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SECOND ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR A CLASS OF ELLIPTIC CONTROL PROBLEMS[1]

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Abstract

In this paper we deal with a class of optimal control problems governed by elliptic equations with nonlinear boundary condition. The case of a boundary control is studied. We consider pointwise constraints on the control and certain equality and inequality constraints on the state. Second order sufficient conditions for local optimality of controls are derived.

1 Introduction

In the paper [4] the authors have established second order sufficient optimality conditions for a class of elliptic boundary control problems with pointwise constraints on the control. Here we extend these results allowing for additional constraints on the state. In this way, we continue the investigations of Casas and Tröltzsch [3] on second order necessary conditions derived for the case of state-constraints. We rely also on general ideas of Maurer and Zowe [5] on first order sufficient optimality conditions and refine them by means of a detailed splitting technique. We should also mention the work [1] by F. Bonnans, who investigates a very weak form of second order sufficient conditions in the case of distributed controls and semilinear elliptic equations without state-constraints.

To establish second order sufficient optimality conditions for problems with pointwise state–constraints given on the whole domain, we have to restrict ourselves to a 2–dimensional domain and controls appearing linearly in the boundary condition. If pointwise state–constraints are imposed on compact subsets of the domain and the other quantities are sufficiently smooth, then arbitrary dimensions can be treated without restrictions on the nonlinearities.

2 The optimal control problem

We consider the problem to *minimize* the functional

$$F_0(y,u) = \int_\Omega f(x,y(x))\,dx + \int_\Gamma g(x,y(x),u(x))\,dS(x) \tag{2.1}$$

subject to the *equation of state*

$$\begin{cases} -\Delta y(x) + y(x) = 0 & \text{in } \Omega \\ \partial_\nu y(x) = b(x,y(x),u(x)) & \text{on } \Gamma, \end{cases} \tag{2.2}$$

to the constraints on the *state* y

$$F_i(y) = 0, \quad 1 \le i \le m_1, \tag{2.3}$$

$$F_i(y) \le 0, \quad m_1 < i \le m, \tag{2.4}$$

and to the constraints on the *control* u

$$u_a(x) \le u(x) \le u_b(x) \quad \text{a. e. on } \Gamma. \tag{2.5}$$

In this setting, $\Omega \subset \mathbb{R}^n$ is a bounded domain (i. e. simply connected and open) with a Lipschitz boundary Γ according to the definition by Nečas [7]. By $b : \Gamma \times \mathbb{R}^2 \to \mathbb{R}$ a sufficiently smooth function is given, ∂_ν is the derivative in the direction of the unit outward normal ν on Γ. The functionals $F_i : C(\overline{\Omega}) \to \mathbb{R}$, $i = 1, \ldots, m$, are supposed to be twice continuously Fréchet differentiable, i. e. being of class C^2. u_a, $u_b : \Gamma \to \mathbb{R}$ are functions of $L^\infty(\Gamma)$ satisfying $u_a(x) \le u_b(x)$ on Γ.

The *control* u is looked upon the control space $\mathcal{U} = L^\infty(\Gamma)$, while the *state* y is defined as *weak solution* of (2.2) in the state space $C(\overline{\Omega}) \cap H^1(\Omega) = Y$, i. e.

$$\int_\Omega (\nabla y \nabla v + yv)\, dx = \int_\Gamma b(\cdot, y, u) v\, dS \quad \forall v \in H^1(\Omega). \tag{2.6}$$

We endow Y with the norm $\|y\|_Y = \|y\|_{C(\overline{\Omega})} + \|y\|_{H^1(\Omega)}$. In the paper, partial derivatives are indicated in the usual way by subscripts. For instance, b_{yu} stands for $\partial^2 b/\partial y \partial u$. By $b'(x,y,u)$ and $b''(x,y,u)$ we denote the gradient and the Hessian matrix of b with respect to (y,u):

$$b'(x,y,u) = \begin{pmatrix} b_y(x,y,u) \\ b_u(x,y,u) \end{pmatrix}, \quad b''(x,y,u) = \begin{pmatrix} b_{yy}(x,y,u) & b_{yu}(x,y,u) \\ b_{uy}(x,y,u) & b_{uu}(x,y,u) \end{pmatrix},$$

$|b'|$ and $|b''|$ are defined by adding the absolute values of all entries.

