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ON SUBOPTIMAL CONTROL STRATEGIES FOR THE NAVIER-STOKES EQUATIONS

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Abstract

We summarize some of the recent developments in the fields of suboptimal control strategies for time dependent incompressible Navier-Stokes Equations. Reduced order modeling (ROM) and instantaneous control strategies are described. Both approaches are applied to compute controls for the laminar backward facing step flow.

1 Introduction

In many applications it can be useful to control fluid flow and to optimize its characteristics. Typical applications include the reduction of drag and the lag or elimination of the transition from laminar to turbulent flow regime. The solutions of the underlying optimal control problems satisfy a system of coupled nonlinear partial differential equations involving the time-dependent Navier-Stokes equations. Solving these systems for realistic flows numerically is unworkable with most currently available computing facilities. Hence there is a need for suboptimal control strategies which incorporate the nonlinear character of the flow and which are amenable to controlling the flow using presently available computing environments.

In the recent past three suboptimal approaches to the control of fluids have been proposed. The concept of instantaneous control was developed in the work [6] of H. Choi, R. Temam, P. Moin & J. Kim and was applied to the control of Navier-Stokes equations in several subsequent papers; see e.g. [5], [15], [7] and the references cited therein. An alternative approach is based on approximating the nonlinear dynamics of the equations of fluid flow by reduced order models (ROM) and carrying out an exact optimization for the reduced system. Various approaches differ in the construction of the basis functions that are used for the reduced models. In the reduced basis approach one uses as basis functions the terms which arise in series expansion of the solution with respect to a parameter. J. Peterson [23], for example, has used the reduced basis method to solve the Navier-Stokes equation for high Reynolds numbers by expansion with respect to the Reynolds number. In K. Ito & S.S. Ravindran [22] the reduced basis method was successfully applied for the control of flow over the backward facing step in the context of stationary Navier-Stokes equations. An alternative approach is based the 'snapshot' form of the proper orthogonal decomposition (POD) described by Sirovich in [24]. The POD approach to suboptimal control of fluids is described in W.R. Graham, J. Peraire & K.Y. Tang [25], for example. Finally in [20] Ito & Kang describe an approach to obtain suboptimal solutions to the Hamilton-Jacobi-Bellman form of the closed loop solution for optimal control problems governed by the Burgers and the Navier-Stokes equations.

Let us also point out some of the recent developments that concentrate on analytical aspects of control for the Navier-Stokes equations. Significant advances have been made in the area of controllability, see [9] and the references given there. Optimality systems for various types of cost functionals are obtained in [1] by F. Abergel & R. Temam and [10] by Fursikov, Gunzburger & Hou, for example.

An excellent overview on diverse aspects of optimal control for fluids can be obtained from [12]. For further references we also refer to [3], [8], [11], [13], and [19].

The paper is organized as follows. In Section 2 we briefly describe a general boundary control problem for the Navier-Stokes equations which serves as a model problem and derive the corresponding first order optimality conditions. Section 3 is devoted to instantaneous control applied to the model problem.

In Section 4 we briefly describe the ROM approach for computing suboptimal controls. Finally in Section 5 we give some numerical examples including optimal and suboptimal control computations for the two dimensional laminar backward facing step flow.

2 An optimal control problem

For the following presentation it will be convenient to refer to Fig. 1 which depicts the spatial domain Ω that constitutes the flow region and the subsets of the boundary that we shall refer to. Let (u_1, u_2, u_3) denote the velocity of

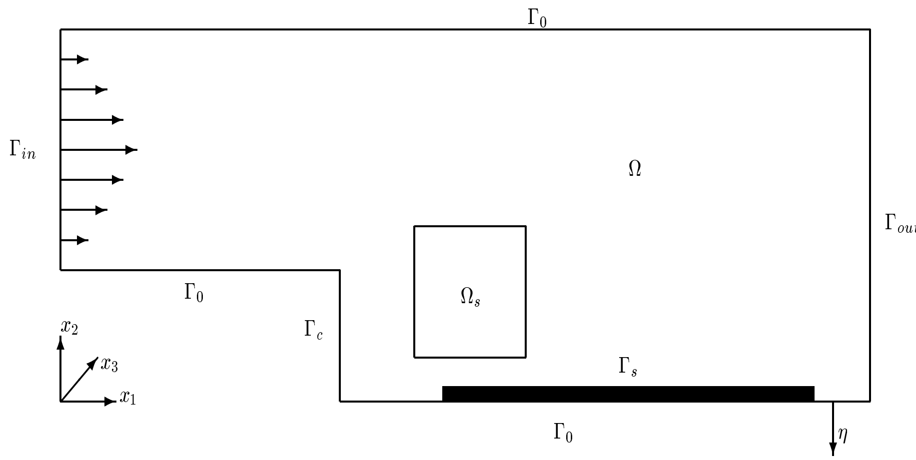

Figure 1: Flow region

the fluid in the directions (x_1, x_2, x_3) and let p denote its pressure. The time-dependent Navier-Stokes equations on the space-time cylinder $Q := \Omega \times (0, T)$, $T > 0$ are given by

$$\left.\begin{array}{rcl} u_t - \frac{1}{Re}\Delta u + (u \cdot \nabla)u + \nabla p & = & \tilde{f} \\ -div\, u & = & 0 \end{array}\right\} \quad \text{in } Q, \tag{1}$$

where $(u \cdot \nabla)u$ stands for the vector with components $\sum u_j \frac{\partial}{\partial x_j} u_i$ and $Re = \frac{UH}{\nu}$ denotes the Reynolds number with U the bulk velocity at the inlet profile, H the step height and ν the kinematic viscosity. At $t = 0$ the initial condition

