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SOME APPLICATIONS OF BV FUNCTIONS IN OPTIMAL CONTROL AND CALCULUS OF VARIATIONS[1]

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Abstract

In this paper we discuss the use of bounded variation functions in the study of some optimal control problems as well as in the calculus of variations. The bounded variation functions are well adapted to the study of parameter identification problems, such as the coefficients of an elliptic or parabolic operator. These functions are also convenient for the image recovery problems. These problems are well formulated in the space $BV(\Omega)$ in the sense that they have a solution under reasonable assumptions. The numerical approximation of these problems is interesting because of the non separability of the space $BV(\Omega)$. A very surprising fact is the influence of the chosen norm, among all the possible equivalent norms in $BV(\Omega)$, on the convergence of the numerical approximations. Indeed the approximation by piecewise constant functions fails if the norm is not properly chosen.

1 A problem of calculus of variations

The image enhancement or image recovery problems, which have recently received a considerable amount of attention, are an example of problems of calculus of variations that can be studied in the space of the functions of bounded variation. To briefly describe this problem let z denote the grey values of an image which extends in a two dimensional domain Ω. The values of z correspond to the values of the initial image u on Ω corrupted by noise and blurring. This relationship is expressed as

$$z = Tu + \eta,$$

where η represents noise and T is a linear operator that describes the blurring. To recover u from z a regularized least squares functional is minimized:

$$(\mathcal{CV}) \qquad \min \int_\Omega |Tu - z|^2 dx + \beta \int_\Omega |\nabla u|,$$

where $\beta > 0$ is the positive regularization parameter and $\int_\Omega |\nabla u|$ denotes the BV-seminorm. The notation $\int_\Omega |\nabla u|$ is formal here and will be made precise in Section 3 where we summarize properties of the space of $BV(\Omega)$ functions that are relevant to the theory of this research. For the moment it suffices to recall that functions in $BV(\Omega)$ are not necessarily continuous. The use of BV-seminorms was, to our knowledge, proposed originally in Osher and Rudin [12] with the goal to dampen excessive oscillations in z without smoothing sharp edges and corners that are contained in the original image u. This would occur if the BV-seminorm was replaced by $\int_\Omega |\nabla u|^2 dx$, for example. Let us point out that besides some technical difficulties related to the space of BV-functions, the main theoretical and numerical difficulty relates to the lack of differentiability of the cost functional. The image reconstruction problem with BV-regularization was further investigated in Dobson and Santosa [5], Ito and Kunisch [8], Li and Santosa [9] and Rudin et al. [14], for example. The contributions in [5], [9] and [14] using variations of the formulation in $(\mathcal{CV})$ provide substantial evidence that BV-regularization can be a very effective numerical technique for image enhancement. In [8] augmented Lagrangian algorithms are analyzed for $(\mathcal{CV})$ under the assumption that an additional regularization term of the form $\gamma \int_\Omega |\nabla u|^2 dx$, with γ small with respect to β, is introduced and an active set strategy based on a duality technique is proposed. In Acar and Vogel [1] BV-regularization is studied and the problem of nondifferentiability is circumvented by replacing $\int_\Omega |\nabla u|$ by $\int_\Omega \sqrt{|\nabla u|^2 + \epsilon}$, for $\epsilon > 0$. Several theoretical aspects have been left open in the abovementioned papers some of which will be discussed here.

2 An optimal control problem

As a special example of optimal control problems we are going to discuss the following problem of identification of parameters. Let us consider the state equation

$$\begin{cases} -div(u(x)\nabla y(x)) = f(x) & \text{in } \Omega \\ y(x) = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where Γ is the boundary of Ω, $f \in L^2(\Omega)$ and u is the parameter to be identified. We consider the control problem

$$(\mathcal{OC}) \qquad \min_{u \in K \cap BV(\Omega)} \int_\Omega |y_u(x) - y_d(x)|^2 dx + \beta \int_\Omega |\nabla u|,$$

where $y_d \in L^2(\Omega)$ and $\beta > 0$ are given. With y_u we denote the solution of (1) and

$$K = \{u \in L^\infty(\Omega) : u_{min} \leq u(x) \leq u_{max}\},$$

where $0 < u_{min} < u_{max} < +\infty$.