The following assumptions are imposed on the given quantities:

(A1) For each fixed $x \in \Gamma$, the function $b = b(x,y,u)$ is of class C^2 with respect to (y,u). For (y,u) fixed, b is Lebesgue measurable with respect to $x \in \Gamma$. It holds

$$b_y(x,y,u) \le 0 \qquad \text{a. e. } x \in \Gamma,\ \forall (y,u) \in \mathbb{R}^2, \tag{2.7}$$

There is a continuous, monotone increasing function $\eta \in C(\mathbb{R}^+ \cup \{0\})$ with $\eta(0) = 0$ and for all $M > 0$ there are constants $C_M > 0$ such that: $b(\cdot, 0) \in L^p(\Gamma)$, for a $p > n-1$,

$$|b'(x,y,u)| + |b''(x,y,u)| \le C_M$$

$$|b''(x,y_1,u_1) - b''(x,y_2,u_2)| \le C_M\, \eta(|y_1 - y_2| + |u_1 - u_2|)$$

for almost all $x \in \Gamma$ and all $|y|, |u|, |y_i|, |u_i| \le M$, $i = 1,2$.

In the next assumption **(A2)**, fixed parameters s, r are used, which depend on n. For the possible (maximal) choices of s and r we refer to the discussion of regularity in (3.9). Roughly speaking, we have for the linearized system (2.2) that $y|_\Gamma \in L^s(\Gamma)$ and $y \in L^r(\Omega)$, if $u \in L^2(\Gamma)$. s' and r' are defined as conjugate numbers, for instance, $1/s' + 1/s = 1$.

(A2) To simplify and shorten the presentation of assumptions, we assume that f and g have the form $f(x,y) = \alpha_\Omega(x)(y - y_\Omega(x))^2$, $g(x,y,u) = \alpha_\Gamma(x)(y - y_\Gamma(x))^2 + \alpha_u(x)u^2$ with fixed functions α_Ω, α_Γ, α_u, y_Ω, y_Γ, such that $\alpha_\Omega \in L^{(r/2)'}(\Omega)$, $\alpha_\Omega y_\Omega \in L^{r'}(\Omega)$, $\alpha_\Gamma \in L^{(s/2)'}(\Gamma)$, $\alpha_\Gamma y_\Gamma \in L^{s'}(\Gamma)$, and $\alpha_u \in L^\infty(\Gamma)$. We do not rely on this particular form in this paper. We shall use only certain continuity and Lipschitz properties, which hold for more general nonlinear functions as well.

(A3) Let us define for $y \in C(\overline{\Omega})$ and a certain measurable compact subset $A \subset \overline{\Omega}$ the norm

$$\|y\|_2 = \|y\|_{C(A)} + \|y\|_{L^r(\Omega)} + \|y\|_{L^s(\Gamma)}.$$

A stands for a subset, where we know $y \in C(A)$ for Neumann boundary data of $L^2(\Gamma)$. In the case $n = 2$, we may take $A = \overline{\Omega}$, for $n > 2$ we need $A \subset \Omega$. We put $\|y\|_{C(A)} = 0$, if $A = \emptyset$. We require for a fixed reference state $\overline{y} \in C(\overline{\Omega})$:

$$|F_i'(\overline{y})y| \le C_F \|y\|_2 \quad \forall y \in C(\overline{\Omega})$$
$$|F_i''(\overline{y})[y_1, y_2]| \le C_F \|y_1\|_2 \|y_2\|_2 \quad \forall y_1, y_2 \in C(\overline{\Omega}),$$

where $C_F > 0$, and

$$|F_i'(y_1)y - F_i'(y_2)y| \le C_M \|y_1 - y_2\|_2 \|y\|_2$$
$$|(F_i''(y_1) - F_i''(y_2))[y, v]| \le C_M\, \eta(\|y_1 - y_2\|_{C(\overline{\Omega})}) \|y\|_2 \|v\|_2$$

for all y_j with $\|y_j\|_{C(\overline{\Omega})} \le M$, $j = 1, 2$, y and v from $C(\overline{\Omega})$ and $i = 1, \ldots, m$.

It should be mentioned that $A = \overline{\Omega}$ cannot be allowed for $n \ge 3$. However, we may approximate associated measures by functions of L–spaces. This motivates the use of L^r– and L^s–spaces in **(A3)**.

