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EXACT BOUNDARY CONTROLLABILITY FOR THE LINEAR KORTEWEG-DE VRIES EQUATION - A NUMERICAL STUDY

LIONEL ROSIER

Laboratoire d'analyse numérique
Université d'Orsay, bât. 425
F91405 Orsay, France
E-mail: Lionel.ROSIER@math.u-psud.fr

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Abstract

The exact boundary controllability of linear and nonlinear Korteweg-de Vries equation on bounded domains was established in [15] by means of Hilbert Uniqueness Method. The aim of these notes is to illustrate this approach by numerical simulations.

1. Introduction

The Korteweg-de Vries (KdV) equation (see [13])

$$y_t + yy_x + y_{xxx} = 0, \quad (x \in I\!R,\ t \geq 0) \tag{1}$$

is a well known instance of a nonlinear dispersive partial differential equation, which may serve as a model for propagation of small amplitude long water waves in a uniform channel: In this model y represents the deflection of the surface from rest position, t is the time variable and x is the space variable. It must be emphasized that in equation (1.1) the coordinates x and y are taken with respect to a *moving* frame. As it was suggested in [5] the extra term y_x should be incorporated in the equation (1.1) in order to obtain an appropriate model for water waves in a uniform channel when coordinates x and y are taken with respect to a *fixed* frame, a fact which is essential when dealing with boundary controllability. In these notes we are concerned with the following controllability problem: Given $L > 0$ (the lenght of the domain) $T > 0$ (the final time), $y_0, y_T \in L^2(0,L)$, does it exist a control function $h \in L^2(0,T)$ such that the solution $y = y(t,x)$ of the Cauchy problem

$$\begin{aligned} y_t + y_x + yy_x + y_{xxx} &= 0 && (2)\\ y(t,0) = y(t,L) &= 0 && (3)\\ y_x(t,L) &= h(t) && (4)\\ y(0,x) &= y_0(x) && (5) \end{aligned}$$

satisfies $y(T,x) = y_T(x)$? It turns out that in the linear case (that is when the term yy_x is dropped) because of the introduction of the extra term y_x, exact controllability holds if and only if the lenght L does not belong to the set

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}};\ k, l \in I\!N^* \right\}. \tag{6}$$

To be more precise, we have the following result:

Theorem 1: [15] *For any $T > 0$ and any $L \in (0, +\infty) \setminus \mathcal{N}$, for any $y_0, y_T \in L^2(0,L)$, there exists $h \in L^2(0,T)$ such that the mild solution $y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ of*

$$\begin{aligned} y_t + y_x + y_{xxx} &= 0 && (7)\\ y(t,0) = y(t,L) &= 0 && (8)\\ y_x(t,L) &= h(t) && (9)\\ y(0,x) &= y_0(x) && (10) \end{aligned}$$

satisfies $y(T,.) = y_T$. If $L \in \mathcal{N}$, if R denotes the space of the states that may be reached from 0, i.e. $R = \{y(T,.);\ y$ solution of $(1.7) - (1.10)$ for $h \in L^2(0,T),\ y_0 = 0\}$, then there exists $u_T \in L^2(0,L) \backslash \{0\}$ such that $R \subset Span(u_T)^\perp$

for the $L^2(0,L)$ scalar product.

Notice that a boundary controllability result for the KdV equation was already mentioned in [16], but the setting was quite different: Indeed it is proved in [16] that the system

$$y_t + y_{xxx} = 0, \quad 0 < x < 2\pi, \quad 0 < t < T \tag{11}$$

$$\frac{\partial^k y}{\partial x^k}(t,0) = \frac{\partial^k y}{\partial x^k}(t,2\pi), \quad k \in \{0,2\}, \tag{12}$$

$$\frac{\partial y}{\partial x}(t,2\pi) - \frac{\partial y}{\partial x}(t,0) = h(t), \quad 0 < t < T, \tag{13}$$

is exactly controllable in the space $\{u \in L^2(0,\pi),\ \int_0^{2\pi} u(x)\,dx = 0\}$. This result is obtained as a consequence of the uniform stabilizability of this time-reversible linear system in both t-directions, whose proof is rather long and technical. (For other controllability results for the KdV equation with periodic boundary conditions, see [12], [15] and [17].) The proof of Theorem 1 rests on Hilbert Uniqueness Method (H.U.M.) and the multiplier method. This approach presents many advantages: It is direct, the control h is explicitly given (moreover this control h is optimal in a certain sense since it minimizes the quadratic cost $\int_0^T h(t)^2 dt$ among all boundary control functions for which $y(T,.) = y_T$) and, as for the wave equation (see [1], [8]) a numerical treatment resting on H.U.M. is quite easy to be achieved. (See sections 2 and 3.) We shall use here a collocation pseudo-spectral method, whereas finite-difference implementation has been successfully applied for the wave equation (see [1], [8]).

