindlin models.

2. Internal approximation of the scaled three-dimensional model

The asymptotic analysis for justifying two-dimensional planar elastic models consists in examining the limit behaviour of the deformations of an elastic cylindrical body when the thickness goes to zero. In the present paper, we restrict our analysis to situations that lead to linear plate models. A thorough description of the method in the linear case, as well as an extensive references list, can be found in [5]. The basis result is a rigorous justification of the Kirchhoff-Love model. It was first proved in [6], and was a thread for intensive subsequent research in the asymptotic procedure. A formal justification of the von Kármán model follows the same lines and can also be found in [5]. For a justification of non linear planar membrane models, and non linear bending models that respect essential features of the three-dimensional equations such as frame-indifference, see [9], [12], [13].

The purpose of this Section is to match the asymptotic procedure with an internal approximation. Indeed, internal approximations in plate theories are found in two settings. First, reducing the dimension of the elastic model from 3 to 2 is classically obtained by projecting the three-dimensional displacement on a closed subspace of functions that are polynomial with respect to the thickness variable, see [1], [3], [8], [20]. The consistency result we prove below gives precise information on minimal polynomial spaces that should be used. Second, internal approximation underlies conformal finite element methods. Consistency results explaining the locking phenomenon in the finite element framework can be found in [16].

2.1. General setting

As usual Latin indices belong to $\{1,2,3\}$ and Greek indices belong to $\{1,2\}$. We consider a plate of thickness 2ε made of an elastic material with Lamé constants λ and μ. The reference configuration of the plate is $\Omega^\varepsilon := \omega\times]-\varepsilon,\varepsilon[$, where ω is a domain in $\mathbb{R}^2$. For the asymptotic analysis to come, data and unknown functions defined on Ω^ε will be transformed on functions defined on $\Omega := \omega\times]-1,1[$. Let γ_0 be a part of $\partial\omega$ such that $meas(\gamma_0) \neq 0$. The plate is clamped along the portion $\Gamma_0^\varepsilon = \gamma_0\times]-\varepsilon,\varepsilon[$ of its lateral boundary and subject to applied body forces with densities $f^\varepsilon = (f_i^\varepsilon) \in L^2(\Omega^\varepsilon)$ and to applied surface forces with densities $g^\varepsilon = (g_i^\varepsilon) \in L^2(\omega \times \pm\varepsilon)$ and on its top and bottom surfaces. Let $V^\varepsilon = (H^1_{\Gamma_0^\varepsilon}(\Omega^\varepsilon))^3 := \{v = (v_i) \in H^1(\Omega^\varepsilon);\ v = 0 \ on\ \Gamma_0^\varepsilon\}$. The linearized strain tensor is given by

$$e(v) = (e_{ij}(v)), \quad e_{ij}(v) := \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

It is well-known that the equilibrium problem in linearized elasticity reads:

Find $u^\varepsilon \in V^\varepsilon$ such that $a^\varepsilon(u^\varepsilon, v) = l^\varepsilon(v)$ for all $v \in V^\varepsilon$,

where

$$a^\varepsilon(u,v) = \int_{\Omega^\varepsilon} \{2\mu\, e_{ij}(u)\, e_{ij}(v) + \lambda\, e_{ii}(u)\, e_{jj}(v)\}\, dx \tag{1}$$

and

$$l^\varepsilon(v) = \int_{\Omega^\varepsilon} f_i^\varepsilon\, v_i\, dx + \int_{\omega\times\{-\varepsilon,+\varepsilon\}} g_i^\varepsilon\, v_i\, dx_1\, dx_2.$$

Appropriate hypotheses on the order of magnitude of the applied loads allow to recover in the limit the linear Kirchhoff-Love model. As the system here is linear, several choices can be made. For definiteness, we consider the choice given in [5], *i.e.*,

$$f^\varepsilon_\alpha(x_1,x_2,\varepsilon x_3) = \varepsilon^2 f_\alpha(x_1,x_2,x_3), \quad f^\varepsilon_3(x_1,x_2,\varepsilon x_3) = \varepsilon^3 f_3(x_1,x_2,x_3) \quad \text{for all } x \in \Omega,$$

$$g^\varepsilon_\alpha(x_1,x_2,\varepsilon x_3) = \varepsilon^3 g_\alpha(x_1,x_2,x_3), \quad g^\varepsilon_3(x_1,x_2,\varepsilon x_3) = \varepsilon^4 g_3(x_1,x_2,x_3) \quad \text{for all } x \in \omega \times \{-1,1\}.$$

