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STABILIZATION OF A HYBRID SYSTEM FOR NOISE REDUCTION

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Abstract

We consider a simple model arising in the control of noise. We assume that the two-dimensional cavity $\Omega = (0,1)\times(0,1)$ is occupied by an elastic, inviscid, compressible fluid. The potential ϕ of the velocity field satisfies the linear wave equation. The boundary of Ω is divided in two parts Γ_0 and Γ_1. The first one, Γ_0 is flexible and occupied by a Bernoulli-Euler beam. On Γ_0 the continuity of the normal velocities of the fluid and the beam is imposed. The subset Γ_1 of the boundary is assumed to be rigid and therefore, the normal velocity of the fluid vanishes. A dissipative term is assumed to act in the one-dimensional beam equation. We prove the existence and the uniqueness of solutions. Each trajectory is proved to converge to an equilibrium as $t \to \infty$. On the other hand we show that the convergence rate of the energy is not exponential. The proof of this r esult uses a perturbation argument allowing to modify the boundary conditions so that separation of variables applies.

1 Introduction and the mathematical model

In this paper we study a simplified model for the problem of the active control of noise, introduced in [3], consisting of a two-dimensional interior cavity with a flexible boundary. The acoustic vibrations of the fluid which fills the cavity are coupled with the mechanical vibrations of a clamped beam located on the boundary of the cavity. This constitutes a hybrid system since two vibrations of different nature interact. For other examples of hybrid systems, such as those coupling strings or beams with rigid bodies, see ̧iteLM or [9].

Let us describe in more detail the mathematical model we shall study.

We consider the two-dimensional cavity $\Omega = (0,1) \times (0,1)$ filled with an elastic, inviscid, compressible fluid, in which the acoustic vibrations are coupled with the mechanical vibration of a clamped beam located in the subset $\Gamma_0 = \{(x,0) : x \in (0,1)\}$ of the boundary of Ω.

To describe the acoustic wave motion let $\vec{v}$ be the velocity, p the pressure and ρ the density of the fluid in our domain. Also, we consider that, at rest, the pressure p_0 and the density ρ_0 are constant. The linearized equations for the propagation of sound in an inviscid, elastic and compressible fluid, describing small disturbances, are (see [11]):

$$\begin{cases} \rho' + \rho_0 \, div\, \vec{v} = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \rho_0 \vec{v}\,' + \nabla p = 0 & \text{in} \quad \Omega \times (0,\infty). \end{cases} \tag{1}$$

We denote by $'$ the time derivative.

Let W be the transversal displacement (in the plane of Ω) of the beam which is assumed to be dissipative. The interior pressure p of the fluid is acting on the beam:

$$\begin{cases} W'' + W_{xxxx} + W' = p - p_0 & \text{on} \quad \Gamma_0 \times (0,\infty), \\ W(0) = W(1) = 0 & \text{for} \quad t \in (0,\infty), \\ W_x(0) = W_x(1) = 0 & \text{for} \quad t \in (0,\infty). \end{cases} \tag{2}$$

On Γ_0 we impose the condition of continuity of the velocity fields which results from the assumption that the beam is impenetrable to the fluid. The part $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ of the boundary of Ω is rigid and impenetrable, thus leading to zero normal velocity. We obtain the following boundary conditions:

$$\begin{cases} \vec{v} \cdot \nu = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \vec{v} \cdot \nu = W' & \text{on} \quad \Gamma_0 \times (0,\infty). \end{cases} \tag{3}$$

By ν we denote the outward unit normal to the boundary.

In studying sound waves, it is usual to assume that $p = f(\rho)$. In the case of small perturbations, we can consider that the relation between p and ρ is linear (see [11]):

$$p - p_0 = c_0^2(\rho - \rho_0) \tag{4}$$

where c_0 is the speed of sound in our fluid.

We obtain the following system in $\vec{v}$, p and W:

$$\begin{cases} \vec{v}' + \nabla p = 0 & \text{in} \quad \Omega \times (0,\infty) \\ p' + \operatorname{div} \vec{v} = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \vec{v} \cdot \nu = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \vec{v} \cdot \nu = W' & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W'' + W_{xxxx} + W' - p = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W(0,t) = W(1,t) = 0 & \text{for} \quad t \in (0,\infty) \\ W_x(0) = W_x(1) = 0 & \text{for} \quad t \in (0,\infty) \\ \vec{v}(0) = \vec{v^0}, \quad p(0) = p^0 & \text{in} \quad \Omega \\ W(0) = W^0, \quad W'(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases} \tag{5}$$

Observe, in particular, that the beam is clamped at its ends $x = 0, 1$.

We define the energy associated with this system by:

$$E(t) = \frac{1}{2}\int_{\Omega} (\mid \nabla \vec{v} \mid^2 + p^2) + \frac{1}{2}\int_{\Gamma_0} ((W_{xx})^2 + (W')^2). \tag{6}$$

In (5), for simplicity, we have normalized all the constants to unity. If this is not done one has to change in a convenient way the definition of the energy but the main results of this paper remain valid.

The system has a dissipative nature. Indeed, multiplying in (5) the first equation by $\vec{v}$, the second equation by p, the fifth equation by W' and integrating by parts, we get, formally, that:

$$dE(t)/dt = -\int_{\Gamma_0} (W')^2 \leq 0.$$

The aim of this article is to study the effect of the damping term, which is concentrated in the beam equation, on the asymptotic dynamics of the whole system. We shall prove that the dissipation can force the strong stabilization but it cannot ensure an uniform decay rate.

We remark that this result is not surprising in view of the structure of the damping region. Indeed, as Bardos, Lebeau and Rauch prove in [6], in the context of the control and stabilization of the wave equation in bounded domains, if one characteristic ray escapes to the dissipative region we can not expect an uniform de cay to hold (see also Ralston [22]). In our case each segment $\{(x,a),\ x \in (0,1)\}$, $0 < a < 1$ is such a ray and therefore the decay rate may not be uniform.