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega \tag{2}$$

is imposed, where u_0 is a given function on Ω. Next we describe the boundary conditions. On the inflow boundary Γ_{in} we prescribe an inhomogeneous Dirichlet condition

$$u(\cdot, t) = u_{in} \quad \text{on } \Gamma_{in}, \tag{3}$$

where u_{in} is chosen to be a parabolic inflow profile in the numerical examples. On the outflow part of the boundary a natural (" do-nothing") boundary condition

$$\frac{1}{Re}\partial_\eta u = p\eta \quad \text{on } \Gamma_{out}, \tag{4}$$

is imposed. Here η denotes the unit normal to $\partial\Omega$ in the outward direction. This boundary condition is used as a downstream condition in [14] and it was observed numerically in [18], e.g. that this choice of boundary condition has little artificial influence on practically observed flow patterns. On the remaining part of the boundary, except for the control portion Γ_c, no-slip boundary conditions are assumed:

$$u = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_c). \tag{5}$$

Finally, in order to influence the flow behind the step a control will be applied along the boundary Γ_c,

$$u = \phi \quad \text{on } \Gamma_c. \tag{6}$$

Here ϕ denotes the control function representing blowing and suction which will be chosen in such a manner that an appropriately defined performance index is minimized. It will be convenient to combine the uncontrolled Dirichlet boundary conditions (3) and (5) into one statement

$$u = g \quad \text{on } \Gamma_d \;:=\; \partial\Omega \setminus (\Gamma_{out} \cup \Gamma_c), \tag{7}$$

where $g = u_{in}$ on Γ_{in} and zero otherwise. The uncontrolled flow over the backward facing step contains a recirculation zone behind the step. The control objective consists in influencing or diminishing this recirculation. The choice of the cost functional appropriate to capture this goal is a delicate one. Here we shall consider functionals of the form

$$J(\phi) = \int_0^T \left[\int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s \;+\; \frac{\gamma}{2} \int_{\Gamma_c} \phi^2 \, d\Gamma_c \right] dt, \tag{8}$$

where $\gamma > 0$ and $F : \mathbb{R}^3 \to \mathbb{R}$ is a given, sufficiently smooth function [7]. Referring to Fig. 1 we have $\eta = (0, -1, 0)^t$ on Γ_s and hence a special case of (8) is given by

$$J(\phi) = \int_0^T \left[\int_{\Gamma_s} \frac{\partial u_1}{\partial x_2} \Big(\frac{\partial u_1}{\partial x_2} - \Big| \frac{\partial u_1}{\partial x_2} \Big| \Big)\, d\Gamma_s \;+\; \frac{\gamma}{2} \int_{\Gamma_c} \phi^2 \, d\Gamma_c \right] dt. \tag{9}$$

The first part of the cost penalizes a negative wall velocity gradient in the sensor area Γ_s which is supposed to contain the re-attachment point. The second summand weights the costs for the control acting along Γ_c. Other functionals involving shear stresses or drag along Γ_s can be expressed as special cases of

(8). We can now state the optimal control problem

$$
(P)\quad \left\{\begin{array}{rcll}
\multicolumn{4}{l}{\min J(\phi) = \int\limits_0^T \left[\int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s \;+\; \frac{\gamma}{2}\int_{\Gamma_c} \phi^2\, d\Gamma_c\right] dt} \\
\multicolumn{4}{l}{\text{s.t.}} \\
u_t - \frac{1}{Re}\Delta u + (u\cdot\nabla)u + \nabla p &=& f & \text{in } Q \\
-div\, u &=& 0 & \text{in } Q \\
u &=& g & \text{on } \Gamma_d \times (0,T] \\
u &=& \phi & \text{on } \Gamma_c \times (0,T] \\
\frac{1}{Re}\partial_\eta u &=& p\eta & \text{on } \Gamma_{out} \times (0,T] \\
u(x,0) &=& u_0(x) & \text{in } \Omega.
\end{array}\right. \tag{10}
$$

The optimality system for this problem can be derived by using a Lagrangian approach as described in [7]. In order to present the result let ϕ^* denote a solution to (10) and let $(u^*, p^*) = (u(\phi^*), p(\phi^*))$ be the corresponding optimal velocity field and pressure, respectively satisfying the Navier-Stokes system. Then formally (u^*, p^*, ϕ^*) satisfies the following system of coupled equations in the 'primal variables' (u, p, ϕ) and the 'adjoint variables' (ξ, π, μ):