Notice that, thanks to a fixed-point argument, we may infer from Theorem 1 the following (local) exact boundary controllability for *nonlinear* KdV equation on a bounded domain:

Theorem 2: **[15]** *Let $T > 0$ and $L > 0$. Then there exists $r_0 > 0$ such that for any $y_0, y_T \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} < r_0$, $\|y_T\|_{L^2(0,L)} < r_0$, there exists $y \in C([0,T],L^2(0,L)) \cap L^2(0,T,H^1(0,L)) \cap W^{1,1}(0,T,H^{-2}(0,L))$ solution of*

$$y_t = -(y_x + yy_x + y_{xxx}) \text{ in } \mathcal{D}'(0,T,H^{-2}(0,L)) \tag{14}$$

$$y(.,0) = 0 \text{ in } L^2(0,T), \tag{15}$$

and such that $y(0,.) = y_0$, $y(T,.) = y_T$. If moreover $L \notin \mathcal{N}$, then we may in addition assume that $y(.,L) = 0$ and take $y_x(.,L)$ in $L^2(0,T)$ as control function.

The paper is organized as follows: In section 2 we describe the numerical scheme that has been implemented for boundary controlling the linear KdV equation and in section 3 we present several numerical simulations and we investigate the relations between

- the precision of the results and the parameters of the code;

• the time T needed to control and the cost of the control.

2. The numerical algorithm based on HUM

In this section we describe the collocation pseudo-spectral method used for numerical experiments. In what follows $L = 2$, but the domain is $(-1, 1)$ instead of $(0, 2)$. The exact boundary controllability problem we want to study from a numerical viewpoint is the following one: Given $T > 0$ and $y_T \in L^2(-1, 1)$ we search for a control function $h \in L^2(0, T)$ such that the solution y of the boundary initial-value problem

$$\begin{aligned} y_t + y_x + y_{xxx} &= 0, \quad x \in (-1, 1) && (1) \\ y(t, -1) = y(t, 1) &= 0 && (2) \\ y_x(t, 1) &= h(t) && (3) \\ y(0, x) &= 0 && (4) \end{aligned}$$

satisfies $y(T, x) = y_T(x)$. Before applying H.U.M. we have to explain how to calculate the solution of this Cauchy problem (for any boundary control h). We first give a variational formulation of this problem. Let $< ., . >$ denote the usual scalar product in $L^2(-1, 1)$. After some integrations by part we get

$\forall \phi \in H^2(-1, 1) \cap H^1_0(-1, 1)$

$$\frac{d}{dt} < y, \phi > + < y_x, \phi_{xx} + \phi > + y_x(t, -1)\phi_x(-1) = h(t)\phi_x(1). \qquad (5)$$

Let $I\!R_N[x]$ denote the space of real polynomial functions of degree at most N. ($N = 20$ or 40 in the sequel.) Let $y^N(t, x)$ denote a function such that for every time t, $y^N(t, .) \in I\!R_N[x]$. If we replace y by its approximation y^N and if we test in (2.5) with $\phi \in I\!R_N[x]$ we are led to estimate integral terms of the form $\int_{-1}^{1} P(x)\,dx$ where $P \in I\!R_{2N-1}[x]$: This can be done using a Gauss Legendre quadrature formula (see [2], [4] and [7])

$$\int_{-1}^{1} P(x)\,dx = \sum_{j=0}^{N} \rho_j\, P(\xi_j). \qquad (6)$$

In this formula, the points ξ_j are the roots of $(1 - x^2){L_N}'$ where L_N denotes the N^{th} Legendre polynomial. The real numbers ρ_j are the weights associated to the ξ_j. (Notice that $-1 = \xi_0 < \xi_1 < ... < \xi_N = 1$.) Let $(e_j)_{j=0,\cdots,N}$ be the Lagrange polynomials associated to the ξ_j, i.e. $e_j \in I\!R_N[x]$ and $e_j(\xi_i) = \delta^j_i$. Using the condition $y(t, -1) = y(t, 1) = 0$ we look for y^N in the form