Defining new functions $u_\alpha(\varepsilon)$ and $u_3(\varepsilon)$ by

$$u^\varepsilon_\alpha(x_1,x_2,\varepsilon x_3) = \varepsilon^2 u_\alpha(\varepsilon)(x_1,x_2,x_3), \quad u^\varepsilon_3(x_1,x_2,\varepsilon x_3) = \varepsilon u_3(\varepsilon)(x_1,x_2,x_3) \quad \text{for all } x \in \Omega,$$

we obtain by mere change of variables the variational system that $u(\varepsilon)$ solves. Indeed, letting $V := (H^1_{\Gamma_0}(\Omega))^3$ where $\Gamma_0 = \gamma_0 \times]-1,1[$, we define three bilinear forms a_0, a_2, a_4 on V by

$$\begin{aligned} a_0(u,v) &= \int_\Omega \{2\mu\, e_{\alpha\beta}(u)\, e_{\alpha\beta}(v) + \lambda\, e_{\gamma\gamma}(u)\, e_{\gamma\gamma}(v)\}\, dx, \\ a_2(u,v) &= \int_\Omega \{4\mu\, e_{\alpha 3}(u)\, e_{\alpha 3}(v) + \lambda\, (e_{\gamma\gamma}(u)\, e_{33}(v) \;+\; e_{33}(u) e_{\gamma\gamma}(v)\})\, dx, \\ a_4(u,v) &= (2\mu+\lambda) \int_\Omega e_{33}(u) e_{33}(v)\, dx, \end{aligned}$$

and we let

$$a(\varepsilon)(u,v) = \frac{1}{\varepsilon^4} a_4(u,v) + \frac{1}{\varepsilon^2} a_2(u,v) + a_0(u,v).$$

With these notations, the system is:

Find $u(\varepsilon) \in V$ such that $a(\varepsilon)(u(\varepsilon),v) = l(v)$ for all $v \in V$ $\qquad (P(\varepsilon,V))$

where

$$l(v) = \int_\Omega f_i\, v_i\, dx + \int_{\omega\times\{-1,1\}} g_i\, v_i\, dx_1\, dx_2.$$

It is proved in [6] that the sequence $u(\varepsilon)$ converges to a limit u, whose vertical component solves an adimensional Kirchhoff-Love problem and whose horizontal components solve an adimensional linear membrane problem. The variational form of the limit problem uses the space V_{KL} of Kirchhoff-Love displacements

$$\begin{aligned} V_{KL} &:= \{v \in V;\; e_{i3}(v) = 0, i = 1,2,3\} \\ &= \{v = (v_i); v_3 \in H^2_{\gamma_0}(\omega),\; v_\alpha = \eta_\alpha - x_3 \partial_\alpha v_3,\; \eta_\alpha \in H^1_{\gamma_0}(\omega)\} \end{aligned} \tag{2}$$

and reads:

Find $u \in V_{KL}$ such that $a^\star_0(u,v) = l(v)$ for all $v \in V_{KL}$ $\qquad (P_{KL}(V))$

where

$$a^\star_0(u,v) := \int_\Omega \{2\mu\, e_{\alpha\beta}(u)\, e_{\alpha\beta}(v) + \lambda^\star\, e_{\gamma\gamma}(u)\, e_{\gamma\gamma}(v)\}\, dx. \tag{3}$$

Note that the limit bilinear form contains a coefficient $\lambda^\star := \dfrac{2\mu\,\lambda}{2\mu+\lambda}$ different from λ.

2.2. Internal approximation

We now perform an internal approximation of problem $(P(\varepsilon, V))$. The approximation space is denoted by W. It is assumed to be a closed subspace of V, that may be finite-dimensional or not. Without loss of generality, we assume that $W = W_1 \times W_2 \times W_3$ with $W_1 = W_2$. The internal approximation of problem $(P(\varepsilon, V))$ is obtained by projecting its solution $u(\varepsilon)$ on W. In other words, we define $w(\varepsilon)$ as the unique solution of

$$\text{Find } w(\varepsilon) \in W \text{ such that } a(\varepsilon)(w(\varepsilon), v) = l(v) \text{ for all } v \in W. \qquad (P(\varepsilon, W))$$

Our purpose is to identify the limit behaviour of the sequence $w(\varepsilon)$ when ε goes to zero. Results will be given in terms of $\kappa_\varepsilon(w(\varepsilon))$ where, for all v in V, we define $\kappa_\varepsilon(v) := \Big(\kappa_{\varepsilon,ij}(v)\Big)$ by

$$\kappa_{\varepsilon,\alpha\beta}(v) = e_{\alpha\beta}(v),\ \kappa_{\varepsilon,\alpha 3}(v) = \varepsilon^{-1} e_{\alpha 3}(v),\ \kappa_{\varepsilon,33}(v) = \varepsilon^{-2} e_{33}(v). \qquad (4)$$