$$
\begin{array}{rcl}
u_t - \frac{1}{Re}\Delta u + (u\cdot\nabla)u + \nabla p &=& f \quad \text{in } Q \\
-div\, u &=& 0 \quad \text{in } Q \\
u &=& \left\{\begin{array}{ll} g & \text{on } \Gamma_d \\ \phi & \text{on } \Gamma_c \end{array}\right. \\
u(0) &=& u_0 \\
-\xi_t - \frac{1}{Re}\Delta\xi + (\nabla u)^t\xi - (u\cdot\nabla)\xi + \nabla\pi &=& 0 \quad \text{in } Q \\
-div\, \xi &=& 0 \quad \text{in } Q \\
\xi &=& \left\{\begin{array}{ll} 0 & \text{on } \partial\Omega \setminus (\Gamma_{out}\cup\Gamma_s)\times (0,T] \\ Re\nabla F(\partial_\eta u) & \text{on } \Gamma_s \times (0,T] \end{array}\right. \\
\xi(T) &=& 0 \\
\frac{1}{Re}\partial_\eta\xi + (u\cdot\eta)\xi &=& \pi\eta \quad \text{on } \Gamma_{out} \times (0,T] \\
\gamma\phi + \frac{1}{Re}\partial_\eta\xi &=& \pi\eta \quad \text{on } \Gamma_c \times (0,T],
\end{array} \tag{11}
$$

where $((u\cdot\nabla)\xi)_i = \sum_j (u_j\partial_{x_j})\xi_i$. The gradient of J at ϕ in direction χ is given by

$$
\nabla J(\phi)\chi = \int\limits_0^T \int_{\Gamma_c} (\gamma\phi + \frac{1}{Re}\partial_\eta\xi - \pi\eta)\, \chi\, d\Gamma_c dt, \tag{12}
$$

where (ξ, π) are computed from (11) with (u^*, p^*, ϕ^*) replaced by (u, p, ϕ). We observe that the primal and dual variables are strongly coupled in system (11). The coupling in the coefficient of the adjoint equation as well as in the boundary condition on Γ_{out} is due to the nonlinear character of the Navier-Stokes equations. The coupling between ϕ and (ξ, π) and the fact that the primal equation is solved forward whereas the adjoint equation for (ξ, π) is solved backwards in time are characteristic for optimal control problems. Since solving (11) numerically is beyond the todays workstation capabilities, suboptimal control strategies that can be realized numerically will be described in the following sections.

For the special case of J given in (9) and the geometry of Fig. 1 we have (note that $\eta = (0, -1, 0)^t$)

$$F(z_1, z_2, z_3) = \frac{1}{2} z_1 (z_1 + |z_1|)$$

and

$$\nabla F(\partial_\eta u) = \begin{pmatrix} \frac{\partial u_1}{\partial_{x_2}} - |\frac{\partial u_1}{\partial_{x_2}}| \\ 0 \\ 0 \end{pmatrix}.$$

3 The instantaneous control strategy

As pointed out in the previous section solving the optimality system (11) is almost unworkable. This suggests the use of suboptimal control strategies. A second motivation for such strategies is given by closed-loop control considerations as will shortly be explained.

The approach that we take is based on a time discretization of (10). To our knowledge it was first advocated in [3],[6]. At each discrete time level t_i a stationary control problem is solved for an optimal control ϕ_i^* and this control is used to steer the system from t_i to t_{i+1}, where a new optimal control is determined. It cannot be claimed that the controls obtained in this manner approximate the optimal control for (10) as the discretization parameter tends to zero. However, this procedure will be justified by the effectiveness that it exhibits for numerical examples and the interpretation that it allows for suboptimal feedback controls.

To commence, let $m > 1$ be fixed and set $\delta t = \frac{T}{m}$, $t_i = i\delta t, i = 0, \ldots, m$. As an intermediate step in the derivation let us consider the case where the Navier-Stokes equations in (11) are approximated by a Crank Nicolson scheme. At the i-th level of the Crank-Nicolson based suboptimal strategy one solves the following stationary optimal control problem, where the variables (u, p, ϕ) correspond to $(u(t_i), p(t_i), \phi(t_i))$.

$$\left.\begin{array}{l} \min J(\phi) = \int_{\Gamma_s} F(\partial_\eta u)\, d\Gamma_s + \frac{\gamma}{2} \int_{\Gamma_c} \phi^2\, d\Gamma_c \\ \text{s.t.} \\ \begin{array}{rcll} \frac{1}{\delta t} u + \frac{1}{2}(u \cdot \nabla)u - \frac{1}{2Re}\Delta u + \nabla p &=& f_i & \text{in } \Omega \\ -div\, u &=& 0 & \text{in } \Omega \\ u &=& g & \text{on } \Gamma_d \\ \frac{1}{2Re}\partial_\eta u &=& p\eta - \frac{1}{2Re}\partial_\eta u_{i-1} & \text{on } \Gamma_{out} \\ u &=& \phi & \text{on } \Gamma_c. \end{array} \end{array}\right\} \quad (13)$$

Here $f_i = \tilde{f}_{i-\frac{1}{2}} - \frac{1}{2Re}\Delta u_{i-1} - \frac{1}{2}(u_{i-1} \cdot \nabla)u_{i-1}$ is a known inhomogeneity. Let $\phi_i = \phi(t_i)$ denote a solution to (13) and set $(u_i, p_i) = (u(\phi_i), p(\phi_i))$. Then the triple (u_i, p_i, ϕ_i) satisfies the optimality system for (13) given by the equality constraints in (13), together with the adjoint equation and the optimality condition

$$\left.\begin{array}{rll} \frac{1}{\delta t}\xi_i - \frac{1}{2}(u_i\cdot\nabla)\xi_i + \frac{1}{2}(\nabla u_i)^t\xi_i = & \frac{1}{2Re}\Delta\xi_i - \nabla\pi_i & \text{in } \Omega \\ -div\,\xi_i = & 0 & \text{in } \Omega \\ \xi_i = & \begin{cases} 0 & \text{on } \partial\Omega\setminus(\Gamma_{out}\cup\Gamma_s) \\ 2Re\nabla F(\partial_\eta u_i) & \text{on } \Gamma_s \end{cases} & \\ \frac{1}{2Re}\partial_\eta\xi_i + \frac{1}{2}(u_i\cdot\eta)\xi_i = & \pi_i\eta & \text{on } \Gamma_{out} \end{array}\right\} \tag{14}$$

$$\gamma\phi_i + \frac{1}{2Re}\partial_\eta\xi_i = \pi_i\eta \quad \text{on } \Gamma_c. \tag{15}$$

