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BOUNDARY CONTROL AND DYNAMICAL RECONSTRUCTION OF VECTOR FIELDS (THE BC-METHOD)

M.I. BELISHEV

St. Petersbourg Department of Steklov
Mathematical Institute
E-mail: belishev@pdmi.ras.ru

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Abstract

An approach to the dynamical inverse problems (IP's) based upon their relations to the Boundary Control Theory (the so-called BC-method) is developed. The method is applied to the problem of reconstruction of a vector field given on a Riemannian manifold via the response operator (the dynamical Dirichlet-to-Neumann map). A peculiarity of the case under consideration is that the operator which governs an evolution of the corresponding dynamical system is nonselfadjoint. The paper announces the results and gives a brief description of technique of the BC-method.

1 Introduction

An approach to the dynamical inverse problems (IP's) based upon their relations to the Boundary Control Theory (the so-called BC-method) is developed. The method is applied to the problem of reconstruction of a vector field given on a Riemannian manifold via the response operator (the dynamical Dirichlet-to-Neumann map). A peculiarity of the case under consideration is that the operator which governs an evolution of the corresponding dynamical system is nonselfadjoint. The paper announces the results and gives a brief description of technique of the BC-method.

2 Statement of the Problem

Let (Ω, g) be a compact $C^\infty-$smooth Riemannian manifold with the border Γ ; $\dim\Omega = n \geq 2$; $b = b^k\frac{\partial}{\partial x^k}$ be a smooth vector field on Ω. Consider *the dynamical system*

$$u_{tt} - \triangle_g u - bu = 0 \text{ in } Q^T, \tag{1}$$

$$u \mid_{t=0} = u_t \mid_{t=0} = 0, \tag{2}$$

$$u \mid_{\Sigma^T} = f, \tag{3}$$

$\left(Q^T := \Omega \times (0,T)\,;\ \Sigma^T := \Gamma \times [0,T]\right)$ with a Dirichlet boundary control $f \in \mathcal{F}^T := L_2\left(\Sigma^T\right)$; let $u = u^f(x,t)$ be a solution ("wave").

The map $R^T : \mathcal{F}^T \to \mathcal{F}^T, Dom R^T = \left\{f \in H^1\left(\Sigma^T\right) : f\mid_{t=0} = 0\right\}$,

$$R^T f := \frac{\partial u^f}{\partial \nu} \mid_{\Sigma^T}$$

(ν is an outward normal to Γ) is said to be *a response operator* of the system. *The dynamical IP is to recover the field b in Ω via given response operator R^T*.

3 Main result

Let $l_{x,\alpha}$ be a geodesic starting from $x \in \Omega$ in direction $\alpha \in S^{n-1}$. Suppose, that the manifold (Ω, g) satisfies *the nontrapping condition* : there exists positive $T_0 < \infty$ such that for any $(x,\alpha) \in \Omega \times S^{n-1}$ the geodesic $l_{x,\alpha}$ reaches the border Γ on time $t_{x,\alpha} \leq T_0$.

Theorem 1 *The operator R^{2T} given for any $T > T_0$ determines b in Ω uniquely.*

Moreover, the BC-method proposes an efficient procedure recovering a field.

As important example of the nontrapping manifold, the bounded domain $\Omega \subset \mathbb{E}^n$ (with Euclidean metric) can be mentioned. Thus, the approach permits to recover arbitrary vector fields in $\mathbb{E}^n$ via dynamical inverse data.

4 Tools of the method

The approach is based upon the scheme of the paper[1], and uses standard tools of the multidimensional BC-method : (i) the semigeodesical coordinates considering "in the large" on Ω, (ii) a controllability of the dynamical system, and (iii) the Geometrical Optics (propagation of discontinuities of waves u^f) (see [4],[5]).

(i) Let $\omega \subset \Omega$ be a separation set (cut locus) in Ω with respect to Γ; $\omega = \overline{\omega}$, *vol* $\omega = 0$. For any $x \in \Omega \setminus \omega$ the pair $(\gamma(x), \tau(x)) \in \Gamma \times [0, T_*]$: $\tau(x) := dist\,(x, \Gamma)$, $\gamma(x)$ is such that $dist\,(x, \gamma(x)) = \tau(x)$ ($T_* := \max_\Omega \tau(\cdot)$) is said to be *the semigeodesical coordinates* of the point x.

(ii) The set of waves $\mathcal{U}^T := \left\{u^f(\cdot, T) : f \in \mathcal{F}^T\right\}$ is called *reachable* (on the moment $t = T$). The procedure recovering a vector field in Ω is based upon the following result : if the manifold satisfies the nontrapping condition then for any $T > T_0$ the equality

$$\mathcal{U}^T = L_2(\Omega)$$

holds, i.e. the system (1) - (3) is *exactly controllable* for large enough T (see [3]).

(iii) The controllability opens a way to visualize the waves. The so-called *visualizing operator* $V^T : \mathcal{F}^T \to \mathcal{F}^T$ is introduced as the map

$$\left(V^T f\right)(\gamma, \tau) := u^f\left(x(\gamma, \tau), T\right),$$

where $x(\gamma, \tau) \in \Omega \setminus \omega$ is a point with semigeodesical coordinates γ, τ . Using the Geometrical Optics Relations one can express the operator V^T via inverse data, i.e. the operator R^{2T} (see [5]).

Possessing operator V^T one can recover a sufficiently rich set of waves u^f , and then find the field b from the wave equation (1) written in semigeodesical coordinates.

5 The strengthening for $\Omega \subset \mathbb{E}^n$

In the case of bounded $\Omega \subset \mathbb{E}^n$ (with Euclidean metric g) the result of Theorem 1 may be strengthened as follows.

Let $\Omega^\xi := \{x \in \Omega : dist\,(x, \Gamma) < \xi\}$, $0 < \xi \leq T$ be a family of subdomains in Ω being captured by waves moving from Γ on the moment $t = \xi$; introduce also the subdomain $B^T \subset \Omega$ determined by the condition : $x \in B^T$ iff for any direction $\alpha \in S^{n-1}$ at least one of the straight rays $l_{x,\alpha}$ and $l_{x,-\alpha}$ reaches the boundary Γ at time which doesn't exceed T.

The following results describe a character of controllability of the system (1)-(3) in captured domains :

(i) (D.Russell, D.Tataru) for any $T > 0$ the relation

$$clos\, \mathcal{U}^T = L_2\left(\Omega^T\right)$$

holds, i.e. the system is *approximately controllable* on any moment ([6], [7]; see also [5]);

(ii) (C.Bardos) the equality

$$\left\{u^f(\cdot, T)\,|_{B^T}\colon f \in \mathcal{F}^T\right\} = L_2\left(B^T\right)$$

is valid for any $T > 0$, i.e. the system turns out to be *exactly controllable* in B^T (see [2]).

Using these facts the BC - method leads to the following.

Theorem 2 *Let $\Omega \subset \mathbb{E}^n$ and $T > 0$ are such that the inclusion $\Omega^\xi \subset B^T$ holds for some $\xi \leq T$; then the response operator R^{2T} determines the field b in Ω^ξ .*

6 Hypothesis

There are reasons to assume that results of Theorems 1,2 are far from optimal. Indeed, in the case of the wave equation $u_{tt} - \triangle u + qu = 0$ with a *scalar potential* q the BC - method recovers $q\,|_{\Omega^T}$ via given R^{2T} for *any* $T > 0$. Our hypothesis is that the same is valid for vector fields : for any $T > 0$ the response operator R^{2T} determines $b\,|_{\Omega^T}$ uniquely.

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