Nevertheless, in our problem, the lack of uniform decay is fundamentally due to the hybrid structure of the system. Indeed, the nature of the coupling between the acoustic and elastic components of the system (i.e. the boundary conditions on Γ_0) allows to build solutions with arbitrarily slow decay rate and with the energy distributed in all of the domain and not only along some particular ray of geometrical optics as in [22].

B. P. Rao in [23] has shown that, in various one-dimensional hybrid systems, the coupling is such that the damping term is a compact perturbation of the underlying conservative dynamics. This kind of arguments does not apply in our problem, since w e are in space dimension two. Actually, in [17], we have proved that, in a similar system, the difference between the semigroup generated by the dissipative system and the one generated by the corresponding conservative system is not compact.

In [1] and [2] the strong stability of the following system is studied:

$$\begin{cases} \Phi'' - \Delta\Phi = 0 & \text{in} \quad D \times (0,\infty) \\ \Phi = 0 & \text{on} \quad \gamma \setminus \gamma_0 \times (0,\infty) \\ \partial\Phi/\partial\nu + \alpha\Phi' = W' & \text{on} \quad \gamma_0 \times (0,\infty) \\ W'' + \Delta^2 W - \Delta W' + \Phi' = 0 & \text{on} \quad \gamma_0 \times (0,\infty) \\ W = \partial W/\partial\nu = 0 & \text{on} \quad \partial\gamma_0 \times (0,\infty) \\ \Phi(0) = \Phi^0, \quad \Phi'(0) = \Phi^1 & \text{in} \quad D \\ W(0) = W^0, \quad W'(0) = W^1 & \text{on} \quad \gamma_0, \end{cases} \tag{7}$$

where D is a bounded open subset of $I\!R^n$ with Lipshitz boundary γ, γ_0 is a segment of γ and $\alpha \geq 0$.

Observe that, since we are dealing with an irrotational fluid, the velocity $\vec{v}$ and pressure p can be written in terms of a potential Φ: $\vec{v} = \nabla\Phi$ and $p = -\Phi_t$. When doing this, system (5) can be rewritten as fol lows:

$$\begin{cases} \Phi'' - \Delta\Phi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \partial\Phi/\partial\nu = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \partial\Phi/\partial\nu = W' & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W'' + W_{xxxx} + W' + \Phi' = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W(0) = W(1) = 0 & \text{for} \quad t \in (0,\infty) \\ W_x(0) = W_x(1) = 0 & \text{for} \quad t \in (0,\infty) \\ \Phi(0) = \Phi^0, \quad \Phi'(0) = \Phi^1 & \text{in} \quad \Omega \\ W(0) = W^0, \quad W'(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases} \tag{8}$$

Let us point out some of the differences between systems (7) and (8). First of all observe that the potential Φ is assumed to vanish on the rigid subset $\gamma \setminus \gamma_0$ of the boundary. This simplifies the set of equilibria of the system that, in this case, is reduced to $(\Phi, W) = (0, 0)$. However, the condition $\Phi = 0$ on $\gamma \setminus \gamma_0$ does not seem to be realistic. On the other hand the continuity condition on the velocity fields has been modified. Indeed, the condition

$$\partial\Phi/\partial\nu = W',$$

has been replaced by:

$$\partial\Phi/\partial\nu + \alpha\Phi' = W', \quad \alpha \geq 0.$$

These boundary conditions introduce an extra dissipation on the system, since

$$\frac{dE}{dt}(t) = -\int_{\gamma_0} |\nabla W'|^2 - \alpha \int_{\gamma_0} |\Phi'|^2.$$

Moreover, the displacement W is assumed to satisfy a strongly damped plate equation whose principal part $W'' + \Delta^2 W - \Delta W'$ is known to generate an analytic semigroup. In this sense, this problem is different from ours. An analogous model in which the strongly damped plate equation is replaced by $W'' - W_{xx} - W'_{xx} + \Phi' = 0$ and $\alpha = 0$ has been analyzed in [18].

In [1], taking $\alpha > 0$ and γ_0 sufficiently large, the exponential stability result is proved by using multipier techniques.

We also remark that the properties of a similar system with a string equation instead of a beam equation has been analyzed in [19].

The rest of the paper is organized as follows.

In Section 2 we present an abstract formulation of the problem and we give a result of existence, uniqueness and stability of solutions. Since we are dealing with a linear system all these results are direct consequences of the classical theory of maximal-monotone operators.

The asymptotic properties of the solutions are studied in Sections 3 and 4.

In Section 3 we prove the convergence of each solution of the system to an equilibrium point uniquely determined by the corresponding initial datum. We do this using classical techniques involving La Salle's Invariance Principle and Holmgren's Uniqueness Theorem.

The rate of the convergence to the equilibrium is studied in Section 4. We prove that the decay rate is not uniform. In order to do this we start from the observation that the same property is true for a system with different boundary conditions for the beam (which allows separation of variables) and next we use the fact that the difference between these two systems is negligible at high frequencies.

2 Existence and uniqueness of solutions

We define the space of finite energy corresponding to (5) by:

$$\mathcal{X}_0 = \mathcal{L} \times L^2(\Omega) \times H_0^2(\Gamma_0) \times L^2(\Gamma_0),$$
$$\mathcal{L} = \left\{ \vec{v} \in L^2(\Omega) \times L^2(\Omega) : \ \text{curl } \vec{v} = 0 \right\} =$$
$$= \left\{ \vec{v} = (v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : \int_\Omega \left(\frac{\partial \varphi}{\partial x} v_2 - \frac{\partial \varphi}{\partial y} v_1 \right) = 0, \, \forall \varphi \in C_c^\infty(\Omega) \right\}.$$

Remark 1 Observe that $\vec{v} \in \mathcal{L}$ if and only if there exists a function $\Phi \in H^1(\Omega)$ such that $\nabla \Phi = \vec{v}$.

$\mathcal{X}_0$ with the natural inner product is a Hilbert space.