Note that (14) requires information of the velocity inside the domain as well as on its boundary. For computations this requires recalculation of the system matrix for the adjoint equation whenever the primal velocity field u_i changes. Let us also note that from the feedback-control point of view, one aims for replacing u_i by observations of the system and computing ϕ_i from (14), (15). This would require knowledge of the velocity field u_i inside Ω which is impractical. We therefore simplify further the discretization by replacing the implicit Crank-Nicolson scheme for the nonlinear equation by a semi-implicit time discretization scheme. The discretization we choose is implicit in the viscous term and explicit in the nonlinear convective term. In this case only measurements of the shear wall stress will be necessary to compute the adjoint variables of the corresponding adjoint system. More precisely, let us consider the minimization problem

$$(P_i)\begin{cases} \min J(\phi) = \int_{\Gamma_s} F(\partial_\eta u)\,d\Gamma_s + \frac{\gamma}{2}\int_{\Gamma_c}\phi^2\,d\Gamma_c \\ \text{s.t.} \\ \begin{array}{rcll} u - c\Delta u + \delta t\,\nabla p & = & f_i & \text{in } \Omega \\ -div\,u & = & 0 & \text{in } \Omega \\ u & = & g & \text{on } \Gamma_d \\ c\partial_\eta u & = & \delta t\,p\eta & \text{on } \Gamma_{out} \\ u & = & \phi & \text{on } \Gamma_c, \end{array} \end{cases} \tag{16}$$

where $c = \frac{\delta t}{Re}$ and $f_i = u_{i-1} - \delta t(u_{i-1}\cdot\nabla)u_{i-1} - \tilde{f}_i$. Note that the information from time level t_{i-1} to t_i is passed solely through the inhomogeneity f_i. Let ϕ_i denote a solution to (P_i) and set again $(u_i,p_i) = (u(\phi_i),p(\phi_i))$. Then (u_i,p_i,ϕ_i) must satisfy an optimality system consisting of the primal equations which are the equality constraints in (16) and an adjoint equation, coupled with an optimality condition which are given next:

$$\left.\begin{array}{rcll} \xi_i - c\Delta\xi_i + \nabla\pi_i & = & 0 & \text{in } \Omega \\ -\delta t\,div\,\xi_i & = & 0 & \text{in } \Omega \\ \xi_i & = & \begin{cases} 0 & \text{on } \partial\Omega\setminus(\Gamma_{out}\cup\Gamma_s) \\ \frac{1}{c}\nabla F(\partial_\eta u_i) & \text{on } \Gamma_s \end{cases} & \\ c\partial_\eta\xi_i & = & \pi_i\eta & \text{on } \Gamma_{out} \end{array}\right\} \tag{17}$$

$$\gamma\phi_i + c\partial_\eta\xi_i = \pi_i\eta \quad \text{on } \Gamma_c. \tag{18}$$

In the optimality system for (P_i) the coupling between primal and adjoint equations for the problems (P_i) occurs only along the observation boundary Γ_s.

The optimal control problem (10) has thus been replaced by a sequence of stationary problems (P_i) which can be realized on workstations. Moreover, if one interprets (18) as a feedback control law, then only observations of the state along the boundary Γ_s are required. To solve (16) numerically several different approaches are possible. Putting speed above accuracy the following gradient algorithm with (almost) optimal step size is proposed to solve (P_i):

Gradient Algorithm

1. Set $k = 0$ and choose ϕ_0,

2. Compute
$$\rho^* = \arg\min_{\rho>0} H(\rho) := J(\phi^k - \rho\nabla J(\phi^k))$$

3. Set
$$\phi^{k+1} = \phi^k - \rho^*\nabla J(\phi^k)$$

4. Set $k = k + 1$ and goto 2.

Here and in what follows the subscript i is dropped. The gradient of J at ϕ^k is found to be
$$\nabla J(\phi^k) = (\gamma\phi^k + c\partial_\eta\xi - \pi\eta)_{|\Gamma_c}, \tag{19}$$
where (ξ, π) is a solution to (17) with (u, p) a solution of the primal equations in (16).

