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DYNAMIC DOMAIN DECOMPOSITION OF OPTIMAL CONTROL PROBLEMS FOR NETWORKS OF EULER-BERNOULLI BEAMS

G. LEUGERING [1]

Institute of Mathematics
University of Bayreuth
D-95440 Bayreuth
Germany
leugering@uni-bayreuth.de

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Abstract

We consider planar networks of Euler-Bernoulli beams subject to Neumann-type boundary controls at simple nodes. The object is to minimize a cost functional along some part of the beam structure by the way of dynamic domain decomposition.

1 Introduction

We consider a simple planar graph $G = (V, E)$ with vertices $V = \{v_J, v_N, v_M \ldots\}$ labeled by capital letters, and edges $E = \{e_i, e_j, \ldots\}$ labeled by lower case letters. We denote by e_i the normalized vector in the direction of the edge i. Further, let d_J denote the edge degree of the vertex v_J. Then we can distinguish between multiple nodes V_M, where the $d_J{}'s$ are > 1, and simple nodes V_S with $d_J = 1$. The simple nodes consist of Dirichlet-nodes V_D, where the beam is clamped and Neumann-nodes $V_N \dot{\cup} V_C$, where the label N signifies a free end, while C signifies a controlled Neumann-node. We denote the deformation of the i-th beam by

$$r_i(x,t) = u_i(x,t)e_i + w_i(x,t)e_i^{\perp} \ ,$$

where u_i, w_i represent the longitudinal and vertical displacement in the local coordinate system. Let ϵ_{iJ} be $= 1$ if edge i ends at the vertex v_J, and $= -1$ if it starts there (one may always consider $\epsilon_{iJ} = 0$ if the i-th edge is not incident at v_J). Further, let $\mathcal{E}_J$ denote the set of indices belonging to edges incident at v_J. For more detailed modelling issues we refer to [7]. We first write down the equations governing the motion of the elastic network of Euler-Bernoulli beams. Rayleigh-beams can also be treated similarly.

$$\begin{cases} \rho_i \ddot{u}_i = Eh_i u_i'', \\ \rho_i \ddot{w}_i + EI_i w_i'''' = 0, \quad \text{in } (0,l_i) \times (0,T), \end{cases} \tag{1.1}$$

$$\begin{cases} r_i(v_J) = r_j(v_J), \\ w_i'(v_J) = w_j'(v_J), \quad \forall\, i,j \in \mathcal{E}_J, v_J \in V_M, \end{cases} \tag{1.2}$$

$$\begin{cases} r_i(v_D) = 0, \\ w_i'(v_D) = 0, \quad \text{if } \epsilon_{iD} \neq 0,\ v_D \in V_D, \end{cases} \tag{1.3}$$

$$\begin{cases} \sum\limits_{i \in \mathcal{E}_J} \epsilon_{iJ} EI_i w_i''(v_J) = 0, \\ \sum\limits_{i \in \mathcal{E}_J} \epsilon_{iJ} (Eh_i, u_i'(v_J) e_i - EI_i w_i'''(v_J) e_i^{\perp}) = f_J, \\ \qquad\qquad \forall\, v_J \in V_M \cup V_C \cup V_N, t \in (0,T), \end{cases} \tag{1.4}$$

$$r_i(\cdot,0) = r_{i0},\ \dot{r}_i(0,0) = r_{i1} \text{ on } (0,l_i). \tag{1.5}$$

Here the first set of equations (1.1) represents the dynamics of the longitudinal and vertical displacements along an individual edge i, (1.2) characterizes the continuity of displacements and rotation angles accross the joint v_J, which is then said to be rigid. Position (1.3) signifies a clamped node, while the two nodal balance laws (1.4) represent the balance of moments and forces at v_J, respectively, $f_J = 0$ for $v_J \notin V_C$. Finally, (1.5) are the initial data for the problem. For definiteness, we assume that $|V_D| \geq 1$.