We define in $\mathcal{X}_0$ the unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ in the following way:

$$\mathcal{A}(\vec{v}, p, W, V) = (\nabla p, \text{div } \vec{v}, -V, W_{xxxx} + V - p),$$
$$\mathcal{D}(\mathcal{A}) = \left\{ U = (\vec{v}, p, W, V) \in \mathcal{X}_0 : \ \mathcal{A}(U) \in \mathcal{X}_0, \ \vec{v} \cdot \nu = 0 \text{ on } \Gamma_1, \ \vec{v} \cdot \nu = V \text{ on } \Gamma_0 \right\}.$$

Remark 2 Let $(\vec{v}, p, W, V) \in \mathcal{D}(\mathcal{A})$. Observe that $\operatorname{div} \vec{v} \in L^2(\Omega)$ and $\vec{v} \in \mathcal{L}$ imply that there exists $\Phi \in H^1(\Omega)$ with $\nabla\Phi = \vec{v}$ such that $\Delta\Phi \in L^2(\Omega)$. Since, in addition, we have $\vec{v} \cdot \nu = 0$ on Γ_1 and $\vec{v} \cdot \nu = V$ on Γ_0 we obtain that

$$\begin{cases} \Delta\Phi \in L^2(\Omega) \\ \partial\Phi/\partial\nu = 0 \text{ on } \Gamma_1, \quad \partial\Phi/\partial\nu = V \in H_0^1(\Gamma_0) \text{ on } \Gamma_0. \end{cases}$$

Since Ω is convex it results that $\Phi \in H^2(\Omega)$ (see [8], Theorem 5.1.3.5, p. 263). It follows that $\mathcal{D}(\mathcal{A}) \subseteq H^1(\Omega) \times H^2(\Omega) \times H^4(\Gamma_0) \cap H_0^2(\Gamma_0) \times H_0^2(\Gamma_0)$ and therefore $\mathcal{D}(\mathcal{A})$ is compact in $\mathcal{X}_0$.

We can consider now the following abstract Cauchy formulation of (5):

$$\begin{cases} U' + \mathcal{A}U = 0 \\ U(0) = U_0 \\ U(t) = (\vec{v}, p, W, W')(t) \in \mathcal{D}(\mathcal{A}). \end{cases} \tag{9}$$

First, we have a classical result of existence, uniqueness and stability for the system (9). The terminology we use is the same as in [7].

Theorem 1 *i) $\mathcal{A}$ is a maximal monotone operator in $\mathcal{X}_0$ generating a strongly continuous semigroup of contractions, $\{S(t)\}_{t\geq 0}$, in $\mathcal{X}_0$.*

ii) Strong solutions: If $U^0 = (\vec{v^0}, p^0, W^0, W^1) \in \mathcal{D}(\mathcal{A})$ then there exists a unique strong solution $S(t)U^0 = U \in \mathcal{C}([0,\infty), \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0,\infty), \mathcal{X}_0)$ of (9).

iii) Weak solutions: If $U^0 = (\vec{v^0}, p^0, W^0, W^1) \in \mathcal{X}_0$ then there exists a unique solution $S(t)U^0 = U \in \mathcal{C}([0,\infty), \mathcal{X}_0)$ of (9).

For any weak solution, the associated energy (6) satisfies:

$$\frac{dE}{dt}(t) = -\int_{\Gamma_0} (W')^2. \tag{10}$$

proof: We prove first that the operator $\mathcal{A}$ is maximal monotone in $\mathcal{X}_0$.

Indeed, if $U^0 = (\vec{v^0}, p^0, W^0, W^1) \in \mathcal{D}(\mathcal{A})$ then
$\langle \mathcal{A}U^0, U^0 \rangle \leq -\int_{\Gamma_0} (W^1)^2 \leq 0$, which means that $\mathcal{A}$ is monotone.

On the other hand, for all $F = (\vec{f_1}, f_2, f_3, f_4) \in \mathcal{X}_0$ we can find a unique solution $U = (\vec{v}, p, W, V) \in \mathcal{D}(\mathcal{A})$ for the equation $(\mathcal{A} + \mathcal{I})U = F$. This is equivalent to solve the following system:

$$\begin{cases} \nabla p + \vec{v} = \vec{f_1} \\ \operatorname{div} \vec{v} + p = f_2, \ \vec{v} \cdot \nu = 0 \text{ on } \Gamma_1 \text{ and } \vec{v} \cdot \nu = V \text{ on } \Gamma_0 \\ V + W = f_3 \\ W_{xxxx} + V - p + V = f_4, \ W(0) = W(1) = 0 \text{ and } W_x(0) = W_x(1) = 0. \end{cases} \tag{11}$$

First, we consider the variational formulation of (11), which consists in finding (p, W) in $H^1(\Omega) \times H_0^2(\Gamma_0)$ such that, for all $(\varphi, u) \in H^1(\Omega) \times H_0^2(\Gamma_0)$:

$$\begin{aligned} &\int_\Omega \nabla p \cdot \nabla \varphi + \int_\Omega p\varphi + \int_{\Gamma_0} W\varphi + \int_{\Gamma_0} W_{xx}u_{xx} - \int_{\Gamma_0} pu + 2\int_{\Gamma_0} Wu = \\ &= \int_\Omega \vec{f_1} \cdot \nabla\varphi + \int_\Omega f_2\varphi + \int_{\Gamma_0} f_3\varphi + \int_{\Gamma_0} (f_4 + 2f_3)u. \end{aligned} \tag{12}$$

The left side of the equation (12) defines a continuous and coercive bilinear form in $(H^1(\Omega) \times H_0^2(\Gamma_0))^2$ while the right side defines a continuous linear form in $H^1(\Omega) \times H_0^2(\Gamma_0)$.

Applying Lax-Milgram's Lemma it results that (12) has a unique solution (p, W) in $H^1(\Omega) \times H_0^2(\Gamma_0)$. Finally, in view of the classical regularity results for Laplace's operator, this implies that $\mathcal{A} + \mathcal{I}$ is maximal.

Since the operator $\mathcal{A}$ is maximal monotone in $\mathcal{X}_0$ we can apply the Hille-Yosida theory (see [7], Theorem 3.1.1, p.37) and obtain the stated results.

3 Strong stabilization

Concerning the asymptotic behavior of solutions we prove first the following theorem.