In the gradient algorithm the expensive part is step 2. We shall see next how to obtain a good approximation to ρ^* that requires only one additional solution of a primal-adjoint system. The optimal step size in the minimization problem of step 2. can be characterized via a Lagrange approach similar to the one of the previous section, see [7]. Using linearization an approximation ρ^*_{app} of ρ^* can be obtained by

$$\rho^*_{app} = \frac{\int_{\Gamma_c}(\gamma\phi^k + c\partial_\eta\xi(\phi^k) - \pi(\phi^k)\eta)s\,d\Gamma_c}{\int_{\Gamma_c}(\gamma s + c\partial_\eta\xi(s) - \pi(s))s\,d\Gamma_c} = \frac{|\nabla J(\phi^k)|^2}{\gamma|\nabla J(\phi^k)|^2 + \int_{\Gamma_c}(c\partial_\eta\xi(s) - \pi(s))s\,d\Gamma_c}, \tag{20}$$

where $s := \nabla J(\phi^k)$ and $(\xi(s), \pi(s))$ is the solution of the system of quasi-Stokes equations

$$\left.\begin{array}{rcl} u(s) - c\Delta u(s) + \delta t\,\nabla p(s) &=& 0 \quad \text{in } \Omega \\ -div\, u(s) &=& 0 \quad \text{in } \Omega \\ u(s) &=& \left\{\begin{array}{l} 0 \text{ on } \Gamma_d \\ s \text{ on } \Gamma_c \end{array}\right. \\ c\partial_\eta u(s) &=& \delta t\, p(s)\eta \quad \text{on } \Gamma_{out} \end{array}\right\} \tag{21}$$

and

$$\left.\begin{array}{rcl} \xi(s) - c\Delta\xi(s) + \nabla\pi(s) &=& 0 \quad \text{in } \Omega \\ -\delta t\, div\,\xi(s) &=& 0 \quad \text{in } \Omega \\ \xi(s) &=& \left\{\begin{array}{ll} 0 & \text{on } \Gamma_d \\ \frac{1}{c}\nabla^2 F(\partial_\eta u(\phi^k))\partial_\eta u(s) & \text{on } \Gamma_s \end{array}\right. \\ c\partial_\eta\xi(s) &=& \pi(s)\eta \quad \text{on } \Gamma_{out}. \end{array}\right\} \tag{22}$$

Every iteration of the gradient algorithm requires the solution of four quasi-Stokes problems: The solution of the state equation in (16) and (17) for the evaluation of the gradient and two additional ones given by (21) and (22) for an approximation ρ^*_{app} to the optimal step length in the line search of the gradient algorithm. We note that in the case that F is quadratic (20) is exact.

For the function $F(z_1, z_2, z_3) = z_1 + |z_1|$ of the costs in (9) the elements of the Hessian must be interpreted as directional derivatives. We obtain

$$\nabla F(\partial_\eta u) = \begin{pmatrix} \frac{\partial u_1}{\partial x_2} - |\frac{\partial u_1}{\partial x_2}| \\ 0 \\ 0 \end{pmatrix}$$

and

$$\nabla^2 F(\partial_\eta u)\partial_\eta w = \begin{pmatrix} -2sgn(\frac{\partial u_1}{\partial x_2})\frac{\partial w_2}{\partial x_2} \\ 0 \\ 0 \end{pmatrix}.$$

4 Reduced Order Modeling

The ROM approach to optimal control problems such as (10) is based on approximating the nonlinear dynamics by a Galerkin technique utilizing basis functions that contain characteristics of the expected controlled flow. This is in contrast to finite element based Galerkin schemes where the basis elements are unrelated to the physical properties of the system that they approximate. Consequently one expects that only a few basis elements will provide a reasonably good description of the controlled flow. Various ROM techniques differ in the choice of the basis functions. We shall briefly describe the reduced basis method that was used by Ito & Ravindran [22] for a variety of optimal control problems. A second approach relies on the Karhunen-Loève Proper Orthogonal Decomposition (POD) of fluids. In [25] Tang, Graham & Peraire apply the snapshot variant of POD introduced by Sirovich in [24] to the control of the unsteady wake flow around a cylinder. In their approach control is introduced into the system via cylinder rotation.

Reduced order models for an approximation to the solution of (P) are based on (variations of) an Ansatz of the form

$$u = u_0 + \sum_{i=1}^{N} \alpha_i(t) u_i + \sum_{i=1}^{M} c_i(t)\varphi_i \tag{23}$$

where in the following u_i, φ_j are functions that are assumed to be independent of t. Here u_0 denotes the flow corresponding to a reference control ϕ_0, i.e. satisfies $u_0 = \phi_0$ on Γ_c and $u_0 = g$ on $\Gamma_d \cup \Gamma_c$. Further let $u_1, \ldots, u_N$ denote solutions of the Navier-Stokes equations with inhomogeneous boundary values at least on the control part of the boundary Γ_c and u_0 denotes the flow corresponding to a reference control ϕ_0, i.e. satisfies

$$\begin{array}{rcll} -\frac{1}{Re}\Delta u + (u\cdot\nabla)u + \nabla p &=& f & \text{in } \Omega \\ -div\, u &=& 0 & \text{in } \Omega \\ u &=& g & \text{on } \Gamma_d \\ u &=& \phi & \text{on } \Gamma_c \\ \frac{1}{Re}\partial_\eta u &=& p\eta & \text{on } \Gamma_{out} \end{array} \tag{24}$$

with $\phi = \phi_0$. Further $u_i, i = 1, \ldots, N$ are solutions to (24) with $g = 0$ and $\phi = \phi_i$ with ϕ_i some preselected control patterns, as for instance blowing or sucking at the wall with a certain intensity. Finally the test functions $\varphi_1, \ldots, \varphi_M$ are chosen as linear combinations of solutions to (24) in such a way that they satisfy homogeneous boundary conditions. For example the $\varphi_1, \ldots, \varphi_M$ could be constructed as linear combinations of $u_1, \ldots, u_N$. Note that by construction every function u of the form in (23) satisfies the continuity equation $div\, u = 0$.