It is well known that in general problem $(\mathcal{OC})$ has no solution if $\beta = 0$; see Murat [10]. The fact that K is bounded in $L^\infty(\Omega)$ is not sufficient to assure the existence of a solution. Additional regularity of u, allowing a compactness argument is necessary to guarantee existence. Previously this was achieved by including the term $\beta \int_\Omega |\nabla u(x)|^p dx$ in the cost functional, $p > 1$, in place of the BV-seminorm considered in the above formulation. Minimization in the Sobolev spaces $W^{1,p}(\Omega)$ was studied by Banks and Kunisch [2]. Alternatively the choice $\beta = 0$ was used together with the constraint $|\nabla u(x)| \leq c$; see Casas [3]. In the latter case, the Sobolev space $W^{1,\infty}(\Omega)$ is taken for the parameter u. The use of Sobolev spaces can lead to overregularization of the function u. For instance if piecewise continuous functions have to be identified, the space $BV(\Omega)$ seems to be more appropriate than Sobolev spaces because one can obtain the necessary compactness to prove existence of a solution and the induced regularity for the parameter u is not too restrictive, since it includes piecewise continuous functions. For a previous paper dealing with BV-functions to identify parameters, the reader is referred as to Gutman [7]. In that paper, theoretical questions remained open, which will be studied in this paper.

3 The space $BV(\Omega)$

Let Ω be an open bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ. With $M(\Omega)$ we denote the space of real Borel measures in Ω, which is a Banach space when endowed with the norm

$$\|\mu\|_{M(\Omega)} = |\mu|(\Omega) = \int_\Omega |\mu|, \tag{2}$$

$|\mu| = \mu^+ + \mu^-$ being the total variation measure associated to μ and then $|\mu|(\Omega)$ is the total variation of μ in Ω. It is well known that $M(\Omega)$ is the dual of the space of the continuous functions in Ω vanishing on Γ, $C_0(\Omega)$, and

$$\int_\Omega |\mu| = \sup \left\{ \int_\Omega v(x) d\mu(x),\ v \in C_0(\Omega),\ \|v\|_\infty \leq 1 \right\}. \tag{3}$$

See, for instance, Rudin [13] for details.

Now let us denote by $BV(\Omega)$ the space of functions of bounded variation in Ω. Let us remind that a function $u \in L^1(\Omega)$ is said to be a bounded variation function if $\partial_{x_i} u \in M(\Omega)$ for every $1 \le i \le n$, where the derivatives are understood in the distributional sense, or equivalently u is a bounded variation function if for every $1 \le i \le n$ we have

$$\sup \left\{ \int_\Omega u \partial_{x_i} v dx,\ v \in C_0^\infty(\Omega),\ \|v\|_\infty \le 1 \right\} = \int_\Omega |\partial_{x_i} u| \ < +\infty. \tag{4}$$

Setting

$$\|u\|_{BV(\Omega)} = \int_\Omega |u| dx + \sum_{i=1}^n \int_\Omega |\partial_{x_i} u| = \int_\Omega |u| dx + \int_\Omega |\nabla u|, \tag{5}$$

we have that $BV(\Omega)$ is a Banach space. Let us note that (5) can be written as

$$\|u\|_{BV(\Omega)} = \int_\Omega |u| dx$$

$$+ \sup \left\{ \int_\Omega u(x) \text{div } v(x) dx,\ v \in C_0^\infty(\Omega)^n,\ |v(x)|_\infty \le 1,\ x \in \Omega \right\}$$

where $|\cdot|_\infty$ denotes the l_∞ norm in $\mathbb{R}^n$:

$$|x|_\infty = \max_{1 \le i \le n} |x_i|.$$

A different norm in $\mathbb{R}^n$ induces another norm on $BV(\Omega)$ that is equivalent to (5). For instance, a frequently used norm, mainly employed in the study of minimal surfaces, is the Euclidean norm, see Giusti [6]. The reader is referred to Temam [15] for the use of the norm (5).

The proof of the next properties of $BV(\Omega)$ can be found in [6].