The problem of exact controllability of (1.1)–(1.5) has been studied in [11] for G being a star-graph with $|V_M| = 1$, $V_C = V_S$. For non collinear in-plane-beam structures this is, to our knowledge, the only result available in the literature, whereas serial (out-of-the-plane) beams have been widely studied [5], [3], [14], to name just a few. The paper [3] contains rotational inertia of cross sections and joint-masses. While we have been able in [7] to show exact controllability of Timoshenko-tree-like networks, we have not succeeded so far in passing to Euler-Bernoulli or Rayleigh-beam networks. As a result, the controllability-theory and hence also the theory of uniform stabilization of such networks is widely open.

This motivates the interest in *optimal* control problems for beam structures. In this note we give a first attempt towards this goal. The main point is that we want to achieve optimal control of such networks by domain decomposition, i.e. we want to reduce the global optimality system to a collection of local ones, which in turn we might solve in parallel.

We have been able to achieve this goal for networks of strings in [8], [9] and Timoshenko-beams [10] by various methods, where we have been inspired by recent work of Benamou [1], [2], and Glowinski et.al. [4], Quarteroni and Valli [13].

For the sake of simplicity, we consider the following cost functional

$$J(f) := \frac{1}{2} \sum_{v_J \in V_C} \int_0^T |f_J|^2 dt + \frac{1}{2} \sum_{i=1}^{n_e} \int_0^T \int_0^{l_i} |r_i|^2 dx\, dt \ , \tag{1.6}$$

where r_i satisfies (1.1)–(1.5). We note that we can also treat the case, where not all edges are involved in the cost. This gives the possibility of silencing just some edges. Other cost functions involving penalized end conditions are currently under investigation. However, as opposed to [9], more subtle arguments seem to be necessary in the beam case. Functionals like (1.6) have been considered by Benamou for serial planar 2–d decompositions without transmission phenomena. Such set-up for transmission problems is the subject of current joint work with J.E. Lagnese.

2 The global optimality system

We introduce the adjoint state $p_i := \varphi_i e_i + \psi_i e_i^\perp$. Then, using standard arguments, we derive the following set of equations describing the backwards running adjoint variables.

$$\begin{cases} \rho_i \ddot{\varphi}_i = E h_i \varphi_i'' + u_i, \\ \rho_i \ddot{\psi}_i + E I_i \psi_i'''' = w_i, \quad \text{on } (0, l_i) \times (0, T), \end{cases} \tag{2.1}$$

$$\begin{cases} p_i(v_J) = p_j(v_J), \\ \psi_i'(v_J) = \psi_j'(v_J), \quad \forall\, i, j \in \mathcal{E}_J, v_J \in V_M, \end{cases} \tag{2.2}$$

$$\begin{cases} p_i(v_D) = 0, \\ \psi_i'(v_D) = 0 \quad \text{if } \epsilon_{iD} \neq 0,\ v_D \in V_D, \end{cases} \tag{2.3}$$

$$\begin{cases} \sum_{i \in \mathcal{E}_J} \epsilon_{iJ} EI_i \psi_i''(v_J) = 0, \\ \sum_{i \in \mathcal{E}_J} \epsilon_{iJ} (Eh_i, \psi_i'(v_J) e_i - EI_i \psi_i'''(v_J) e_i^{\perp}) = 0, \\ \qquad\qquad \forall\, v_J \in V_M \cup V_N \cup V_C, t \in (0,T), \end{cases} \tag{2.4}$$

$$\begin{cases} p_i(T) = 0, \\ \dot{p}_i(T) = 0 \quad \text{ on } (0, l_i)\ . \end{cases} \tag{2.5}$$

$$f_J = -p_i(v_J),\ v_J \in V_C, i \in \mathcal{E}_C,\ t \in (0,T)\ . \tag{2.6}$$

3 Well-posedness

We introduce the space setting for problems like (1.1)–(1.5), (2.1)–(2.6).