The ROM control problem can then be expressed as

$$\text{(ROM)} \left\{ \begin{array}{l} \min J(\vec{\alpha}, u) = \frac{l}{2}\int_0^T \vec{\alpha}^t M^u \vec{\alpha}\, dt + \int_0^T \int_{\Gamma_{obs}} F(\partial_\eta u)\, d\Gamma dt \\ \text{s.t.} \\ (u_t, \varphi_j) + \frac{1}{\mathrm{Re}}(\nabla u, \nabla\varphi_j) + ((u\nabla)u, \varphi_j) = 0 \quad (j = 1, \ldots, M), \end{array} \right.$$

where $M^u = (m_{ij})_{ij=1,\ldots,M}$, $m_{ij} = (u_i, u_j)_{\Gamma_c}$ is the mass matrix of the basis controls on Γ_c. The resulting optimality system is a forward-backward in time non-linear dynamical system with dimension $2M + N$. Details can be found in [17].

The crucial point in this approach is the choice of the test functions $\varphi_1, \ldots, \varphi_M$ since they must be chosen such that the expansion in (23) describes as much of the nonlinear dynamics as possible.

Guidelines for the choice of the functions u_i, φ_j in (23) are provided by the POD and the Reduced Basis Method which we now describe.

In order to present the POD approach let now $v_1, \ldots, v_M$ denote *snapshots* of the uncontrolled flow at time instances $t_1 < t_1 < \ldots < t_M$ and define the mean value of these snapshots by

$$v_m := \frac{1}{M}\sum_{i=1}^{M} v_i.$$

Furthermore denote by $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{iN})^t$ the i-th eigenvector of the correlation matrix

$$K = [k_{ij}]_{i,j=0}^{M}, \quad k_{ij} = \int_\Omega (v_i - v_m)(v_j - v_m)\, dx.$$

The basis functions (or modes) then are defined by

$$\varphi_i = \sum_{j=0}^{M} \epsilon_{ij}(v_j - v_m)$$

and it follows from this definition that $\varphi_{i_{|\Gamma_d \cup \Gamma_c}} = 0$ since the boundary values of the snapshots are assumed to be independent of t.

The basis functions constructed in this way are mutually orthogonal and optimal in terms of their ability to represent the flow kinetic energy (see Sirovich [24]). Using the POD approach to the control problem (ROM) the function u_0 in the Ansatz (23) should be replaced by the mean of the snapshots v_m.

To describe the reduced basis approach we proceed formally and write $E(y) = 0$ for the Navier-Stokes equations in problem (10), where $y = (u, p)$. Let us further introduce the parameter λ into the equation. This could represent the Reynolds number as in [23] or a factor multiplying a fixed shape function ζ for the control so that $u = \lambda\zeta$ on $\Gamma_c \times (0, T]$. Then, introducing λ into the notation for E, the equation $E(y, \lambda) = 0$ defines a branch of solutions $y(\lambda)$, for λ in an appropriate interval Λ. Reduced basis subspaces are constructed by interpolating this branch. The Taylor subspace is defined by

$$X_T = \text{span}\{y_j = \frac{\partial^j y}{\partial \lambda^j}_{|\lambda=\lambda^*},\ j = 0, \dots, N\}.$$

We observe that y_1 is the solution to

$$E_y(y_0, \lambda^*) y_1 = -E_\lambda(y_0, \lambda^*).$$

The Lagrange subspace is obtained by selecting values $\lambda_1, \dots, \lambda_N$ and defining

$$X_L = \text{span}\{y_j; E(y_j, \lambda_j) = 0\}.$$

The Hermite space finally is given by a combination of the former choices.

Clearly it is difficult to obtain elements beyond y_1 in the Taylor subspace. Concerning the construction according to Lagrange it has to be kept in mind that λ_j need to be chosen such that $y(\lambda_j)$ are 'sufficiently linearly independent', and that the elements $y(\lambda_j)$ still need to be modified so that linear combinations satisfy appropriate boundary conditions. In [22] Ito & Ravindran proceed as follows (we consider $N = 1$, for simplicity). Let u_0 denote the velocity field corresponding to the reference control $\lambda_0\zeta$ and let u_1 denote a second solution to the Navier-Stokes equations with inhomogeneous initial and boundary conditions and control $\lambda_1\zeta$ on $\Gamma_c \times (0, T]$. Then u is expanded as

$$u = u_0 + \frac{\alpha(t) - \lambda_0}{\lambda_1 - \lambda_0}(u_1 - u_0) + \sum_{j=1}^{M} c_j(t)\varphi_j,$$

which, as far as representing the terms that realize the control action in concerned, corresponds to a Taylor expansion at λ_0 with the differential $\frac{\partial u}{\partial \lambda}$ replaced by a finite difference quotient. Note that, just as with POD, the elements in the reduced order basis subspaces are divergence free and hence we do not keep the pressure component of y. The elements φ_j can be obtained, for example by sampling λ at further values λ_i evaluating the Navier-Stokes equations to obtain

$u(\lambda_i\zeta)$ and taking for φ_j linear combinations of the $u(\lambda_i\zeta)$ so that homogeneous initial and boundary conditions are satisfied. This can be done in such a way that finite difference approximations to $\frac{\partial^i u}{\partial\lambda^i}$, $i = 1, 2$, for example are realized.