$$y^N(t, x) = \sum_{j=1}^{N-1} v_j(t) e_j(x). \qquad (7)$$

Testing in (2.5) with $\phi = e_i,\ i = 1, ..., N - 1$ we get the linear differential equation

$$A\frac{d}{dt}\begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} + B\begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} + C\begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} = h(t)\begin{pmatrix} e_1'(1) \\ \vdots \\ e_{N-1}'(1) \end{pmatrix} \tag{8}$$

where $A = (a_{ij})_{i,j=1,\cdots,N-1}$, $B = (b_{ij})_{i,j=1,\cdots,N-1}$ and $C = (c_{ij})_{i,j=1,\cdots,N-1}$ are given as follows:

- $a_{ij} = \int_{-1}^{1} e_i(x)e_j(x)dx$; (Since the degree of the polynomial e_ie_j is $2N$, we have to use a Gauss Legendre quadrature formula at the level $N+1$.)
-

$$\begin{aligned} b_{ij} = \int_{-1}^{1} {e_j}'(x)({e_i}''(x) + e_i(x))dx &= \sum_{q=0}^{N} \rho_q {e_j}'(\xi_q)\left({e_i}''(\xi_q) + e_i(\xi_q)\right) \\ &= \sum_{q=0}^{N} \rho_q {e_j}'(\xi_q){e_i}''(\xi_q) + \rho_i {e_j}'(\xi_i); \end{aligned}$$

- $y_x^N(t,-1) = \sum_{j=1}^{N-1} v_j(t){e_j}'(-1)$ hence $c_{ij} = {e_i}'(-1){e_j}'(-1)$.

The numbers ρ_j, ξ_j and ${e_j}'(i)$ are given by standart routines. (See [6].) The following calculation is needed to estimate the ${e_i}''(\xi_q)$. Since ${e_i}'$ is a polynomial of degree $\leq N$, we may write

$${e_i}'(x) = \sum_{k=0}^{N} {e_i}'(\xi_k)e_k(x) \tag{9}$$

hence

$${e_i}''(\xi_j) = \sum_{k=0}^{N} {e_i}'(\xi_k){e_k}'(\xi_j), \tag{10}$$

i.e. the matrix of the ${e_i}''(\xi_j)$ is the square of the matrix of the ${e_i}'(\xi_j)$.

For time discretization we adopt an implicit scheme:

$$A\frac{v^{k+1} - v^k}{\Delta t} + Bv^{k+1} + Cv^{k+1} = h((k+1)\Delta t)\begin{pmatrix} {e_1}'(1) \\ \vdots \\ e_{N-1}'(1) \end{pmatrix} \tag{11}$$

where Δt denotes the time step and v^k is the vector $(v_1, \cdots, v_{N-1})'$ at time $k\Delta t$. (Here $'$ means transposition.)

Let us now come back to H.U.M. We consider the map $\Lambda:\ u_T \in L^2(0,L) \mapsto y(T,.) \in L^2(0,L)$ defined in the following way: Firstly we calculate the solution

u of the (backward) Cauchy problem

$$(I)\left\{\begin{array}{c} u_t+u_x+u_{xxx}=0 \\ u(t,-1)=u(t,1)=u_x(t,-1)=0 \\ u(T,x)=u_T(x) \end{array}\right. \tag{12}$$

and, after $u_x(t,1)$ is known, we search for the solution y of the boundary (forward) Cauchy problem

$$(II)\left\{\begin{array}{c} y_t+y_x+y_{xxx}=0 \\ y(t,-1)=y(t,1)=0 \\ y_x(t,1)=u_x(t,1) \\ y(0,x)=0 \end{array}\right. \tag{13}$$

Actually the backward Cauchy problem (I) is solved by performing the change of (independant) variables $s=T-t$, $z=-x$ which transforms (I) into

$$(I')\left\{\begin{array}{c} u_s+u_z+u_{zzz}=0 \\ u(s,-1)=u(s,1)=u_z(s,1)=0 \\ u(0,z)=u_T(-z) \end{array}\right. \tag{14}$$