Theorem 2 *For each initial data* $U^0 = (\vec{v^0}, p^0, W^0, W^1)$ *in* $\mathcal{X}_0$ *the corresponding weak solution of (9) tends asymptotically towards the equilibrium point* $(\vec{0}, b, ba(x), 0)$ *where* $b = \frac{120}{121}\left(\int_\Omega p^0 + \int_{\Gamma_0} W^0\right)$ *and* $a(x) = \frac{1}{4}(x^4 - 2x^3 + x^2)$.

Remark 3 We obtain that the velocities of the fluid and the beam go to zero whereas the pressure of the fluid and the position of the beam tend to some functions that are uniquely determined by the initial data. Notice that the pressure stabilizes around a suitable constant while the asymptotic deformation of the beam describes a quadratic function.

proof: The main tools of our analysis are an extension of the well known Invariance Principle of La Salle and Holmgren's Uniqueness Theorem.

Observe first that it is sufficient to consider only initial data $U^0 = (\vec{v^0}, p^0, W^0, W^1)$ in $\mathcal{D}(\mathcal{A})$. A standard density argument and the property of stability (10) enable us to complete the proof. In this case Theorem 1 gives an unique strong solution $U(t) = (\vec{v}, p, W, W')(t) = S(t)U^0$ for the equation (5), with $\{U(t)\}_{t\geq 0}$ bounded in $\mathcal{D}(\mathcal{A})$. Since $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{X}_0$ with compact inclusion, we have that $\{U(t)\}_{t\geq 0}$ is relatively compact in $\mathcal{X}_0$.

We now describe the equilibrium points corresponding to our problem. These are elements $Z = (\vec{u}, r, X, Y) \in \mathcal{D}(\mathcal{A})$ with $S(t)Z = Z$ for all $t \geq 0$. It follows that the equilibrium points are characterized by the system:

$$\begin{cases} \nabla r = 0 & \text{in } \Omega \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega \\ \vec{u} \cdot \nu = 0 & \text{on } \partial\Omega \\ X_{xxxx} - r = 0 & \text{on } \Gamma_0 \\ X(0) = X(1) = 0 & \\ X_x(0) = X_x(1) = 0. & \end{cases} \tag{13}$$

From (13) we deduce that the equilibrium points are $(\vec{0}, b, b\, a(x), 0)$, where b is a real constant and $a(x)$ is the solution of the differential equation:

$$\begin{cases} a_{xxxx} - 1 = 0, \ x \in (0,1) \\ a(0) = a(1) = 0 \\ a_x(0) = a_x(1) = 0. \end{cases}$$

On the other hand we remark that the energy function defined by (6) is a Lyapunov function for the dynamical system defined by $S(t)U^0 = U(t)$ since it satisfies relation (10). We prove now that $E(t)$ is a strict Lyapunov function. To do this let $Z^0 = (\vec{u^0}, r^0, X^0, Y^0) \in \mathcal{X}_0$, $Z(t) = (\vec{u}, r, X, Y)(t) = S(t)Z^0$ for all $t > 0$ and suppose that the energy of the solution $Z(t)$ is constant. Hence $Y(t) = 0$, by (10).

It follows that $(\vec{u}, r, X, Y)$ satisfies:

$$\begin{cases} \vec{u}' + \nabla r = 0 & \text{in} \quad \Omega \times (0,\infty) \\ r' + \operatorname{div} \vec{u} = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \vec{u} \cdot \nu = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ X_{xxxx} - r = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ X(0,t) = X(1,t) = 0 & \text{for} \quad t \in (0,\infty) \\ X_x(0,t) = X_x(1,t) = 0 & \text{for} \quad t \in (0,\infty). \end{cases} \tag{14}$$

Therefore:

$$\begin{cases} r'' - \Delta r = 0 & \text{in } \Omega \\ \partial r / \partial \nu = 0 & \text{on } \partial\Omega \\ r' = 0 & \text{on } \Gamma_0. \end{cases} \tag{15}$$

We can apply now Holmgren's Uniqueness Theorem (see [10], Theorem 8.6.5, p. 309 and [12], Theorem 8.1, p. 88) which implies that $r' = 0$ in $\Omega \times (1,\infty)$ and so $r(t,x,y) = r(x,y)$ in $\Omega \times (1,\infty)$.

From (15) we can deduce that $r = b$ in $\Omega \times (1,\infty)$ where b is a real constant.

Moreover, from (14), it follows that $\vec{u} = 0$ in $\Omega \times (1,\infty)$ and X is solution of the equation:

$$\begin{cases} X_{xxxx} - b = 0 & \text{on } \Gamma_0 \times (1,\infty) \\ X(0,t) = X(1,t) = 0 & \text{for } t \in (0,\infty) \\ X_x(0,t) = X_x(1,t) = 0 & \text{for } \quad t \in (0,\infty). \end{cases}$$

Taking into account the uniqueness of solutions of the system (14) we obtain that $Z^0 = (\vec{u^0}, r^0, X^0, Y^0) = (\vec{0}, b, ba(x), 0)$. Hence Z^0 is an equilibrium. Therefore $E(t)$ is a strict Lyapunov function.

We are now in conditions to apply La Salle's Invariance Principle.

Let now $U^0 = (\vec{v^0}, p^0, W^0, W^1)$ be the initial data for (5). By La Salle's Invariance Principle it follows that the trajectory tends to the set of the equilibrium points when the times goes to infinity. Let us prove that, in fact, the trajectory converges to a unique point.

Integrating the second equation of (5) in Ω we deduce that the quantity $\displaystyle\int_\Omega p^0 + \int_{\Gamma_0} W^0$ is constant along the trajectory. Since the equilibrium points are of the form $(\vec{0}, b, ba(x), 0)$ it follows that the corresponding solution of (5) tends to an unique equilibrium point, the one for which $b = \dfrac{120}{121}\left(\displaystyle\int_\Omega p^0 + \int_{\Gamma_0} W^0\right)$.