Proposition 3.1 *1) If $\{u_j\}_{j=1}^\infty \subset BV(\Omega)$ and $u_j \to u$ in $L^1(\Omega)$, then*

$$\int_\Omega |\partial_{x_i} u| \le \liminf_{j \to \infty} \int_\Omega |\partial_{x_i} u_j|. \tag{6}$$

2) For every $u \in BV(\Omega) \cap L^r(\Omega), r \in [1, +\infty)$, there exists a sequence $\{u_j\}_{j=1}^\infty \subset C^\infty(\bar{\Omega})$ such that

$$\lim_{j \to \infty} \int_\Omega |u - u_j|^r dx = 0 \ \ \text{and} \ \ \lim_{j \to \infty} \int_\Omega |\partial_{x_i} u_j| = \int_\Omega |\partial_{x_i} u|,\ 1 \le i \le n. \tag{7}$$

3) For every bounded sequence $\{u_j\}_{j=1}^\infty \subset BV(\Omega)$ there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ and a function $u \in BV(\Omega)$ such that $u_{j_k} \to u$ in $L^1(\Omega)$.

Concerning *2)* one can generalize Giusti [6, Theorem 1.17] from $r = 1$ to $r \in [1, +\infty)$ to assert that $C^\infty(\Omega) \cap L^r(\Omega)$ approximates $BV(\Omega) \cap L^r(\Omega)$ in the sense of (7). This implies that $W^{1,1}(\Omega) \cap L^r(\Omega)$ approximates $BV(\Omega) \cap L^r(\Omega)$ in the sense of (7) as well. Since Γ is Lipschitz, $C^\infty(\bar{\Omega})$ is dense in $W^{1,1}(\Omega) \cap L^r(\Omega)$ (see, for instance, Nečas [11]), and we conclude that $C^\infty(\bar{\Omega})$ approximates $BV(\Omega) \cap L^r(\Omega)$ as claimed.

Let us remark that (7) can not be modified to have $\|\partial_{x_i} u_j - \partial_{x_i} u\|_{M(\Omega)} \to 0$. Indeed the closure of $C^\infty(\bar{\Omega})$ in the norm (5) is the Sobolev space $W^{1,1}(\Omega)$, which is obviously a strict subspace of $BV(\Omega)$, with

$$\|u\|_{W^{1,1}(\Omega)} = \|u\|_{BV(\Omega)} \text{ for all } u \in W^{1,1}(\Omega). \tag{8}$$

4 Existence of solution

In this section we study the existence of a solution for problem

$$\text{(P)} \qquad \min_{u \in K} F(u) = J(u) + \beta \int_\Omega |\nabla u|,$$

where $\beta > 0$, K is a convex closed subset of $L^p(\Omega)$ $(1 \le p < +\infty)$, and $J : K \longrightarrow \mathbb{R}$ is a weakly lower semicontinuous function. We also assume that $K \cap BV(\Omega)$ is nonempty and that J is bounded from below.

The following theorem provides the necessary conditions to guarantee the existence of a solution for (P).

Theorem 4.1 *Let us suppose that either K is bounded in $L^p(\Omega)$ or J is coercive in the following sense: If $\{u_j\}_{j=1}^\infty \subset K$, $\left\{\int_\Omega |\nabla u_j|\right\}_{j=1}^\infty$ is bounded and $\|u_j\|_{L^p(\Omega)} \to +\infty$, then $J(u_j) \to +\infty$. Then* (P) *has at least one solution $\bar{u}$. Moreover, if J is a strictly convex function, then the solution is unique.*

Proof: Let $\{u_j\}_{j=1}^\infty \subset K$ be a minimizing sequence, i.e. $F(u_j) \searrow \inf(\mathrm{P})$. Since J is bounded from below it follows that

$$\beta \int_\Omega |\nabla u_j| = F(u_j) - J(u_j) \le F(u_1) - C < +\infty, \text{ for all } j$$

and some appropriately chosen constant C, and hence $\left\{\int_\Omega |\nabla u_j|\right\}_{j=1}^\infty$ is bounded. If K is bounded or if J satisfies the coercivity assumption of the above statement, we have that $\{u_j\}_{j=1}^\infty$ is bounded in $L^p(\Omega)$ as well.