Hence we are led to apply the implicit scheme two times: the first time (with $h=0$) to solve (I'), the second time (with $h=u_x(.,1)$) to solve (II). If $u_T^N=\sum_{j=1}^{N-1}u_je_j$ is given we denote by $\Lambda^N(u_T^N)$ the state $y_T^N=\sum_{j=1}^{N-1}y_je_j$ that we get by numerical integration of (I) followed by (II). We do not intend to prove here that Λ^N is invertible, a fact that seems to be true provided Δt is small enough. Since N is assumed to take small values ($N=20$ or 40) it is possible to calculate the coefficients of the matrix associated to the operator Λ^N in the basis $(e_j)_{1\leq j\leq N-1}$, and to invert this matrix by a standart Gauss elimination method. Finally the problem (2.1)-(2.4) is numerically solved by calculating $u_T^N=\left(\Lambda^N\right)^{-1}(y_T^N)$ where $y_T^N=\sum_{j=1}^{N-1}y_T(\xi_j)e_j$ and then solving (I)-(II).

3. Numerical experiments

The code has been first tested with smooth, slow oscillating and boundary vanishing target functions, namely $y_T(x)=\exp(-10x^2)-\exp(-10)$ (see figure 3.1) and $y_T(x)=\sin(2\pi x)$ (see figure 3.2). Each animation is composed of nine curves $y=y(t_i,x)$, $1\leq i\leq 9$, associated with the times $2\,10^{-2}=t_1<t_2<\ldots<t_8<t_9=10^{-1}=T$. Notice that the representation of the (null) initial state is omitted and that we use an adaptative reference scale for the y variable. These numerical experiments have been made with $T=10^{-1}$, $N=20$ and $\Delta t=5\,10^{-6}$. As expected, waves are generated at the right endpoint. As a wave propagates to the left endpoint, its amplitude is rising, except at the end of the control process. The first generated waves cross over the domain $(-1,1)$ and are absorbed at the left endpoint.