By Korn's inequality, this tensor satisfies

$$C\,\|v\|_1^2 \le |e(v)|^2 \le |\kappa_\varepsilon(v)|^2 \text{ for all } v \in V, \qquad (5)$$

where $|\cdot|$ (resp. $\|\cdot\|_1$) denotes the usual norm of $(L^2(\Omega))^9$ (resp. $(H^1(\Omega))^3$) and $C > 0$. We introduce three symmetric bilinear forms A_0, A_2, A_4 on $(L^2(\Omega))^9_s$ defined by

$$\begin{aligned} A_0(\kappa, \kappa') &= \int_\Omega \{2\mu\, \kappa_{\alpha\beta}\, \kappa'_{\alpha\beta} + \lambda\, tr(\kappa)\, tr(\kappa')\}\, dx, \\ A_2(\kappa, \kappa') &= 4\mu \int_\Omega \kappa_{\alpha 3}\, \kappa'_{\alpha 3}\, dx, \\ A_4(\kappa, \kappa') &= 2\mu \int_\Omega \kappa_{33}\, \kappa'_{33}\, dx, \end{aligned}$$

These notations allow to rewrite the bilinear form $a(\varepsilon)$ as

$$a(\varepsilon)(u, v) = A\Big(\kappa_\varepsilon(u), \kappa_\varepsilon(v)\Big) \qquad (6)$$

where

$$A = A_0 + A_2 + A_4.$$

From (6), one easily deduces the coerciveness inequality

$$2\mu\, |\kappa_\varepsilon(v)|^2 \le a(\varepsilon)(v, v) \text{ for all } v \in V. \qquad (7)$$

Associated with W are a subspace $\bar{D}_3$ of $L^2(\Omega)$ and an operator $\Lambda^\star$ in $L^2(\Omega)$ that play a fundamental role in the limit behaviour of $(P(\varepsilon, W))$. We let

$$D_3 := \partial_3 W_3 = \{\partial_3 w_3,\ w_3 \in W_3\}. \qquad (8)$$

Obviously, D_3 is a linear subspace of $L^2(\Omega)$, but it is not necessarily a closed subspace. We denote by $\bar{D}_3$ its closure in $L^2(\Omega)$ and by Q the inner projection in $L^2(\Omega)$ on $\bar{D}_3$. We define $\Lambda^\star$ by

$$\Lambda^\star := \lambda(Id - \frac{\lambda}{2\mu + \lambda} Q) \qquad (9)$$

In the particular case when $W_3 = H^1_{\Gamma_0}(\Omega)$ (no internal approximation on the vertical component), then $D_3 \supset \mathcal{D}(\Omega)$. Therefore, $\bar{D}_3 = L^2(\Omega)$, $Q = Id$ and $\Lambda^\star = \lambda^\star\, Id$.

Cases of practical interest are

i) the finite element approximation. Then, W_3 and D_3 are finite dimensional spaces.
ii) the polynomial approximation. In this approximation, we suppose that W consists of functions $w = (w_i)$ whose components are polynomial with respect to the vertical variable. In other words, introducing the notation

$$P_k(H) := \{w : \Omega \mapsto R;\ w(x_1, x_2, x_3) := \sum_{j=0}^{j=k} x_3^j\, w_j(x_1, x_2),\ w_j \in H\}$$

with $H = L^2(\omega),\ H^1_{\gamma_0}(\omega) \cdots$, we assume that

$$W = P_m(H^1_{\gamma_0}(\omega)) \times P_m(H^1_{\gamma_0}(\omega)) \times P_n(H^1_{\gamma_0}(\omega)),$$

or $W = P_{m,n}(H^1_{\gamma_0}(\omega))$ for short. Then, $D_3 = P_{n-1}(H^1_{\gamma_0}(\omega))$. We leave it to the reader to check that $\bar{D}_3 = P_{n-1}(L^2(\omega))$. We will return to this setting in Subsection 2.3 and Section 3.

Finally, we introduce a bilinear form $b_0^\star$ on W and a linear subspace W_{KL} of W by

$$b_0^\star(w, v) := \int_\Omega \{2\mu\, e_{\alpha\beta}(w)\, e_{\alpha\beta}(v) + \Lambda^\star\, (e_{\gamma\gamma}(w))\ e_{\gamma\gamma}(v)\}\, dx \tag{10}$$

and

$$W_{KL} := \{v \in W;\ e_{i3}(v) = 0,\ i = 1, 2, 3\}. \tag{11}$$

It is readily seen that if D_3 is dense into $L^2(\Omega)$ (if $W_3 = V_3$, for instance), then $\Lambda^\star = \lambda^\star\, Id$, and the bilinear forms $a_0^\star$ and $b_0^\star$ coincide. A preliminary version of the following result valid when the approximation space is finite dimensional, and that is not applicable for instance to the polynomial approximation defined in case ii) above, can be found in [16].