We have proved that $\{u_j\}_{j=1}^\infty$ is bounded in $BV(\Omega) \cap L^p(\Omega)$. Therefore by using Proposition 3.1, we can take a subsequence that we will denote in the same way, and an element $\bar{u} \in BV(\Omega) \cap L^p(\Omega)$ such that

$$\lim_{j \to \infty} u_j = \bar{u} \text{ weakly in } L^p(\Omega) \quad \text{and} \quad \int_\Omega |\nabla \bar{u}| \le \liminf_{j \to \infty} \int_\Omega |\nabla u_j|.$$

Now using the weak lower semicontinuity of J, we get

$$F(\bar{u}) \le \liminf_{j \to \infty} F(u_j) = \inf(\mathrm{P}).$$

Since K is closed in the weak topology of $L^p(\Omega)$, $\bar{u} \in K$, which along with the previous inequality proves that $\bar{u}$ is a solution of (P).

The uniqueness of the solution under the strict convexity of J follows by using the classical argument.

Let us remark that the continuity of J is enough to conclude the existence of a solution if K is a bounded subset of $L^\infty(\Omega)$. Indeed, the boundedness of $\{u_j\}_{j=1}^\infty$ in $BV(\Omega)$ implies (Proposition 3.1) the existence of a subsequence, denoted in the same way, and an element $\bar{u} \in BV(\Omega)$ such that $u_j \to \bar{u}$ strongly in $L^1(\Omega)$. This together with the boundedness of $\{u_j\}_{j=1}^\infty$ in $L^\infty(\Omega)$ implies strong convergence $u_j \to \bar{u}$ in $L^p(\Omega)$, so that $J(u_j) \to J(\bar{u})$.

It is an easy exercise to prove that the above theorem can be applied to the study of problems $(\mathcal{CV})$ and $(\mathcal{OC})$ to deduce the existence of a solution.

5 Approximation of Problem (P)

Two issues are going to be addressed in this section. The first is concerned with the approximation of the space $BV(\Omega)$ by finite dimensional spaces. Secondly we will consider the approximation of problem (P). To deal with the approximation of $BV(\Omega)$ we introduce a decomposition of Ω in subdomains. Since Ω is bounded we can take for every $1 \le k \le n$

$$a_k = \min\{x_k : x = (x_j)_{j=1}^n \in \bar{\Omega}\} b_k = \max\{x_k : x = (x_j)_{j=1}^n \in \bar{\Omega}\}. \tag{9}$$

Now we define

$$D = \prod_{k=1}^n (a_k, b_k) \supset \Omega. \tag{10}$$

Every interval $[a_k, b_k]$ is divided in m_k subintervals:

$$a_k = t_{k0} < t_{k1} < \cdots < t_{km_k} = b_k,$$

and we take

$$D_{k_1 \cdots k_n} = (t_{1k_1-1}, t_{1k_1}) \times \cdots \times (t_{nk_n-1}, t_{nk_n}).$$

In order to simplify the notations, we denote $\tilde{m} = m_1 \times \cdots \times m_n$ and rename the $\tilde{m}$ parallelepipeds $D_{k_1 \cdots k_n}$ as D_j, $1 \le j \le \tilde{m}$. Furthermore we put

$$D_j = \prod_{k=1}^n (a_{kj}, b_{kj}), \tag{11}$$

where every interval (a_{kj}, b_{kj}) corresponds to some interval (t_{ki_k-1}, t_{ki_k}). Thus we have

$$\bar{D} = \bigcup_{j=1}^{\tilde{m}} \bar{D}_j \quad \text{and} \quad D_i \bigcap D_j = \emptyset \text{ if } i \neq j. \tag{12}$$