Remark 4 We can decompose the space $\mathcal{X}_0$ as $\mathcal{X}_0 = \mathcal{X}_0^0 \oplus \mathcal{X}_0^1$, where:

$$\mathcal{X}_0^0 = \left\{(\vec{v}^0, p^0, W^0, V^0) \in \mathcal{X}_0 : \textstyle\int_\Omega p^0 + \int_{\Gamma_0} W^0 = 0\right\},$$
$$\mathcal{X}_0^1 = \left\{(\vec{0}, b, b\,a(x), 0) \in \mathcal{X}_0,\ b \in \mathbb{R}\right\}.$$

The projection of the solution $U(t)$ of (5) on $\mathcal{X}_0^1$ is a constant function in time whereas, by Theorem 2, the projection on $\mathcal{X}_0^0$ tends to zero as t goes to infinity.

4 The lack of uniform decay

In this paragraph we prove that the rate of decay is not uniform. Results like this are typical for linear hybrid systems in which the dissipation is very weak: it can force the strong stabilization but it cannot ensure the uniform decay.

First of all we recall that a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ has exponential decay if there are two constants $\omega > 0$ and $M > 0$ such that

$$||S(t)|| \leq M \exp(-\omega\, t), \quad \forall t \geq 0. \tag{16}$$

We also remark that, in the case of linear semigroups, the exponential decay is equivalent to the uniform decay. Therefore, if a linear semigroup $\{S(t)\}_{t\geq 0}$ does not have exponential decay then there are initial data U^0 such that $S(t)U^0$ decays arbitrarily slowly to zero. More precisely, if $\psi : [0, \infty) \longrightarrow \infty$ is a continuous decreasing function such that $\psi(t) \to 0$ as $t \to \infty$ then there exist an initial data $U^0 \in \mathcal{X}_0$ and a sequence $(t_k)_{k\geq 0}$ tending to infinity such that $||S(t_k)U^0|| > \psi(t_k)$ (see [13]).

When the boundary conditions of W in (5) are replaced by the following ones:

$$\begin{cases} W_x(0,t) = W_x(1,0) = 0, & t > 0 \\ W_{xxx}(0,t) = W_{xxx}(1,0) = 0, & t > 0. \end{cases} \tag{17}$$

this result is easy to show. Indeed, under these boundary conditions one can find a sequence of solutions $\{(\vec{v_n}, p_n, W_n)\}_{n\in I\!N}$ of the type $(\vec{v_n}, p_n, W_n) = e^{-\lambda_n t}(\vec{v_n}, p_n, W_n)$ in separated variables such that $\mathcal{R}e\,\lambda_n \to 0$ as $n \to \infty$. However, the separation of variables does not apply with the boundary conditions we are considering.

In order to prove that, for our system, there is no uniform decay we analyze first a conservative problem. Next, using the fact that these two systems are very close one from another (in a way that we shall make precise later on) we prove the desired property.

We consider now the following undamped system in $\vec{v}$, p and W:

$$\begin{cases} \vec{v}' + \nabla p = 0 & \text{in} \quad \Omega \times (0,\infty) \\ p' + \operatorname{div} \vec{v} = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \vec{v}\cdot\nu = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \vec{v}\cdot\nu = W' & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W'' + W_{xxxx} - p = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & \text{for} \quad t \in (0,\infty) \\ W_{xxx}(0,t) = W_{xxx}(1,t) = 0 & \text{for} \quad t \in (0,\infty) \\ \vec{v}(0) = \vec{v^0}, \quad p(0) = p^0 & \text{in} \quad \Omega \\ W(0) = W^0, \quad W'(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases} \tag{18}$$

Remark 5 Since we have dropped the dissipative term W' in the beam equation the system (18) is conservative. On the other hand we remark that the boundary conditions for the beam in (5) have been replaced by the boundary conditions (17). This will allow us to use the separation of variables and to obtain useful informations about the eigenvalues and eigenfunctions of the system. We do this in Lemmas 1 and 2.

The initial data $(\vec{v^0}, p^0, W^0, W^1)$ is considered in the space of finite energy:

$$\mathcal{X} = \mathcal{L} \times L^2(\Omega) \times V \times L^2(\Gamma_0),$$

where $V = \left\{ v \in H^2(\Gamma_0) : \; v_x(0) = v_x(1) = 0 \right\}$.

We define the energy associated to this system in the same way as in (6). We also define in $\mathcal{X}$ the unbounded operator $(\mathcal{D}(\mathcal{B}), \mathcal{B})$:

$$\begin{aligned} \mathcal{B}(\vec{v}, p, W, V) &= (\nabla p, \operatorname{div} \vec{v}, -V, W_{xxxx} - p), \\ \mathcal{D}(\mathcal{B}) = \{U &= (\vec{v}, p, W, V) \in \mathcal{X} : \; \mathcal{B}(U) \in \mathcal{X}, \; \vec{v}\cdot\nu = 0 \text{ on } \Gamma_1, \\ & \vec{v}\cdot\nu = V \text{ on } \Gamma_0, \; W_{xxx}(0) = W_{xxx}(1) = 0\}. \end{aligned}$$

Lemma 1 The operator $\mathcal{B}$ has a sequence of purely imaginary eigenvalues $(\lambda_n\, i)_{n\in I\!N}$ where λ_n are the roots of the equation:

$$\zeta \;\tan\zeta = 1. \tag{19}$$

proof We look for a sequence of solutions $\{(\vec{v_n}, p_n, W_n)\}_{n\in \mathbb{N}}$ for (18) of the type $(\vec{v_n}, p_n, W_n) = e^{-\lambda_n\, i\, t}(\vec{u_n}, r_n, v_n)$ where $\vec{u_n} = \vec{u_n}(y)$, $r_n = r_n(y)$ and $v_n \in \mathbb{R}$.

We can see that (18) has solutions of this form if $(\vec{u_n}, r_n, v_n)$ satisfies

$$\begin{cases} -\lambda_n\, i\, \vec{u_n} + \nabla r_n = 0 \text{ for } y \in (0,1) \\ -(\lambda_n)^2 r_n - (r_n)_{yy} = 0 \text{ for } y \in (0,1) \\ (r_n)_y(1) = 0,\ (r_n)_y(0) = -(\lambda_n)^2 v_n \\ (\lambda_n)^2 v_n + r_n(0) = 0. \end{cases} \tag{20}$$

It follows that $r_n(y) = \cos(\lambda_n(y-1))$, $\vec{u_n} = \dfrac{1}{\lambda_n\, i}\nabla r_n$ and $v_n = -\dfrac{1}{\lambda_n}\sin\lambda_n$ solves (20) if λ_n is solution of the algebraic equation (19).