$$H = \prod_{i=1}^{n_e} L^2(0, l_i)^2 \tag{3.1}$$

$$\begin{aligned} V = \{ & v \in H | v_i \cdot e_i \in H^1(0, l_i), v_i \cdot e_i^{\perp} \in H^2(0, l_i) \\ & v_i(v_D) = 0,\ v_i' \cdot e_i^{\perp}(v_D) = 0,\ \epsilon_{iD} \neq 0,\ v_D \in V_D\ , \\ & v_i(v_J) = v_j(v_J),\ v_i' \cdot e_i^{\perp}(v_J) = v_j' \cdot e_j^{\perp}(v_J), \quad \forall\, i, j \in \mathcal{E}_J,\ v_J \in V_M \}\ . \end{aligned} \tag{3.2}$$

The space V is a Hilbert space compactly embedded in H. Moreover, if $|V_D| \geq 1$, then the total potential energy

$$E_{pot} := \frac{1}{2} \sum_{i=1}^{n_e} \int_0^{l_i} Eh_i u_i'^2 + EI_i w_i''^2 \, dx \tag{3.3}$$

is seen to be equivalent to the norm of the Sobolev space

$$H^1 \times H^2 := \prod_{i=1}^{n_e} H^1(0, l_i; e_i) \otimes H^2(0, l_i; e_i^{\perp})\ , \tag{3.4}$$

where the appearance of $e_i, e_i^{\perp}$ indicates longitudinal and vertical parts.

We may now define the operator A in H as follows

$$Ar = (-Eh_i u_i'';\ EI_i w_i'''')_{i=1}^{n_e} \tag{3.5}$$

$$D(A) = \{ r \in V | r_i = u_i e_i + w_i e_i^{\perp},\ u_i \in H^2(0, l_i), w_i \in H^4(0, l_i), \tag{3.6}$$

$$\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} EI_i w_i''(v_J) = 0, \tag{3.7}$$

$$\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} (Eh_i u_i'(v_J) e_i - EI_i w_i'''(v_J) e_i^{\perp}) = 0,$$

$$v_J \in V_M \cup V_N \cup V_C \}\ .$$

We have

Theorem 3.1 *A defined by (3.5), (3.6) is self-adjoint and nonnegative. If $V_D \neq \emptyset$, then A is positive definite. A has discrete spectrum.* ■

Theorem 3.2 *Let $V_D \neq \emptyset$, $f \in \prod_{v_J \in V_C} L^2(0,T)^2$, $r_0 \in H$, $r_1 \in V^\star$. Then there exists a unique weak solution r of (1.1)–(1.5) such that $r \in C(0,T,H)$ $\cap C^1(0,T,V^\star) \cap C^2(0,T,D(A)^\star)$.* ■

Remark i) More regular, i.e. finite energy, strong solutions etc. are obtained in the standard way for more regular data. However, a sharp regularity theory for beam networks of Euler-Bernoulli-type does not seem to be known at this point.
ii) The global optimality system (1.1)–(1.5), (2.1)–(2.5), (2.6) has a unique (weak, mild or strong) solution, according to the regularity of the data.

4 Domain decomposition

As an exemplaric situation, we condsider an edge connecting a multiple node v_J to a controlled simple node v_C. The other cases are to be treated in a similar fashion.

In order to derive the decoupling node conditions we utilize the observation made by Glowinski and Le Tallec in [6] that the non-overlapping Schwarz-iteration considered by P.L. Lions [12] is in fact equivalent to an augmented Lagrangian approach combined with a standard saddle-point algorithm. This observation has lead us to derive dynamic domain decompositions of string and Timoshenko beam networks, see [9], [10]. Because of the limited space, we cannot go into details, but rather present the result. The detailed analysis will be published in a forthcoming paper.