Let us notice that some of the parallelepipeds D_j can have empty intersection with Ω. Let us enumerate the family of parallelepipeds $\{D_j\}_{j=1}^{\tilde{m}}$ in such a way that $D_j \cap \Omega \neq \emptyset$ for $1 \leq j \leq m$ and $D_j \cap \Omega = \emptyset$ if $j > m$, with $m \leq \tilde{m}$. Now we take $\Omega_j = D_j \cap \Omega$ for $j = 1, \cdots, m$. As a consequence of (12) we have

$$\bar{\Omega} = \bigcup_{j=1}^{\tilde{m}} \bar{\Omega}_j \quad \text{and} \quad \Omega_i \bigcap \Omega_j = \emptyset \text{ if } i \neq j. \tag{13}$$

In the one dimensional case the situation is much simpler:

$$\bar{\Omega} = [a,b] = \bigcup_{j=1}^{m} [a_j, b_j] = \bigcup_{j=1}^{m} \bar{\Omega}_j, a = a_1, \quad b = b_m, \quad a_j = b_{j-1}, \; 2 \leq j \leq m. \tag{14}$$

We associate with these domains $\{\Omega_j\}_{j=1}^m$ a finite dimensional space

$$V_m = \left\{ u_m = \sum_{j=1}^{m} u^j \chi_{\Omega_j} : u^j \in \mathbb{R}, \; 1 \leq j \leq m \right\},$$

where χ_{Ω_j} denotes the characteristic function of Ω_j. V_m is the space of piecewise constant functions associated with the partition $\{\Omega_j\}_{j=1}^m$ of Ω. The following theorem states that any element of V_m is a function of bounded variation and characterizes its total variation. Before stating the theorem we need to introduce some additional notation.

For $n > 1$ and each j we denote by $\partial\Omega_j$ the boundary of Ω_j and $\nu_j(x)$ stands for the outward normal vector (in the sense of the Euclidean scalar product) to $\partial\Omega_j$ at the point $x \in \partial\Omega_j$, which exists for almost every point of the boundary. We set $|\Omega_j|$ (resp. $|\partial\Omega_j|$) to indicate the n-dimensional (resp. $(n-1)$-dimensional) Lebesgue measure of Ω_j and with S_j we represent the $(n-1)$-measure on the manifold $\partial\Omega_j \cap \Omega$, so that $|\partial\Omega_j \cap \Omega| = S_j(\partial\Omega_j \cap \Omega)$.

Now given two domains Ω_i and Ω_j such that $|\partial\Omega_j \cap \partial\Omega_i| > 0$, we denote by S_{ij} the $(n-1)$-measure on the manifold $\partial\Omega_j \cap \partial\Omega_j$ and by ν_{ij} the unit normal vector to $\partial\Omega_i \cap \partial\Omega_j$ pointing from Ω_i into Ω_j. It is obvious that $\nu_{ij}(x) = \nu_i(x)$ for every $x \in \partial\Omega_i \cap \partial\Omega_j$ and that $\nu_{ij} = -\nu_{ji}$. The following theorem provides the total variation of the functions of V_m.

Theorem 5.1 *For every $m \in \mathbb{N}$ the inclusion $V_m \subset BV(\Omega)$ holds and for each $u_m \in V_m$*

$$\int_\Omega |\nabla u_m| = \begin{cases} \sum_{j=2}^{m} |u^j - u^{j-1}| & \text{if } \ n = 1, \\ \\ \sum_{i<j} |u^i - u^j||\partial\Omega_i \cap \partial\Omega_j| & \text{if } \ n > 1. \end{cases} \tag{15}$$

The next theorem establishes that $BV(\Omega)$ can be conveniently approximated by the spaces V_m. Let us define

$$h_m = \max\{b_{kj} - a_{kj} : 1 \leq j \leq m, \quad 1 \leq k \leq n\}. \tag{16}$$

Theorem 5.2 *Let us assume that $h_m \to 0$ when $m \to \infty$. Then for every $u \in BV(\Omega) \cap L^r(\Omega)$, $1 \leq r < +\infty$, we can find a sequence $\{u_m\}$, with $u_m \in V_m$, such that*

$$\lim_{m\to\infty} \int_\Omega |u - u_m|^r dx = 0 \quad \textit{and} \quad \lim_{m\to\infty} \int_\Omega |\nabla u_m| = \int_\Omega |\nabla u|. \tag{17}$$