It is well known that, for each $n \in \mathbb{N}$, there is a root of this equation which belongs to the interval $\left(n\pi - \dfrac{\pi}{2},\, n\pi + \dfrac{\pi}{2}\right)$. This concludes the proof.

Remark 6 A very similar proof allows us to show that, if in the beam equation in system (18) we introduce the dissipative term W', there is a sequence of eigenvalues such that $\mathcal{R}e\,(\lambda_n) \to 0$ as $j \to \infty$ as we mentioned before. This implies that the decay rate of the associated semigroup is not uniform. In the case of system (5) under consideration it is difficult to show directly the existence of such solutions since we can not use separation of variables.

Remark 7 The roots $(\lambda_n)_n$ of the equation (19) have the following asymptotic development:

$$\lambda_n = n\pi + \frac{1}{n\pi} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

For details see [20], p. 12.

To each eigenvalue $\lambda_n\, i$ given by Lemma 1 it corresponds an eigenfunction ξ_n defined by:

$$\xi_n = \begin{pmatrix} \dfrac{1}{\lambda_n\, i}\,\nabla\cos(\lambda_n(y-1)) \\ \cos(\lambda_n(y-1)) \\ -\dfrac{\sin\lambda_n}{\lambda_n} \\ i\sin\lambda_n \end{pmatrix}. \tag{21}$$

We shall denote by ξ_n^j, $j \in \{1, ..., 4\}$ the components of ξ_n.

Lemma 2 If $(\xi_n)_n$ is the sequence of eigenfunctions of system (18) corresponding to the eigenvalues $(\lambda_n\, i)_n$ given by Lemma 1 then:

i) The last two components of ξ_n tend to zero when n tends to infinity.

ii) The sequence $(\xi_n)_n$ does not tend to zero in $\mathcal{X}$ when n tends to infinity.

proof i) Since $\lambda_n = n\pi + \dfrac{1}{n\pi} + \mathcal{O}\left(\dfrac{1}{n^3}\right)$ it follows that $(\sin\lambda_n)_n$ tends to zero in $I\!\!R$ when n tends to infinity.

ii) We simply remark that
$||\xi_n||^2_{\mathcal{X}_0} \geq ||\xi_n^2||^2_{L^2(\Omega)} = \dfrac{1}{2} - \dfrac{\sin 2\lambda_n}{4\lambda_n} \longrightarrow \dfrac{1}{2}$ as $n \to \infty$.

Remark 8 Lemma 2 shows that there are solutions of (18) in which the effect of the vibrating beam vanishes asymptotically. This indicates that the boundary conditions for the beam are not very important at high frequencies. Since the system with boundary conditions (17) does not have an exponential decay (see Remark 6) we can expect that this will be the case for system (5) too. Indeed, a s the proof of following Theorem shows, the solutions of (18) can be slightly modified in order to obtain solutions of (5) with arbitrarily small exponential decay rate.

The main result of this paper is the following:

Theorem 3 *The decay rate of the semigroup $\{S(t)\}_{t\geq 0}$ is not exponential in the space $\mathcal{X}_0^0$.*

proof We shall prove the theorem by contradiction. Suppose that $\{S(t)\}_{t\geq 0}$ has exponential decay in $\mathcal{X}_0^0$, i.e. there are two constants $\omega > 0$ and $M > 0$ such that:

$$||S(t)||_{\mathcal{X}_0^0} \leq M \,\exp\,(-\omega\, t), \quad \forall t \geq 0.$$

Let $\mathcal{R}(\mathcal{A} : \mu)$ be the resolvent of $\mathcal{A}$ in μ, $\mathcal{R}(\mathcal{A} : \mu) = (\mathcal{A} - \mu\mathcal{I})^{-1}$, where μ is a complex number in the resolvent set of $\mathcal{A}$. We recall that $\mathcal{R}(\mathcal{A} : \mu) = \int_0^\infty e^{\mu\, t} S(t)\, dt$ (see [21], Theorem 3.1, p. 8). Hence

$$||\mathcal{R}(\mathcal{A} : \mu)||_{\mathcal{X}_0^0} \leq \int_0^\infty e^{\mathcal{R}e\,\mu\, t} ||S(t)||_{\mathcal{X}_0^0}\, dt \leq \int_0^\infty M\, e^{(\mathcal{R}e\,\mu - \omega)\, t}\, dt. \tag{22}$$

Since the operator $\mathcal{A}$ is dissipative we have that the resolvent is well defined from $\mathcal{X}_0^0$ to $\mathcal{D}(\mathcal{A})$ for all imaginary numbers μ (with $\mathcal{R}e\,\mu = 0$). In this case we obtain from (22) that the resolvents are uniformly bounded:

$$||\mathcal{R}(\mathcal{A} : \mu)||_{\mathcal{X}_0^0} \leq \frac{M}{\omega} \quad \text{for all } \mu \text{ with } \mathcal{R}e\,\mu = 0. \tag{23}$$

We shall prove that there exist a sequence of imaginary numbers $(\lambda_n\, i)_{n\in I\!\!N}$, $\lambda_n \in I\!\!R$, and a sequence of functions $(\Phi_n)_{n\in I\!\!N} \subset \mathcal{X}_0^0$, $||\Phi_n||_{\mathcal{X}_0^0} = 1$, such that

$$||\mathcal{R}(\mathcal{A} : \mu)\Phi_n||_{\mathcal{X}_0^0} \longrightarrow \infty \text{ when } n \longrightarrow \infty. \tag{24}$$

This contradicts (23) and the proof is completed.

In order to do this let $(\lambda_n\, i)_{n\in I\!N}$ be the sequence of eigenvalues of the problem (18) given in Lemma 1 and let $(\xi_n)_{n\in I\!N}$ be the corresponding eigenfunctions given by (21).

Observe that $\xi_n \notin \mathcal{X}_0^0$ because the third component, which is a constant, does not belong to $H_0^1(\Gamma_0)$. We shall "cut-off" this constant function in order to get a slightly modified one in $H_0^1(\Gamma_0)$.