We consider the system on edge i:

$$\begin{aligned}
\rho_i \ddot{u}_i^{n+1} - Eh_i (u_i^{n+1})'' &= 0, \\
\rho_i \ddot{w}_i^{n+1} + EI_i (w_i^{n+1})'''' &= 0, \\
\rho_i \ddot{\varphi}_i^{n+1} - Eh_i (\varphi_i^{n+1})'' &= u_i, \\
\rho_i \ddot{\psi}_i^{n+1} + EI_i (\psi_i^{n+1})'''' &= w_i,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\epsilon_{iC}(Eh_i (u_i^{n+1})'(v_C) e_i - EI_i (w_i^{n+1})'''(v_C) e_i^\perp &= -p_i^{n+1}(v_C), \\
\epsilon_{iC}(Eh_i (\varphi_i^{n+1})'(v_C) e_i - EI_i (\psi_i^{n+1})'''(v_C) e_i^\perp &= 0, \\
\epsilon_{iC}(EI_i (w_i^{n+1})''(v_C) = \epsilon_{iC} EI_i (\psi_i^{n+1})''(v_C) &= 0,
\end{aligned} \tag{4.2}$$

$$\epsilon_{iJ}Eh_i(u_i^{n+1})'(v_J)+\varphi_i^{n+1}(v_J)=\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}p_j^n(v_J)e_i-\varphi_i^n(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}\{Eh_j(u_j^n)'(v_J)e_j-EI_j(w_j^n)'''(v_J)e_j^{\perp}\}e_i \qquad (4.3)$$
$$-\epsilon_{iJ}Eh_i(u_i^n)'(v_J)\bigg),$$

$$-\epsilon_{iJ}EI_i(w_i^{n+1})'''(v_J)+\psi_i^{n+1}(v_J)=\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}p_j^n(v_J)e_i^{\perp}-\psi_i^n(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}\{Eh_j(u_j^n)'(v_J)e_j-EI_j(w_j^n)'''(v_J)e_j^{\perp}\}e_i \qquad (4.4)$$
$$+\epsilon_{iJ}EI_i(w_i^n)'''(v_J)\bigg),$$

$$\epsilon_{iJ}EI_i(w_i^{n+1})''(v_J)+(\psi_i^{n+1})'(v_J)=\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}(\psi_j^n)'(v_J)-(\psi_i^n)'(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}EI_j(w_j^n)''(v_J)-\epsilon_{iJ}EI_i(w_i^n)''(v_J)\bigg), \qquad (4.5)$$

$$\epsilon_{iJ}Eh_i(\varphi_i^{n+1})'(v_J)-u_i^{n+1}(v_J)=-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}r_j^n(v_J)e_i-u_i^n(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}\{Eh_j(\varphi_j^n)'(v_J)e_j-EI_j(\psi_j^n)'''(v_J)e_j^{\perp}\}e_i \qquad (4.6)$$
$$-\epsilon_{iJ}Eh_i(\varphi_i^n)'(v_J)\bigg),$$

$$-\epsilon_{iJ}EI_i(\psi_i^{n+1})'''(v_J)-(w_i^{n+1})(v_J)=-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}r_j^n(v_J)e_i^{\perp}-w_i^n(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}\{Eh_j(\varphi_j^n)'(v_J)e_j-EI_j(\psi_j^n)'''(v_J)e_j^{\perp}\}e_i \qquad (4.7)$$
$$+\epsilon_{iJ}EI_i(\psi_i^n)'''(v_J)\bigg),$$

$$\epsilon_{iJ}EI_i(\psi_i^{n+1})''(v_J)-(w_i^{n+1})'(v_J)=-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}(w_j^n)'(v_J)-(w_i^n)'(v_J)\bigg)$$
$$-\bigg(\frac{2}{d_J}\sum_{j\in\mathcal{E}_J}\epsilon_{jJ}EI_j(\psi_j^n)''(v_J)-\epsilon_{iJ}EI_i(\psi_i^n)''(v_J)\bigg), \qquad (4.8)$$

$$r_i(\cdot,0) = r_{i0},\ \dot r_i(\cdot,0) = r_{i1},\ p_i(\cdot,T) = \dot p_i(\cdot,T) = 0\ . \tag{4.9}$$