Remark 5.3 As we mentioned in §1, sometimes the following norm is considered in $BV(\Omega)$

$$\|u\| = \int_\Omega |u(x)|dx + \int_\Omega |\nabla u|_2, \tag{18}$$

where

$$\int_\Omega |\nabla u|_2 = \sup\left\{ \int_\Omega u(x)\text{div } v(x)dx, \ v \in C_0^\infty(\Omega)^n, \ |v(x)|_2 \leq 1, \ x \in \Omega \right\}, \tag{19}$$

with $|\cdot|_2$ denoting the Euclidean norm in $\mathbb{R}^n$. Though (5) and (18) are equivalent norms, the approximation property (17) does not hold in general for the seminorm (19). Let us prove this fact. First we remark that

$$\int_\Omega |\nabla u|_2 \leq \sum_{k=1}^{n} \int_\Omega |\partial_{x_k} u| \quad \text{for all } \ u \in BV(\Omega), \tag{20}$$

the inequality being strict in general. This can be easily checked for the functions of $W^{1,1}(\Omega)$ because

$$|x|_2 < \sum_{k=1}^{n} |x^k| \quad \text{for every } x \in \mathbb{R}^n$$

assuming that x is not collinear with any vector e_k of the canonical base. Every function of $BV(\Omega)$ can be approximated by functions of $W^{1,1}(\Omega)$ in the sense of (7). The property

$$\int_\Omega |\nabla u|_2 \leq \liminf_{m\to\infty} \int_\Omega |\nabla u_m|dx$$

holds whenever $\{u_m\} \subset BV(\Omega)$ and converges to u in $L^1(\Omega)$, see Giusti [6]. This leads to (20), with the inequality being strict in general.

Now we show that the approximation property (17) does not hold, in general, for the seminorm of (19). Let us take $u \in BV(\Omega)$ such that (20) holds strictly and arguing by contradiction let us assume that there exist a sequence $\{u_m\}$, with $u_m \in V_m$, such that

$$\lim_{m\to\infty} \int_\Omega |u(x) - u_m(x)|dx = 0 \quad \text{and} \quad \lim_{m\to\infty} \int_\Omega |\nabla u_m|_2 = \int_\Omega |\nabla u|_2. \qquad (21)$$

We compute the total variation of u_m in the sense of (19). To do this we take $v \in C_0^\infty(\Omega)^n$, with $|v(x)|_2 \leq 1$ for every $x \in \Omega$. Then we have

$$\int_\Omega u_m(x)\text{div } v(x)dx = \sum_{j=1}^m u^j \int_{\Omega_j} \text{div } v(x)dx =$$

$$\sum_{j=1}^m u^j \int_{\partial\Omega_j} v(x)\nu_j(x)dS_j(x) = \sum_{i<j}(u^i - u^j) \int_{\partial\Omega_j\cap\partial\Omega_i} v(x)\nu_{ij}(x)dS_{ij}(x).$$

Taking the supremum in v, we get from the previous identity and (15) that

$$\int_\Omega |\nabla u_m|_2 = \sum_{i<j} |u_i - u_j||\partial\Omega_i \cap \partial\Omega_j| = \sum_{k=1}^n \int_\Omega |\partial_{x_k} u_m|. \qquad (22)$$

By (6), we deduce form (21) and (22) that

$$\sum_{k=1}^n \int_\Omega |\partial_{x_k} u| \leq \lim_{m\to\infty} \int_\Omega |\nabla u_m|_2 = \int_\Omega |\nabla u|_2 < \sum_{k=1}^n \int_\Omega |\partial_{x_k} u|,$$

which gives the desired contradiction.