THEOREM 2.1. The sequence $w(\varepsilon)$ strongly converges in $(H^1(\Omega))^3$ to w when ε goes to zero, where w is the unique solution of the variational problem

$$\text{Find } w \in W_{KL} \text{ such that } b_0^\star(w, v) = l(v) \text{ for all } v \in W_{KL}. \qquad (P_{KL}(W))$$

Moreover, the sequences $\kappa_{\varepsilon,\alpha 3}(w(\varepsilon)), \alpha = 1, 2$, strongly converge to zero in $L^2(\Omega)$ and the sequence $\kappa_{\varepsilon,33}(w(\varepsilon))$ strongly converges to $-\dfrac{\lambda}{2\mu+\lambda} Q(e_{\gamma\gamma}(w))$ in $L^2(\Omega)$.

Proof. From (P(ε), W) and (6), we get that

$$A\Big(\kappa_\varepsilon(w(\varepsilon)), \kappa_\varepsilon(v)\Big) = l(v) \text{ for all } v \in W. \tag{12}$$

Letting $v = w(\varepsilon)$ and using (5) and (7), this implies that sequence $\kappa_\varepsilon(w(\varepsilon))$ is bounded in $(L^2(\Omega))^9$. Therefore, there exists κ in $(L^2(\Omega))^9$ such that (upon extracting a subsequence) $\kappa_\varepsilon(w(\varepsilon))$ weakly converges in $(L^2(\Omega))^9$ toward κ. Returning to (4), there exists w in W such that $w(\varepsilon)$ weakly converges in V toward w. Moreover w necessarily satisfies $e_{i3}(w) = 0,\ i = 1, 2, 3,$ *i.e.*, $w \in W_{KL}$.

Let us now identify w. First we multiply (12) by ε^2. Then,

$$\int_\Omega 2\,\mu\,\kappa_{\varepsilon,33}(w(\varepsilon))\,e_{33}(v)\,dx + \int_\Omega 4\,\mu\,\kappa_{\varepsilon,\alpha 3}(w(\varepsilon))\,\varepsilon\,e_{\alpha 3}(v)\,dx +$$
$$\int_\Omega \{2\mu\,\kappa_{\varepsilon,\alpha\beta}(w(\varepsilon))\,\varepsilon^2 e_{\alpha\beta}(v) + \lambda\,tr\kappa_\varepsilon(w(\varepsilon))\,\Big(\varepsilon^2 e_{\gamma\gamma}(v) + e_{33}(v)\Big)\}\,dx \;=\; \varepsilon^2 l(v).$$

Letting ε go to zero, we obtain

$$\int_\Omega \Big(2\,\mu\,\kappa_{33} + \lambda\,tr(\kappa)\Big)\,\partial_3 v_3\,dx = 0$$

for all v_3 in W_3. In other words, $2\mu\,\kappa_{33} + \lambda\,tr(\kappa)$ is orthogonal to D_3, which also reads $Q\,(2\mu\,\kappa_{33} + \lambda\,tr(\kappa)) = 0$. By definition κ_{33} is the weak limit of $\partial_3(\varepsilon^{-2}w(\varepsilon))$. Therefore it belongs to $\bar{D}_3$. We conclude that

$$\kappa_{33} = -\frac{\lambda}{2\mu+\lambda} Q(e_{\gamma\gamma}(w)). \tag{13}$$

Let us remark that this implies the identity

$$2\mu\,e_{\alpha\beta}(w) + \lambda(e_{\gamma\gamma}(w) + \kappa_{33})\,\delta_{\alpha\beta} = 2\mu\,e_{\alpha\beta}(w) + \Lambda^\star(e_{\gamma\gamma}(w))\,\delta_{\alpha\beta}. \tag{14}$$

We now return to (12) that we restrict to v in W_{KL}. Then

$$\begin{aligned} A_0\Big(\kappa_\varepsilon(w(\varepsilon)), \kappa_\varepsilon(v)\Big) &= \int_\Omega \{2\mu\,e_{\alpha\beta}(w(\varepsilon)) + \lambda\,tr\kappa_\varepsilon(w(\varepsilon))\,\delta_{\alpha\beta}\}\,e_{\alpha\beta}(v)\,dx \\ &= l(v) \text{ for all } v \in W_{KL}. \end{aligned}$$

Letting ε go to zero and using (14), we obtain problem $(P_{KL}(W))$. From the uniqueness of the solution of this problem, we conclude that the whole sequence $w(\varepsilon)$ (resp. $\kappa_{\varepsilon,33}(w(\varepsilon))$) converges weakly in $(H^1(\Omega))^3$ (resp. $(L^2(\Omega))^9$) toward w (resp. κ_{33}). To prove the strong convergence results, we introduce $\tilde{\kappa} = (\tilde{\kappa}_{ij})$ where

$$\tilde{\kappa}_{\alpha\beta} := \kappa_{\alpha\beta},\; \tilde{\kappa}_{\alpha 3} := 0,\; \tilde{\kappa}_{33} := \kappa_{33}.$$