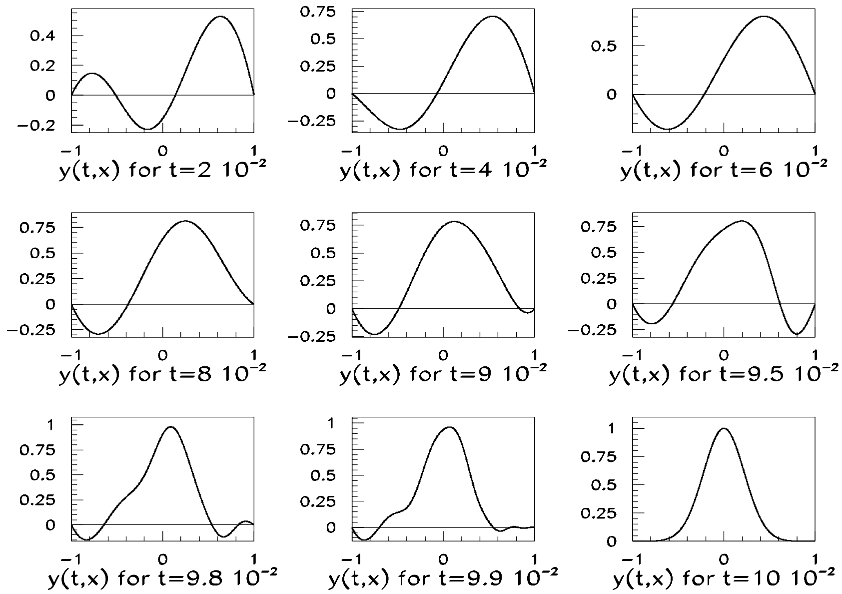

Figure 1: $y_T(x) = \exp(-10x^2) - \exp(-10)$

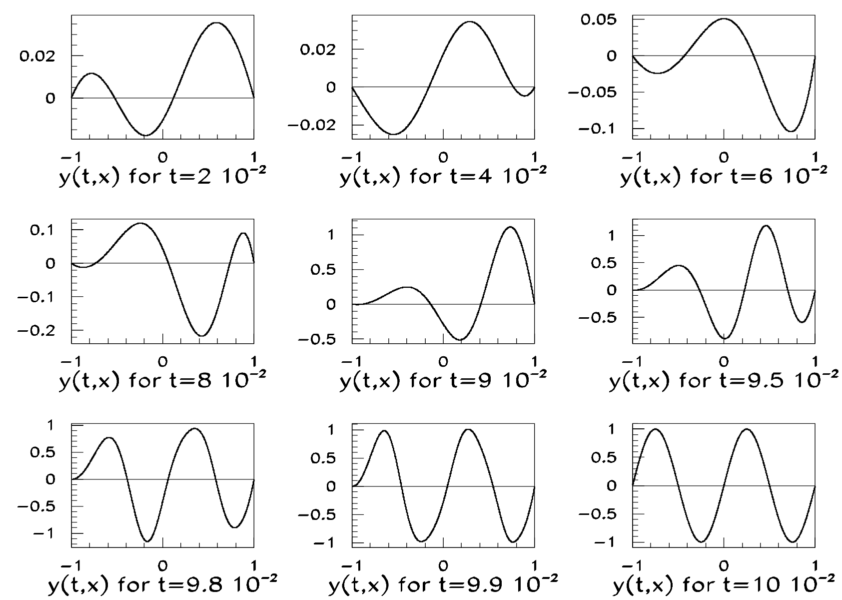

Figure 2: $y_T(x) = \sin(2\pi x)$

Table 1: Relative error

	$y_T(x) =$		
Parameters	$\exp(-10x^2) - \exp(-10)$	$\sin(2\pi x)$	$\sin(8\pi x)$
$N = 20,\ \Delta t = 5\,10^{-6}$	$\epsilon \approx 1.41\,10^{-4}$	$\epsilon \approx 8.86\,10^{-10}$	$\epsilon \approx 1.12$
$N = 40,\ \Delta t = 5\,10^{-7}$	$\epsilon \approx 1.80\,10^{-10}$	$\epsilon \approx 2.49\,10^{-11}$	$\epsilon \approx 1.80\,10^{-6}$

Remarks. • In the animation plotted in figure 3.2 (see also the figure 3.3) we observe that the evolution of the state is not uniform: The main part of the control is performed at the last times of the control process.
• If we set

$$\epsilon = \frac{\|y_T - y^N(T,.)\|_{L^2(-1,1)}}{\|y_T\|_{L^2(-1,1)}} \tag{1}$$

we get satisfactory values of the (relative) error ϵ ($\epsilon \approx 10^{-4}$ for $T = 10^{-1}$, $N = 20$, $\Delta t = 5\,10^{-6}$, see table 3.1) with a reasonable computing time (a few minutes on an Alpha workstation).

If we now are concerned with (smooth) rapidly oscillating or discontinuous target functions - recall that Theorem 1 holds true for $y_0 = 0$ and any $y_T \in L^2(-1,1)$ - the above choice of parameters leads to inadequate results. This is not surprising, since the state which is really reached is the function $y_T^N(x) = \sum_{i=1}^{N-1} y_T(\xi_i)e_i(x)$ instead of the target function y_T. The contribution of the error term $\frac{\|y_T^N - y_T\|}{\|y_T\|}$ in (3.1) cannot be neglected for small values of N. Such a situation occurs when $N = 20$ and $y_T(x) = \sin(8\pi x)$. In table 3.1 estimates for the error ϵ are tabulated for various values of the parameters and of the target functions. The animation corresponding to the rapidly oscillating target function $y_T(x) = \sin(8\pi x)$ is plotted in figure 3.3. The new choice of parameters ($N = 40$, $\Delta t = 5\,10^{-7}$) leads to an error of 10^{-6}. Finally the code has been tested with the discontinuous (step) function $y_T = \chi_{(-0.5\,,\,0.5)}$. Notice that the projection $\hat{y}_T^N$ of y_T onto $I\!R_N[x] \subset L^2(-1,1)$ has been substituted to y_T^N in order to improve the control process. Thanks to this adjustement the value of ϵ is divided by 2. (We find $\epsilon \approx 0.12$.)

To close this numerical study we plot in figure 3.5 the L^2 norm of the control (that is $\left(\int_0^T h^2(t)dt\right)^{\frac{1}{2}}$) as a function of the final time T. In these simulations, $N = 20$ and $y_T(x) = \exp(-10x^2) - \exp(-10)$. Since the control h given by HUM provides the optimal control with respect to the quadratic cost $\int_0^T h^2(t)dt$ we expect that the L^2 norm of h is a nonincreasing function of the final time T, a fact which is valided by these numerical experiments. Moreover we note that the L^2 norm takes a practically constant value (≈ 0.63) for $T > 0.2$. (Compare with [1], figures 3 and 5). So we do not need to take large final time values if