5 A local optimal control problem

It can be shown that the system (4.1)–(4.9) is, in fact, an optimality system for an optimal control problem on the edge i:

$$\begin{aligned} \min_{f_C,f_J,m_J} \quad & \frac{1}{2}\int_0^T\int_0^{l_i} |r_i^{n+1}|^2 dxt + \frac{1}{2}\int_0^T \left\{|f_J|^2 + |f_C|^2 + |m_J|^2\right\} dt \\ & +\frac{1}{2}\int_0^T \left\{|r_i^{n+1}(v_J) + \mu_{iJ}^n|^2 + |(w_i^{n+1})'(v_J) + \eta_{i,J}^n|^2\right\} dt \end{aligned} \tag{5.1}$$

subject to

$$\begin{aligned} \rho_i \ddot u_i^{n+1} = Eh_i(u_i^{n+1})'', \rho_i \ddot w_i^{n+1} + EI_i(w_i^{n+1})'''' &= 0, \\ \epsilon_{iC}(Eh_i(u_i^{n+1})'(v_C)e_i - EI_i(w_i^{n+1})'''(v_C)e_i^\perp &= f_C, \\ \epsilon_{iJ}(Eh_i(u_i^{n+1})'(v_J)e_i - EI_i(w_i^{n+1})'''(v_J)e_i^\perp) &= f_J + \lambda_{iJ}^n, \\ \epsilon_{iC}(EI_i(w_i^{n+1})''(v_C) = 0,\ \epsilon_{iJ}EI_i(w_i^{n+1})''(v_J) &= m_J + \sigma_{iJ}^n, \\ r_i^{n+1} = r_{i0},\ \dot r_i^{n+1} &= r_{i1}\ . \end{aligned} \tag{5.2}$$

Here $\lambda_{iJ}^n = \lambda_{iJ}^n \cdot e_i e_i + \lambda_{iJ}^n \cdot e_i e_i^\perp$ and $\lambda_{iJ}^n \cdot e_i, \lambda_{iJ}^n \cdot e_i^\perp, \sigma_{iJ}^n$ are given by the right hand sides of (4.3)–(4.5). Accordingly, $\mu_{iJ}^n \cdot e_i, \mu_{iJ}^n \cdot e_i^\perp, \eta_{iJ}^n$ are given by the right hand sides of (4.6)–(4.8).

Now, decomposing the initial and boundary data into its local $(e_i, e_i^\perp)$-coordinates, the optimal control problem (5.1), (5.2) separates in two optimal control problems, one for the longitudinal and one for the vertical displacement. Each of the problems can be shown to possess a unique set of minimizers. In fact, these are quadratic problems with well-posed dynamics for finite energy initial data and L^2-boundary inputs.

6 Convergence

In order to prove convergence of the local pairs, which we call $(\hat r_i^n, \hat p_i^n)$, satisfying (4.1)–(4.9) (or (5.1), (5.2)) to the local restrictions (r_i, p_i) of the global optimality system (r, p) satisfying (2.1)–(2.6), we introduce the errors $\tilde r_i^n = r_i - \hat r_i^n,\ \tilde p_i^n = p_i - \hat p_i^n$. It can be seen that $\tilde r_i^n, \tilde p_i^n$ satisfy (4.1)–(4.9) with zero initial data for $\tilde r_i^n$ (and of course zero final data for $\tilde p_i^n$).