Let us estimate $\kappa_\varepsilon(w(\varepsilon)) - \tilde{\kappa}$ in $(L^2(\Omega))^9$. We have

$$\begin{aligned} &A\Big(\kappa_\varepsilon(w(\varepsilon)) - \tilde{\kappa},\, \kappa_\varepsilon(w(\varepsilon)) - \tilde{\kappa}\Big) = \\ &\quad = A\Big(\kappa_\varepsilon(w(\varepsilon)), \kappa_\varepsilon(w(\varepsilon))\Big) + A\Big(\tilde{\kappa} - 2\,\kappa_\varepsilon(w(\varepsilon)), \tilde{\kappa}\Big) \\ &\quad = l(w(\varepsilon)) + A_4\Big(\tilde{\kappa} - 2\,\kappa_\varepsilon(w(\varepsilon)), \tilde{\kappa}\Big) + A_0\Big(\tilde{\kappa} - 2\,\kappa_\varepsilon(w(\varepsilon)), \tilde{\kappa}\Big) \end{aligned}$$

since $\tilde{\kappa}_{\alpha 3} = 0$ by definition. From the weak convergence results, we know that

$$[\tilde{\kappa} - 2\,\kappa_\varepsilon(w(\varepsilon))]_{\alpha\beta} \rightharpoonup -\kappa_{\alpha\beta} \text{ and } [\tilde{\kappa} - 2\,\kappa_\varepsilon(w(\varepsilon))]_{33} \rightharpoonup -\kappa_{33}$$

Therefore the right hand-side of the above inequalities converges to the term $l(w) - A_4(\kappa,\kappa) - A_0(\kappa,\kappa)$. From the definition of A_4 and A_0 and from (14) one gets that

$$\begin{aligned} &A_4(\kappa,\kappa) + A_0(\kappa,\kappa) = \\ &\quad = \int_\Omega \{2\mu\,\kappa_{\alpha\beta}\,\kappa_{\alpha\beta} + \lambda\,tr(\kappa)\,tr(\kappa) + 2\mu\,\kappa_{33}\,\kappa_{33}\}\,dx \\ &\quad = \int_\Omega \{2\mu\,\kappa_{\alpha\beta}\,\kappa_{\alpha\beta} + \lambda\,tr(\kappa)\,e_{\gamma\gamma}(w) + \Big(\lambda\,tr(\kappa) + 2\mu\,\kappa_{33}\Big)\,\kappa_{33}\}\,dx \\ &\quad = \int_\Omega \{2\mu\,\kappa_{\alpha\beta} + \Lambda^*(e_{\gamma\gamma}(w))\,\delta_{\alpha\beta}\}e_{\alpha\beta}(w)\,dx \end{aligned}$$

since $\lambda\, tr(\kappa) + 2\mu\, \kappa_{33}$ is orthogonal to D_3. Therefore

$$A_4(\kappa,\kappa) + A_0(\kappa,\kappa) = l(w).$$

The strong convergence of $\kappa_\varepsilon(w(\varepsilon))$ toward $\tilde{\kappa}$ follows. □

2.3. Consistency condition. Application to the polynomial approximation

We are now interested in determining conditions ensuring that the limit u of the scaled three-dimensional displacements when ε goes to zero and the limit w of their projections coincide. In such a case, the internal approximation is said to be consistent.

THEOREM 2.2. Let u (resp.w) be the solution of problem $(P_{KL}(V))$ (resp. $(P_{KL}(W))$. They coincide for any linear form $l(.)$ in the dual space of V_{KL} if and only if the following two conditions are satisfied:

i) $V_{KL} \subset W$,

ii) $e_{\gamma\gamma}(\eta) - x_3\, \Delta\eta_3 \in \bar{D}_3$ for all $\eta_1, \eta_2 \in H^1_{\gamma_0}(\omega), \eta_3 \in H^2_{\gamma_0}(\omega)$.

Proof. Let us establish the necessary conditions. Let u be an abitrary element in V_{KL}. Define l on V_{KL} by $l(v) := a_0^\star(u,v)$. By construction, u solves $(P_{KL}(V))$. For u to solve problem $(P_{KL}(W))$ as well, it is necessary that $u \in W_{KL}$. Therefore condition *i)* is required. Notice that the equality $V_{KL} = W_{KL}$ then holds.