5 Numerical results

Here we present numerical computations related to the approaches presented in the previous paragraphs. As the flow configuration we choose the two dimensional backward facing step. The optimization objective is the reduction of the recirculation bubble behind the step and thus of the re-attachment length of laminar Navier-Stokes flow at Reynolds number Re= 300. Instantaneous control is applied to problem (10) for the cost functional given by (9). For the discretization of the quasi-Stokes problems the Taylor-Hood finite element is used. The numerical solution of the various problems is performed using an extension of the Navier-Stokes solver developed by Bänsch in [2]. ROM is used to compute a suboptimal control for tracking type control with target given by the Stokes flow.

5.1 Instantaneous Control

We present numerical investigations on the influence of sensor area Γ_s on the feedback control mechanism. The functional to be minimized at each time instance is related to the functional J in (9) and is given by

$$J(\phi) = \frac{1}{2}\int_{\Gamma_s} \frac{\partial u_1}{\partial x_2}(\frac{\partial u_1}{\partial x_2} - |\frac{\partial u_1}{\partial x_2}|)\, d\Gamma + \frac{\gamma}{2}\int_{\Gamma_c} \phi^2\, d\Gamma,$$

where ϕ denotes the boundary control applied at the upper part of the back wall Γ_c and u_1 denotes the first flow component. Consequently, the first part in the cost functional vanishes whenever the flow near the wall at Γ_s is in downstream direction.

In Fig 2 the flows corresponding to different-sized Γ_s at $T = 100$ ($= 10000$ time steps) are shown. It can be observed that the larger the distance between the control boundary Γ_c and the sensor area Γ_s, the more becomes the control mechanism periodic. This is due to the fact that without control the flow evolves to the uncontrolled Navier-Stokes flow and the time span passing till the evolving flow affects the sensor area is larger for larger distances between Γ_c and Γ_s. This can also be observed from the evolution of the costs in Fig. 3(top). Of course, the controlled flow also depends on the size of Γ_s but numerical experiments show that it plays a less important role.

5.2 Reduced Order Modeling

Here we present numerical computations for the ROM approach applied to the in-stationary Navier-Stokes solution of the backward facing step flow at Re= 300. The control problem that we consider here is related to the reconstruction of the Stokes flow u_{stokes} in a given observation volume $\Omega_{obs} \subset \Omega$ behind the step,

$$(\text{ROM}) \begin{cases} \min J(\vec{g}, u) = \frac{\gamma}{2} \|\vec{g}\|^2 \, dt + \int_{\Omega_{obs}} |u - u_{stokes}|^2 \, dx \\ \text{s.t.} \\ \frac{1}{\text{Re}} (\nabla u, \nabla \varphi_j) + ((u\nabla) u, \varphi_j) = 0 \quad (j = 1, \ldots, M), \end{cases}$$

where M denotes the number of basis functions and

$$u = u_0 + \sum_{i=1}^{N} g_i u_i + \sum_{j=1}^{M} c_j \varphi_j .$$

Control is introduced into the system using blowing and/or sucking at the upper part of the back-wall of the step. Our aim is to investigate the contribution of sucking to the control action. As is indicated by several numerical experiments the influence of sucking on the optimal solution should be negligible.

For the numerical results presented in Fig. 4 we have used controls and basis functions based on the simulation data shown in Tab. 1. The value 1 for Inflow stands for parabolic inflow at the inflow boundary Γ_{in}, Inflow equal to 0 means no inflow at Γ_{in}. Control is introduced into the problem using control functions that have inhomogeneous boundary values at the upper half of control part Γ_c. Control equal to 0, positive or negative corresponds to no control, blowing or sucking control basis functions. For the solution of the control problem we make

	u_0	u_1	u_2	u_3	u_4	u_5	u_6
Inflow	1	0	0	0	1	1	1
Control	0	1	0.1	-0.1	1	0.1	-0.1

Table 1: Simulation data for ROM

the Ansatz

$$u = u_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + g_1 u_1 + g_2 u_2 + g_3 u_3,$$

where the basis functions $\varphi_1, \varphi_2, \varphi_3$ are chosen as

$$\varphi_1 = u_4 - u_0 - u_1, \; \varphi_2 = u_5 - u_0 - u_2 \text{ and } \varphi_3 = u_6 - u_0 - u_3,$$

i.e. $M = 3, N = 3$ and the control action is a combination of blowing and sucking the latter introduced into the control problem by u_3. The time interval is chosen as $[0, T]$ with $T = 70$. In Fig. 4 we present the numerical results for the penalty parameter $\gamma = 0.1$. The numerical solution of the optimization problem is computed using Newton's method. The costs of the uncontrolled flow are $J = 7.231$. The final value is $J = 1.75$, so that the cost reduction is about 70%. The flows presented in Fig. 4 from top to bottom are the ROM solution, the streamlines of the ROM solution, the Navier-Stokes solution with ROM optimal boundary values and the streamlines of the latter. As one can see there is good agreement of the flows in the observation region $\Omega_s = [4,6] \times [-1,0]$. Moreover the recirculation bubble is reduced significantly when compared to the uncontrolled flow and the re-attachment length for the ROM-solution and the controlled Navier-Stokes solution are close.

The computations presented were performed by A. Köhler at the Technical University Berlin.