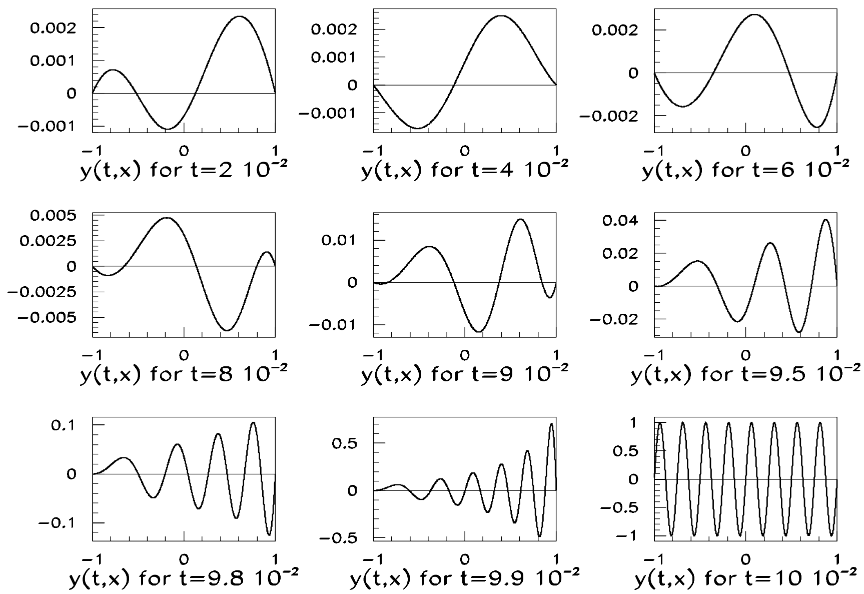

Figure 3: $y_T(x) = \sin(8\pi x)$

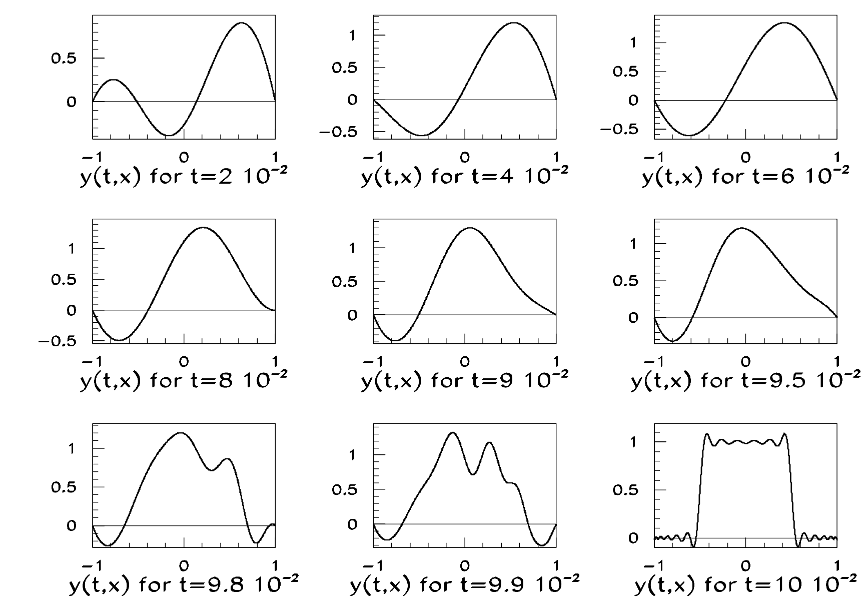

Figure 4: $y_T = \chi_{(-0.5\,,\,0.5)}$

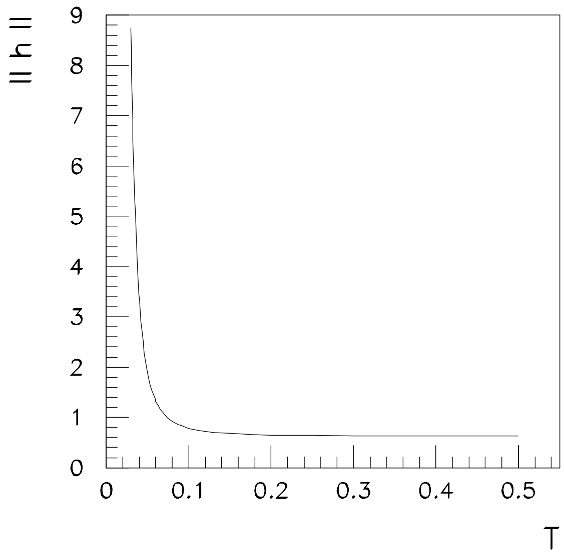

Figure 5: The L^2 norm versus the final time

we also have in mind to minimize the cost of the control.

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