Define now w as the solution of problem $(P_{KL}(W))$ where $l(v) := a_0^\star(u,v)$. If $u = w$, necessarily $a_0^\star(u,v) = b_0^\star(u,v)$ for all $v \in V_{KL}$. Comparing definitions (3) and (10) of $a_0^\star$ and $b_0^\star$, we see that necessarily

$$\int_\Omega \Big((I-Q)e_{\gamma\gamma}(u)\Big)\, e_{\gamma\gamma}(v) = 0 \text{ for all } v \in V_{KL}.$$

This means that $e_{\gamma\gamma}(u)$ belongs to the closure of D_3. Conversely, if conditions *i)* and *ii)* are satisfied, then $V_{KL} = W_{KL}$ and $a_0^\star = b_0^\star$. □

In Section 3 we will focus on the polynomial approximation. Recall that in that case $W = P_{m,n}(H^1_{\gamma_0}(\omega))$. Corollary (2.3) belows gives the minimal degrees required for consistency. A noteworthy remark is that $m = 1,\ n = 0$ is not enough although the solution u of the limit problem belongs to $P_{1,0}(H^1_{\gamma_0}(\omega))$.

COROLLARY 2.3. If $W = P_{m,n}(H^1_{\gamma_0}(\omega))$, the internal approximation is consistent if and only if $m \geq 1, n \geq 2$.

Proof. Condition *i)* requires that $m \geq 1, n \geq 0$. The closure of D_3 is $P_{n-1}(L^2(\omega))$. Therefore condition *ii)* requires that $n - 1 \geq 1$. □

3. The lower degrees models

Our aim in this section is to compare the Reissner-Minlin model with the first models that can be obtained by internal approximation on polynomial spaces. For the sake of brevity, we assume from now on that the horizontal components of the body forces are odd with respect to the vertical variable, and that the vertical component is even. Similarly, we assume that $g_\alpha^{\varepsilon+} = -g_\alpha^{\varepsilon-}$, $\alpha = 1,2$ and $g_3^{\varepsilon+} = g_3^{\varepsilon-}$. Then, it is an easy matter to prove that the solution of the three-dimensional system of elasticity has odd horizontal components and that its vertical component is even.

3.1. The Reissner-Minlin model

In the engineering literature, the Reissner-Minlin model is used to account for transverse shear in moderarely thin plates. The classical construction is performed on the strong formulation of the system of elasticity, which reads:

$$\begin{aligned} -\partial_j \Sigma_{ij} &= f_i^\varepsilon \quad \text{in} \quad \Omega^\varepsilon && (15)\\ \Sigma_{ij} &= 2\mu\, e_{ij}(U) + \lambda\, tr(e(U))\, \delta_{ij} && (16)\\ \Sigma_{i\beta}\, n_\beta &= 0 \text{ on } \Gamma_1^\varepsilon,\ \Sigma_{i3} = \pm\, g_i^\varepsilon \text{ on } \omega^{\pm\varepsilon}, && (17)\\ U &= 0 \text{ on } \Gamma_0^\varepsilon, && (18)\end{aligned}$$

where $\Gamma_1^\varepsilon := \{\partial\omega - \gamma_0\}\times] - \varepsilon, \varepsilon[$. Suppose that

i) the component Σ_{33} of the stress tensor is zero. Replacing in the constitutive law (16), one gets the reduced constitutive law

$$\Sigma_{\alpha\beta} = 2\mu\, e_{\alpha\beta}(U) + \lambda^\star\, e_{\gamma\gamma}(U)\, \delta_{\alpha\beta}.$$

ii) the transverse components $\Sigma_{\alpha 3}, \alpha = 1, 2$ satisfy the modified constitutive law

$$\Sigma_{\alpha 3} = 2\mu\, k\, e_{\alpha 3}(U)$$

where k is a so-called shear correction factor whose value is not clearly determined. Usually k is supposed to be equal to 2/3, 5/6, see [15], [17], [18]. For a determination of k based on optimality criteria and leading to expressions incorporating Young's modulus, see [3].

iii) the horizontal components of U are affine with respect to y_3 and the vertical component is constant with respect to y_3. From the symmetry hypotheses, we get that, for all $(x_1, x_2, y_3) \in \Omega^\varepsilon$,

$$U_\alpha(x_1, x_2, y_3) = y_3\, \psi_\alpha(x_1, x_2),\ U_3(x_1, x_2, y_3) = \zeta_3(x_1, x_2)$$

where $(\psi_1, \psi_2, \zeta_3) \in (H^1_{\gamma_0}(\omega))^3$.