For each $n \in I\!N$ we define the function $u_n : [0,1] \longrightarrow [-1,1]$ by:

$$u_n = \begin{cases} e^{\frac{|\lambda_n| x^2}{1-|\lambda_n| x^2}}, & \text{if } x \in \left[0, \frac{1}{\sqrt{|\lambda_n|}}\right) \\ e^{\frac{|\lambda_n|(1-x)^2}{1-|\lambda_n|(1-x)^2}}, & \text{if } x \in \left(1-\frac{1}{\sqrt{|\lambda_n|}}, 1\right] \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

and the function h_n as the solution of:

$$\begin{cases} -\Delta h_n + h_n = 0 & \text{in } \Omega \\ \dfrac{\partial h_n}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial h_n}{\partial \nu} = -i \sin(\lambda_n)\, u_n & \text{on } \Gamma_0. \end{cases} \tag{26}$$

Let now

$$\psi_n = \begin{pmatrix} \nabla h_n \\ -\frac{\sin\lambda_n}{\lambda_n} \int_0^1 u_n \\ \frac{\sin\lambda_n}{\lambda_n} u_n \\ -i \sin\lambda_n\, u_n \end{pmatrix}$$

and $\varphi_n = \xi_n + \psi_n$.

From the definitions of the functions u_n and h_n it follows that $\varphi_n \in \mathcal{D}(\mathcal{A}) \cap \mathcal{X}_0^0$ for all $n \in I\!N$.

Finally let $\Phi_n = \dfrac{(\mathcal{A} - \lambda_n\, i\, \mathcal{I})\varphi_n}{||(\mathcal{A} - \lambda_n\, i\, \mathcal{I})\varphi_n||_{\mathcal{X}_0^0}}$.

We obtain that:

$$\mathcal{R}(\mathcal{A} : \lambda_n\, i)\, \Phi_n = \frac{\varphi_n}{||(\mathcal{A} - \lambda_n\, i\, \mathcal{I})\varphi_n||_{\mathcal{X}_0^0}},$$

and we want to prove that (24) holds.

Since we need more information about the norms of φ_n and $(\mathcal{A} - \lambda_n\, i\mathcal{I})\varphi_n$ we shall prove first some properties of the functions u_n and h_n.

Lemma 3 The functions u_n and h_n defined by (25) and (26) respectively have the following properties:

i) $||u_n||^2_{L^2} = \mathcal{O}\left(\dfrac{1}{\sqrt{|\lambda_n|}}\right)$.

ii) $||(u_n)_{xxxx}||^2_{L^2} = \mathcal{O}\left(|\lambda_n|^{\frac{7}{2}}\right)$.

iii) $||h_n||^2_{H^1} = \mathcal{O}\left(\dfrac{1}{|\lambda_n|^{\frac{5}{2}}}\right)$.

iv) $||\Delta h_n||^2_{L^2} = ||h_n||^2_{L^2} = \mathcal{O}\left(\dfrac{1}{|\lambda_n|^{\frac{5}{2}}}\right)$.

proof i) From (25) we obtain

$$||u_n||^2_{L^2(0,1)} = 2\int_0^{\frac{1}{\sqrt{|\lambda_n|}}} e^{\frac{2|\lambda_n|x^2}{1-|\lambda_n|x^2}} = \frac{2}{\sqrt{|\lambda_n|}}\int_0^1 e^{\frac{2y^2}{1-y^2}}\,dy = \mathcal{O}\left(\frac{1}{\sqrt{|\lambda_n|}}\right).$$

ii) We have that

$$||(u_n)_{xxxx}||^2_{L^2(0,1)} = 2|\lambda_n|^4\int_0^{\frac{1}{\sqrt{|\lambda_n|}}} v_{yyyy}(\sqrt{|\lambda_n|}x)^2\,dx =$$

$$= 2|\lambda_n|^{\frac{7}{2}}\int_0^1 v_{yyyy}(y)^2\,dy = \mathcal{O}\left(|\lambda_n|^{\frac{7}{2}}\right)$$

where $v(y) = e^{\frac{2y^2}{1-y^2}}$.

iii) From (26) we deduce that, for all $\delta > 0$,

$$||h_n||^2_{H^1(\Omega)} + ||h_n||^2_{L^2(\Omega)} = \left|-\sin\lambda_n\int_{\Gamma_0} u_n\,h_n\right| \leq$$

$$\leq \frac{|\sin\lambda_n|}{2}\left(\frac{1}{\delta}\int_{\Gamma_0}|u_n|^2 + \delta\int_{\Gamma_0}|h_n|^2\right) \leq$$

$$\leq \frac{|\sin\lambda_n|}{2}\left(\frac{1}{\delta}\int_{\Gamma_0}|u_n|^2 + \delta c||h_n||^2_{H^1(\Omega)}\right).$$

Taking $\delta = \frac{1}{|\sin\lambda_n|\,c}$ we obtain that

$$||h_n||^2_{H^1(\Omega)} \leq c|\sin\lambda_n|^2\int_{\Gamma_0}|u_n|^2.$$

Here c is a generic positive constant that may vary from line to line.

Since $\sin\lambda_n = \mathcal{O}\left(\dfrac{1}{\lambda_n}\right)$ and $\displaystyle\int_{\Gamma_0}|u_n|^2 = \mathcal{O}\left(\frac{1}{\sqrt{|\lambda_n|}}\right)$ by i), iii) follows.

iv) We simply observe that

$$||-\Delta h_n||_{L^2(\Omega)} = ||h_n||_{L^2(\Omega)} \leq ||h_n||_{H^1(\Omega)}$$

and use iii). The proof of the Lemma is now completed.

In order to complete the proof of the theorem we estimate $||(\mathcal{A}-\lambda_n\,i\,\mathcal{I})\varphi_n||_{\mathcal{X}_0^0}$ and $||\varphi_n||_{\mathcal{X}_0^0}$ when n tends to infinity.