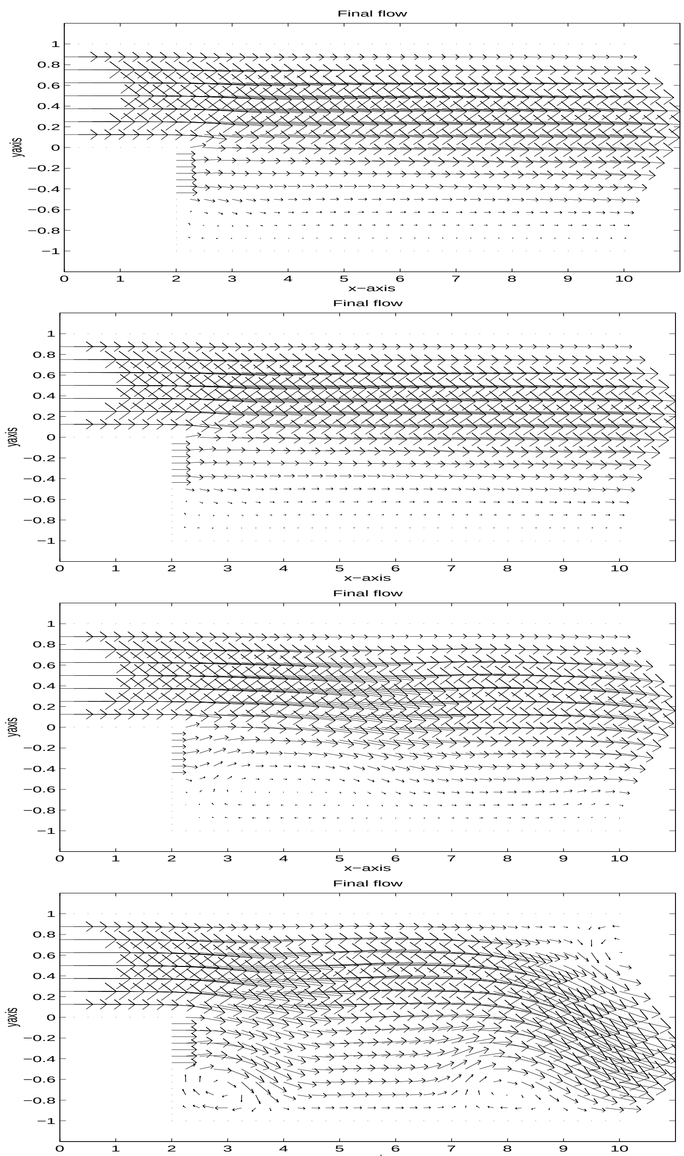

Figure 2: Boundary observation at Re= 300: $\Gamma_s = [2,9]$, $[3,9]$, $[4,9]$, $[5,9]$

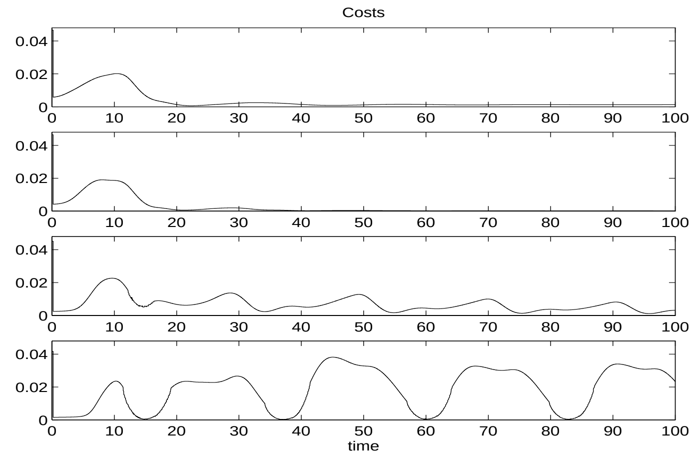

Figure 3: Boundary observation at Re=300, costs(top): $\Gamma_s = [2, 9]$, $[3, 9]$, $[4, 9]$, $[5, 9]$

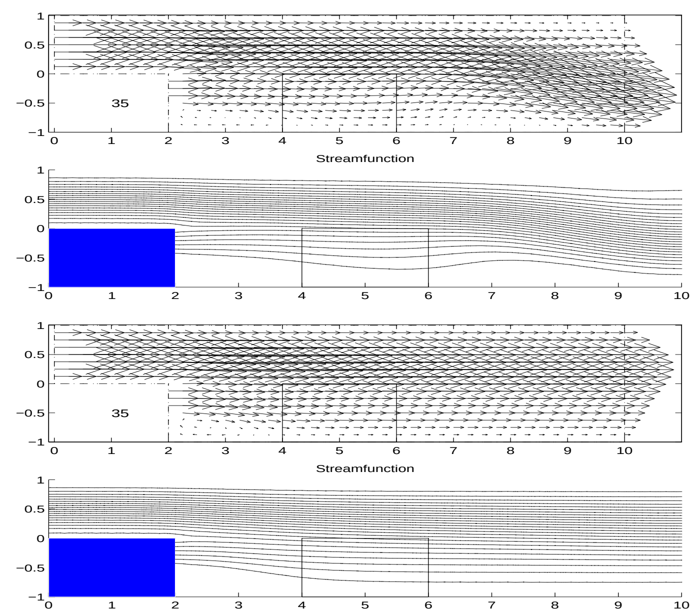

Figure 4: ROM at Re=300, from top to bottom: ROM solution, streamlines of ROM solution, numerical simulation using ROM result as boundary values, streamlines of the former

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