These ad hoc hypotheses are clearly incompatible, since, for instance, plugging $e_{33}(U) = 0$, which is a consequence of iii), in (16) would lead to the "wrong" reduced constitutive law $\Sigma_{\alpha\beta} = 2\mu\, e_{\alpha\beta}(U) + \lambda e_{\gamma\gamma}(U)\, \delta_{\alpha\beta}$. Then, by mere integration of the equilibrium equations (15) and of the reduced constitutive law, one obtains a system of partial differential equations whose variational form is the following:

Find $(\psi_1, \psi_2, \zeta_3)$ in $(H^1_{\gamma_0}(\omega))^3$ such that for any (χ_1, χ_2, η_3) in $(H^1_{\gamma_0}(\omega))^3$

$$\frac{2}{3}\varepsilon^3 \int_\omega (2\mu\, e_{\alpha\beta}(\psi)\, e_{\alpha\beta}(\chi) + \lambda^\star\, e_{\gamma\gamma}(\psi)\, e_{\gamma\gamma}(\chi))\, dx_1 dx_2 \;+$$

$$2\mu\, k\, \varepsilon \int_\omega (\psi + \nabla\zeta)\cdot(\chi + \nabla\eta)\, dx_1 dx_2 = \int_\omega (s_\alpha \chi_\alpha + p_3 \eta_3)\, dx_1 dx_2, \qquad (19)$$

where $s_\alpha := \int_{-\varepsilon}^{+\varepsilon} y_3 f_\alpha^\varepsilon dy_3\ + g_\alpha^{\varepsilon +} - g_\alpha^{\varepsilon -}$ and $p_3 := \int_{-\varepsilon}^{+\varepsilon} f_3^\varepsilon dy_3\ + g_3^{\varepsilon +} + g_3^{\varepsilon -}$.

It is well known that this model is consistent with the Kirchhoff-Love model for any k ([11], [19]). Its qualitative properties, in particular its boundary layer, as well as its numerical approximation have been studied in several works, among them [4], [2].

3.2. The $P_{1,2}$ model

In the Reissner-Mindlin model, the displacement belongs to $P_{1,0}(H^1_{\gamma_0}(\omega))$. The abstract results obtained in Section 2 explain why the naive idea of projecting on $W = P_{1,0}(H^1_{\gamma_0}(\omega))$ with respect to the bilinear form of elasticity to get a simple model does not lead to satisfactory results. The above Subsection has been devoted to the description of a trick which allows, with no mathematical justification, to project on this space in spite of its lack of consistency.

In this Subsection our aim is to identify the lower degrees models that are consistent with the asymptotic behaviour. Let us examine the $P_{1,2}$ case. With the notations of Section 2, we are searching for the solution $w(\varepsilon)$ of $(P(\varepsilon, W))$ when $W = P_{1,2}(H^1_{\gamma_0}(\omega))$. The computations and the interpretation of the results are easier when using the natural Legendre basis for polynomials of the single variable x_3 rather than the basis of monomials ([1], [7]). To avoid a subscript in the computations, we change the name of the vertical variable. From now on, z stands for x_3. The Legendre polynomials L_n are orthogonal in $L^2(-1,1)$ and are normalized by $L_n(1) = 1$. The solution $w(\varepsilon)$ of $(P(\varepsilon, W))$ with $W = P_{1,2}(H^1_{\gamma_0}(\omega))$ can be decomposed as

$$w_\alpha(\varepsilon)(x_1, x_2, z) = w^1_\alpha(\varepsilon)\, L_1(z), \tag{20}$$

$$w_3(\varepsilon)(x_1, x_2, z) = w^0_3(\varepsilon)\, L_0(z) + w^2_3(\varepsilon)\, L_2(z), \tag{21}$$

where $w^1_\alpha(\varepsilon)$, $w^0_3(\varepsilon)$, $w^2_3(\varepsilon)$ belong to $H^1_{\gamma_0}(\omega)$. Similarly any v in $P_{1,2}(H^1_{\gamma_0}(\omega))$ with odd horizontal components and an even vertical component can be written as

$$v_\alpha(x_1, x_2, z) = v^1_\alpha\, L_1(z), \tag{22}$$

$$v_3(x_1, x_2, z) = v^0_3\, L_0(z) + v^2_3\, L_2(z), \tag{23}$$

where v^1_α, v^0_3, v^2_3 belong to $H^1_{\gamma_0}(\omega)$. We now let for w, v in $(H^1(\omega))^3$

$$\bar{a}_0(w, v) = [2\mu\, e_{\alpha\beta}(w) + \lambda\, e_{\gamma\gamma}(w)\, \delta_{\alpha\beta},\, e_{\alpha\beta}(v)] \tag{24}$$

where $[\cdot,\cdot]$ denotes the inner product in $L^2(\omega)$, and

$$\bar{a}^\star_0(w, v) = [2\mu\, e_{\alpha\beta}(w) + \lambda^\star\, e_{\gamma\gamma}(w)\, \delta_{\alpha\beta},\, e_{\alpha\beta}(v)]. \tag{25}$$

Then, it is easy to prove the following lemma.