We define

$$\begin{aligned} E^{n+1} \quad := \quad & \sum_J \sum_{i\in\mathcal{E}_J} \int_0^T \left\{|Eh_i(\tilde\varphi_i^{n+1})'(v_J)|^2 + |Eh_i(\tilde u_i^{n+1})'(v_J)|^2\right. \\ & +|EI_i(\tilde\psi_i^{n+1})'''(v_J)|^2 + |EI_i(\tilde w_i^{n+1})'''(v_J)|^2 + \\ & +|EI_i(\tilde\psi_i^{n+1})''(v_J)|^2 + |EI_i(\tilde w_i^{n+1})''(v_J)|^2 + \end{aligned} \tag{6.1}$$

$$+|\tilde r_i^{n+1}(v_J)|^2 + |\tilde p_i^{n+1}(v_J)|^2 + |(\tilde\psi_i^{n+1})'(v_J)|^2$$
$$+|(\tilde w_i^{n+1})'(v_J)|^2 \ \} \, dt \ .$$

Then it can be shown by massive calculations that the following recursion formula holds

$$\begin{aligned} E^{n+1} &= E^n - 2\left(\sum_{i=1}^{n_e}\int_0^T\int_0^{l_i}|\tilde r_i^{n+1}|^2 dx\,dt + \sum_{v_C\in V_C}\int_0^T|\tilde p_i^{n+1}(v_C)|^2 dt\right) \quad (6.2)\\ &\quad - 2\left(\sum_{i=1}^{n_e}\int_0^T\int_0^{l_i}|\tilde r_i^{n}|^2 dx\,dt + \sum_{v_C\in V_C}\int_0^T|\tilde p_{i_C}^{n}(v_C)|^2 dt\right)\\ &= E^0 - 2{\sum_{k=0}^{n+1}}'\sum_{i=1}^{n_e}\int_0^T\int_0^{l_i}|\tilde r_i^k|^2 dx\,dt - 2{\sum_{k=0}^{n+1}}'\sum_{v_C\in V_C}\int_0^T|\tilde p_{i_C}^k(v_C)|^2\,dt \end{aligned}$$

(with ${\sum_{i=0}^{n}}' a_i = \frac{1}{2}a_0 + \frac{1}{2}a_n + \sum_{i=1}^{n-1} a_i$).

Now, letting n tend to infinity we see that $(E^n)_n$ remains bounded, while :

$$\sum_{k=0}^{\infty}\sum_{i=1}^{n_e}\int_0^T\int_0^{l_i}|\tilde r_i^k|^2\,dx\,dt < \infty$$

implies the convergence to zero of the direct error

$$\int_0^T\int_0^{l_i}|\tilde r_i^k|^2\,dx\,dt \longrightarrow 0 \quad k\to\infty \qquad \forall\, i=1,\dots,n_e \ ,$$

and

$$\sum_{k=0}^{\infty}\sum_{v_C\in V_C}\int_0^T|\tilde p_{i_C}^k(v_C)|^2\,dt < \infty$$

implies

$$\int_0^T|\tilde p_{i_C}^k|^2\,dt \longrightarrow 0 \qquad k\to\infty \qquad \forall\, i_C : \epsilon_{i_C C}\neq 0 \ .$$

The limiting adjoint system $p=(p_i)_i$ can be shown to admit the zero solution only. We can, therefore, show

Theorem 6.1 *Let E^0 be finite. Then the sequence of pairs $(\hat r_i^n, \hat p_i^n)$ satisfying (4.1)–(4.9) converge to (r_i, p_i) such that (r,p) satisfies the global optimality system (2.1)–(2.6). The convergence takes place in $L^2(0,T,H)$.* ■

Remark. i) More can be said about the sense of convergence as the boundedness of $(E^n)_n$ implies weak-$L^2(0,T)$ convergence on subsequences of all traces. We defer also this analysis to a

: forthcoming paper.

ii) We can also consider masses at the joints. We have developed algorithms which iteratively decouple the beams from the motion of the masses, so that the local problems which have to be computed numerically reduce to classical Euler-Bernoulli-problems rather than to hybrid ones.
iii) We currently consider this procedure also for networks of large deflection beams.
iv) As in [1] one can derive local feedback controls by the way of Riccati equations.
v) As to how the approach presented here can be used to study controllability problems is subject to current research.

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