Observe first that, by Lemma 3 i), we have

$$||\varphi_n||_{\mathcal{X}_0^0} \geq ||\xi_n^2||_{L^2(\Omega)} - \left|\frac{\sin\lambda_n}{\lambda_n}\int_0^1 u_n\right| \longrightarrow \frac{1}{2} \text{ as } n \longrightarrow \infty. \tag{27}$$

On the other hand

$$(\mathcal{A}-\lambda_n\, i\,\mathcal{I})\varphi_n = \begin{pmatrix} -\lambda_n\, i\,\nabla h_n \\ \Delta h_n + i\sin\lambda_n \displaystyle\int_0^1 u_n \\ 0 \\ \dfrac{\sin\lambda_n}{\lambda_n}(u_n)_{xxxx} + \dfrac{\sin\lambda_n}{\lambda_n}\displaystyle\int_0^1 u_n - i\sin\lambda_n\, u_n - \lambda_n\sin\lambda_n\, u_n \end{pmatrix}.$$

We obtain that

$$||(\mathcal{A}-\lambda_n\, i\,\mathcal{I})\varphi_n||^2_{\mathcal{X}_0^0} \leq |\lambda_n|^2||h_n||^2_{H^1(\Omega)} + 2||\Delta h_n||^2_{L^2(\Omega)} +$$

$$+\left|\frac{4\sin\lambda_n}{\lambda_n}\right|^2\left(||(u_n)_{xxxx}||^2_{L^2(\Gamma_0)} + ||u_n||^2_{L^2(\Gamma_0)}\right) +$$

$$+6|\sin\lambda_n|^2||u_n||^2_{L^2(\Gamma_0)} + 4|\lambda_n\sin\lambda_n|^2||u_n||^2_{L^2(\Gamma_0)}.$$

Taking into account the results of Lemma 3 and the fact that $\sin\lambda_n = \mathcal{O}\left(\frac{1}{\lambda_n}\right)$ we obtain that

$$||(\mathcal{A}-\lambda_n\, i\,\mathcal{I})\varphi_n||^2_{\mathcal{X}_0^0} \longrightarrow 0 \text{ when } n \longrightarrow \infty. \tag{28}$$

The last result together with (27) contradicts (23). So the assumption that $\{S(t)\}_{t\geq 0}$ has exponential decay must be false and the proof is completed.

Remark 9 Analyzing the exponential stability of the classical wave equation with dissipation on the boundary

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega\times(0,\infty) \\ \dfrac{\partial u}{\partial\nu} + u' = 0 & \text{on } \Gamma_0\times(0,\infty) \\ u = 0 & \text{on } \Gamma_1\times(0,\infty) \end{cases} \tag{29}$$

Bardos, Lebeau and Rauch in [6] prove that if one characteristic ray escapes to the dissipative region Γ_0 we can construct solutions with an arbitrary decay rate and with the energy concentrated along this ray. In our case every segment $\{(x, y_0) : x \in (0,1)\}$, for any $y_0 \in (0,1)$, constitutes a ray with such a property and their argument could be applied as well.

Nevertheless the proof of Theorem 3 shows that we can find a sequence of solutions of (5) with the energy uniformly distributed in all Ω and with arbitrarily small exponential decay rate. Indeed, if $(\Phi_n)_n$ is the sequence considered in the proof, let $(S(t)\Phi_n)_n$ be the sequence of corresponding solutions of (5). By (22) we have that

$$||\mathcal{R}(\mathcal{A} : \lambda_n\, i)\Phi_n||_{\mathcal{X}_0^0} \leq \int_0^\infty ||S(t)\Phi_n||_{\mathcal{X}_0^0}\, dt.$$

If $(S(t)\Phi_n)_n$ had an uniform exponential decay rate, for example, $||S(t)\Phi_n||_{\mathcal{X}_0^0} \leq M\exp(-\omega t)$, then

$$||\mathcal{R}(\mathcal{A} : \lambda_n\, i)\Phi_n||_{\mathcal{X}_0^0} \leq \frac{M}{\omega}$$

which is not true since (24) holds.

Therefore the lack of uniform decay of our system is of a different nature and is related not only to the support of the dissipative mechanism but also to the nature of the boundary conditions or of the coupling between the different components of the system.

Remark 10 We mention that in the proof of Theorem 3 we may start with solutions $(\vec{v_n}, p_n, W_n)_{n\in\mathbb{N}}$ of (18) of the type $(\vec{v_n}, p_n, W_n) = e^{-\lambda_n\, i\, t}(\vec{u_n}, r_n, v_n)$ $\cos(m\pi x)$ where $\vec{u_n} = \vec{u_n}(y)$, $r_n = r_n(y)$, $v_n \in \mathbb{R}$ and an arbitrary $m \in \mathbb{N}$. Therefore we can find a sequence of solutions of (5) with arbirary exponential decay rate and with a fixed frequency of vibration in the $x-$direction ($m \in \mathbb{N}$ fixed). This is due to the fact that the one-dimensional problems obtained by separating the variable x do not have an exponential decay for m fixed. This is an important difference with respect to system (29) in which the exponential decay holds if the frequency of vibration in the $x-$direction is fixed, but with a decay rate that vanishes as $m \to \infty$.

5 Comments

The results of Sections 2 and 3 can be generalized to similar models in other domains. For instance, if Ω is a bounded open set in $\mathbb{R}^2$ with smooth boundary and Γ_0 is an open subset of the boundary of the domain, one can replace in (5) the beam equation satisfied by W by

$$W'' - \frac{d^4W}{d\tau^4} + W' - p = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty)$$

where $\frac{d}{d\tau}$ is the derivative in the tangential direction.

The results of Section 4 may be extended to some particular geometries. For instance, in [15] we analyze the case in which Ω is a ball of $\mathbb{R}^2$ and the dissipative term acts on the whole boundary of Ω. We obtain that the corresponding system does not have exponential decay. This indicates something we already pointed out in Remark 9: the lack of uniform decay in this type of systems is due to the hybrid structure and not to the localization of the dissipation in a relatively small part of the boundary. Although this model does not have much physical meaning, all the techniques we used there can be adapted to the case of a cavity enclosed by a thin cylindrical shell which is much more realistic (see [4]).

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