Lemma 3.1. The functions $w^1_\alpha(\varepsilon)$, $w^0_3(\varepsilon)$, $w^2_3(\varepsilon)$ solve the variational system

$$\begin{aligned}\varepsilon^{-2}\,\{2\mu\,[\partial_\alpha w^0_3(\varepsilon) + w^1_\alpha(\varepsilon),\, \partial_\alpha v^0_3 + v^1_\alpha] + 2\lambda\,[w^2_3(\varepsilon),\, \partial_\alpha v^1_\alpha]\}\\ +\tfrac{2}{3}\bar{a}_0(w^1(\varepsilon), v^1) = l\,(v^1_\alpha\, L_1,\, v^0_3),\end{aligned} \tag{26}$$

$$\begin{aligned}\varepsilon^{-4}\, 6\,(2\mu+\lambda)\,[w^2_3(\varepsilon),\, v^2_3] + \varepsilon^{-2}\,\{\tfrac{2\mu}{5}[\partial_\alpha w^2_3(\varepsilon),\, \partial_\alpha v^2_3] + 2\lambda\,[\partial_\alpha w^1_\alpha(\varepsilon),\, v^2_3]\}\\ = l\,(0, 0, v^2_3\, L_2).\end{aligned} \tag{27}$$

We describe in the sequel a formal procedure, still to be justified by means of an error estimate, that links model (26, 27) to the Reissner-Mindlin model.

Let us first define

$$p_3^0 := \int_{-1}^{1} f_3(z)\, dz + g_3^+ + g_3^-, \tag{28}$$

$$s_\alpha^1 := \int_{-1}^{1} z\, f_\alpha\, dz + g_\alpha^+ - g_\alpha^-,\ \alpha = 1, 2, \tag{29}$$

$$p_3^2 := \int_{-1}^{1} L_2(z) f_3(z)\, dz + g_3^+ + g_3^-. \tag{30}$$

In (27), let us express the unknown function $w_3^2(\varepsilon)$ in terms of the unknown function $w_\alpha^1(\varepsilon)$. We get, upon multiplication by ε^4,

$$\begin{aligned}\varepsilon^2\, \tfrac{2\mu}{5}[\partial_\alpha w_3^2(\varepsilon),\, \partial_\alpha v_3^2] + 6\,(2\mu+\lambda)\,[w_3^2(\varepsilon),\, v_3^2] = \\ -2\lambda\, \varepsilon^2\, [\partial_\alpha w_\alpha^1(\varepsilon),\, v_3^2] + \varepsilon^4\, [p_3^2,\, v_3^2].\end{aligned} \tag{31}$$

Equation (31) is a singular perturbation problem. Ignoring the boundary conditions, we expect that an approximation of $w_3^2(\varepsilon)$ can be obtained by cancelling the term with partial derivatives in the left-hand side (see [10], [14]). Therefore, we approximate $w_3^2(\varepsilon)$ which belongs to $H^1_{\gamma_0}(\omega)$ by

$$\tilde{w}_3^2(\varepsilon) := -\frac{\lambda}{3(2\mu+\lambda)}\, \varepsilon^2\, \partial_\alpha w_\alpha^1(\varepsilon) + \varepsilon^4\, \frac{1}{6(2\mu+\lambda)} p_3^2. \tag{32}$$

Note that we can only say that $\tilde{w}_3^2(\varepsilon)$ belongs to $L^2(\omega)$.

STATEMENT 3.2. If the equality $w_3^2(\varepsilon) = \tilde{w}_3^2(\varepsilon)$ were true, then the functions $w_\alpha^1(\varepsilon)$, $w_3^0(\varepsilon)$ would solve the variational system

$$\begin{aligned}\tfrac{2}{3}\bar{a}_0^\star(w^1(\varepsilon), v^1) + \varepsilon^{-2}\, 2\mu\, [\partial_\alpha w_3^0(\varepsilon) + w_\alpha^1(\varepsilon),\, \partial_\alpha v_3^0 + v_\alpha^1] = \\ [s_\alpha, v_\alpha^1] + [p_3^0, v_3^0] - \varepsilon^2\, \tfrac{\lambda}{3(2\mu+\lambda)}[p_3^2,\, \partial_\alpha v_\alpha^1].\end{aligned} \tag{33}$$

Proof. The result is easily obtained by replacing $w_3^2(\varepsilon)$ in (26) by $\tilde{w}_3^2(\varepsilon)$ and by checking that $l(v_\alpha^1\, L_1,\, v_3^0) = [s_\alpha, v_\alpha^1] + [p_3^0, v_3^0]$. □

Comment:

- If the ε^2 loading term is dropped in the above equation, the Reissner-Mindlin model with $k = 1$ is recovered. The unknown $w_3^0(\varepsilon)$ of the Reissner-Mindlin model obtained that way is a perturbation of the mean value of $w_3(\varepsilon)$ which, by definition, is quadratic in z.
- The question arises to know whether increasing the degrees of the polynomials in the internal approximation leads to other choices of k. For example, using a similar statement to simplify the equations of model $P_{3,2}$, it is again possible to recover a Reissner-Mindlin model. In this case, k is equal to $\frac{5}{6}$.

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