The key point in the above proof is (22), which holds because the vectors ν_{ij} coincide with the vectors of the canonical base of $\mathbb{R}^n$. Consequently $|\nu_{ij}|_p = 1$ for every $1 \leq p \leq +\infty$, where $|\cdot|_p$ denotes the l_p norm in $\mathbb{R}^n$. In fact we have that

$$\int_\Omega |\nabla u|_{p'} = \sup\left\{\int_\Omega u(x)\text{div } v(x)dx,\ v \in C_0^\infty(\Omega)^n,\ |v(x)|_p \leq 1,\ x \in \Omega\right\} \leq$$

$$\int_\Omega |\nabla u|_1 = \sum_{k=1}^n \int_\Omega |\partial_{x_k} u|,$$

the inequality being strict in general. Here p' denotes the conjugate of p. Moreover

$$\int_\Omega |\nabla u_m|_p = \sum_{i<j} |u_i - u_j||\partial\Omega_i \cap \partial\Omega_j|, \quad \text{for every } 1 \leq p \leq \infty. \qquad (23)$$

Summarizing the discussion we found that among all the equivalent norms in $BV(\Omega)$ induced by the l_p norms of $\mathbb{R}^n$, only the norm (5) considered in this paper has the approximation property (17) in the spaces V_m. $\square$

We refer to[4] for additional results concerning the approximation of $BV(\Omega)$ by finite element subspaces.

Now we consider the approximation of problem (P). In the remaining part of this section V_m will denote a finite dimensional subspace of $BV(\Omega) \cap L^p(\Omega)$. For every m we take $K_m = K \cap V_m$ and we make the following assumption

Assumption (A) For every $u \in K$ there exists a sequence $\{u_m\}$, with $u_m \in K_m$, such that

$$\lim_{m\to\infty} \int_\Omega |u(x) - u_m(x)|^p dx = 0 \quad \text{and} \quad \lim_{m\to\infty} \int_\Omega |\nabla u_m| = \int_\Omega |\nabla u|. \tag{24}$$

If

$$K = \{u \in L^p(\Omega) : u_{min} \le u(x) \le u_{max}, \quad \text{a.e. } x \in \Omega\}, \tag{25}$$

where $u_{min} < u_{max}$ are constants, then $K_m = K \cap V_m$ satisfies *(A)* when $\{V_m\}$ is given as in Theorem 5.2.

Now we formulate the finite dimensional optimization problems

$$(\mathrm{P}_m) \qquad \min_{u_m \in K_m} F(u_m) = J(u_m) + \beta \int_\Omega |\nabla u_m|.$$

The next theorem deals with the existence of a solution for these problems and with their convergence.

Theorem 5.4 *Let us assume that (A) holds and that either K is bounded in $L^p(\Omega)$ or J is coercive in the sense of Theorem 4.1. Then* (P_m) *has at least one solution $\bar{u}_m$ for every m. The sequence $\{\bar{u}_m\}$ is bounded in $L^p(\Omega) \cap BV(\Omega)$ and the limit of every subsequence converging weakly in $L^p(\Omega)$ also converges strongly in $L^1(\Omega)$ and it is a solution of* (P)*. If J is strictly convex, then the solution of* (P_m) *is unique and the whole sequence $\{\bar{u}_m\}$ converges to the solution of* (P)*.*

For the proof we refer to [4].

6 Application to image enhancement

We give a brief description and some numerical results of (BV)-seminorm regularization to the image denoising problem. Thus we consider the discretized form of $(\mathcal{P})$ of Section §1 with $T = I$. The unperturbed image u^* extends over a square consisting of 80×80 pixels and the noisy image z is obtained from

$$z_{ij} = u^*_{ij} + \eta_{ij}, \quad i, j = 1, \dots, 80,$$

with η_{ij} uniformly distributed random numbers in $[-\delta, \delta]$. The use of the (BV)-seminorm regularization is expected to be especially effective for blocky images and thus as test example we considered, among others, $u^* = u^*_1 + u^*_2 + u^*_3$, where $u^*_{1,i,j} = 1$ for all i, j,

$$u^*_{2,i,j} = \begin{cases} 1 & \text{if } 13 \le i \le 68 \text{ and } 13 \le j \le 68, \\ 0 & \text{otherwise;} \end{cases}$$

$$u^*_{3,i,j} = \begin{cases} 1 & \text{if } \ 25 \le i \le 56 \ \text{ and } \ i \le j \le 56, \\ 0 & \text{otherwise.} \end{cases}$$

After reshaping the matrices z and u^* as $m \times 1$ vectors, $m = 6400$, the resulting minimization problem can be expressed as

$$\min_{u_m \in \mathbb{R}^m} \frac{1}{2m} \sum_{i=1}^{m} |u_i - z_i|^2 + \beta \sum_{i<j} |\partial\Omega_i \cap \partial\Omega_j||u^i - u^j|. \qquad (26)$$

We considered a MATLAB code to solve (26). In Figure 1 we show the noisy image with $\delta = 1$ and Figure 2 give the corresponding enhanced image obtained by the code.

FIG. 1. The noisy image.

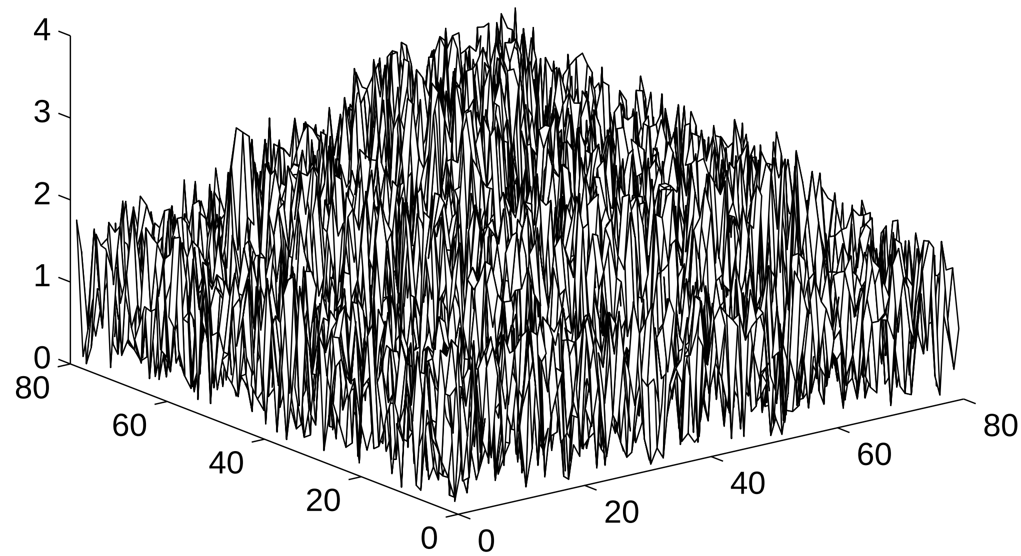
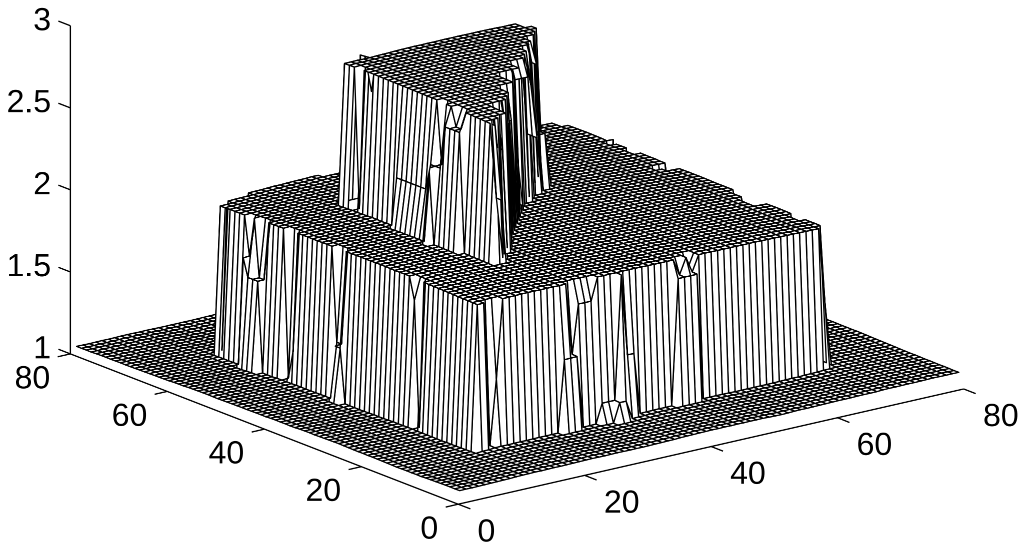


FIG. 2. Enhanced image.



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