3 The state equation, first order necessary optimality conditions

It can be shown that the equation (2.2) admits for each $u \in \mathcal{U}^{ad}$ a unique weak solution $y = y(u) \in Y$, where $\mathcal{U}^{ad} = \{u \in L^\infty(\Gamma) \,|\, u_a(x) \le u(x) \le u_b(x)$ a. e. on $\Gamma\}$. Moreover, there is a constant M such that $\|y(u)\|_Y \le M \quad \forall u \in \mathcal{U}^{ad}$, in particular $\|y\|_{C(\overline{\Omega})} \le M$. In [3] it was shown that the mapping $G : u \mapsto y(u)$ is of class C^2 from $L^\infty(\Gamma)$ into Y. Furthermore, there is a constant C_2 such that the Lipschitz property $\|y(u_1) - y(u_2)\|_2 \le C_2 \|u_1 - u_2\|_{L^2(\Gamma)}$ holds for all u_1, $u_2 \in \mathcal{U}^{ad}$ ($\|\cdot\|_2$ was defined in **(A3)**). For fixed $u \in \mathcal{U}^{ad}$ we have $b(\cdot, y, u) \in L^p(\Gamma)$, hence the weak solution $y \in Y$ of (2.2) belongs to the space $Y_{q,p} = \{y \in H^1(\Omega) \,|\, -\Delta y + y \in L^q(\Omega), \partial_\nu y \in L^p(\Gamma)\}$, which is known to be continuously embedded into $Y = C(\overline{\Omega}) \cap H^1(\Omega)$ for every $q > n/2$ and every $p > n - 1$. In all what follows we assume that a *reference pair* $(\overline{y}, \overline{u}) \in Y \times \mathcal{U}^{ad}$ is given, satisfying together with an associated *adjoint state* $\overline{\varphi} \in W^{1,\sigma}(\Omega)$, $\forall \sigma < \frac{n}{n-1}$ and a vector of *Lagrange multipliers* $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_m)^T \in \mathbb{R}^m$, $\overline{\lambda}_i \ge 0, i = m_1 + 1, ..., m$, the associated standard *first order necessary optimality conditions* derived in [3]. The first order optimality system to be satisfied by $(\overline{y}, \overline{u})$ consists of the state equations (2.2), the constraint $\overline{u} \in \mathcal{U}^{ad}$, the *adjoint equation*

$$-\Delta\overline{\varphi} + \overline{\varphi} = f_y(\cdot, \overline{y}) + \sum_{i=1}^m \overline{\lambda}_i F_i'(\overline{y})|_\Omega \qquad \text{in } \Omega \tag{3.1}$$

$$\partial_\nu \overline{\varphi} = b_y(\cdot, \overline{y}, \overline{u})\overline{\varphi} + g_y(\cdot, \overline{y}, \overline{u}) + \sum_{i=1}^m \overline{\lambda}_i F_i'(\overline{y})|_\Gamma \quad \text{on } \Gamma \tag{3.2}$$

for the adjoint state $\overline{\varphi}$, the *complementary slackness condition*

$$\sum_{i=m_1+1}^{m} \overline{\lambda}_i F_i'(\overline{y}) = 0 \tag{3.3}$$

and the *variational inequality*

$$\int_\Gamma (g_u(x,\overline{y}(x),\overline{u}(x)) + \overline{\varphi}(x) b_u(x,\overline{y}(x),\overline{u}(x))(u(x)-\overline{u}(x))\, dS(x) \geq 0 \tag{3.4}$$

for all $u \in \mathcal{U}^{ad}$. We have $F_i'(\overline{y}) \in C(\overline{\Omega})^*$, $i = 1, \ldots, m$, hence these quantities can be identified with real Borel measures on $\overline{\Omega}$. Owing to Casas [2], the adjoint equation admits for given real Borel measures μ_Ω and μ_Γ concentrated on Ω and Γ, respectively, a unique solution $\varphi \in W^{1,\sigma}(\Omega)$ for all $\sigma < n/(n-1)$, since $b_y \in L^\infty(\Gamma)$ is nonnegative. Moreover, $\overline{\varphi}$ satisfies a formula of integration by parts in $Y_{q,p}$ which permits to verify that the optimality conditions can be expressed by means of the *Lagrange function*

$$\mathcal{L}(y,u,\varphi,\lambda,z^*) = F_0(y,u) - \int_\Omega (-\Delta y + y)\, \varphi\, dx - \int_\Gamma (\partial_\nu y - b(\cdot,y,u))\varphi\, dS + \sum_{j=1}^{m} \lambda_j F_j(y),$$

$\mathcal{L} : Y_{q,p} \times \mathcal{U} \times W^{1,\sigma}(\Omega) \times \mathbb{R}^m \to \mathbb{R}$. The regularity of y and φ fit together, as $\varphi \in W^{1,\sigma}(\Omega)$ for all $\sigma < n/(n-1)$ ensures $\varphi \in L^s(\Omega)$ for all $s < n/(n-2)$ (cf. Nečas [7], Thm. 3.4, p. 69) and $\varphi|_\Gamma \in L^r(\Gamma)$ for all $r < 1 + 1/(n-2)$ ([7], Thm. 4.2, p.84). Hence this definition makes sense. It is obvious that $\mathcal{L}$ is of class C^2 with respect to (y,u) for fixed φ, λ. The optimality system can be written in the form (2.6), $u \in \mathcal{U}^{ad}$, and

$$\mathcal{L}_y(\overline{y},\overline{u},\overline{\varphi},\overline{\lambda},\overline{z}^*)y = 0 \quad \forall y \in Y \tag{3.5}$$

$$\mathcal{L}_u(\overline{y},\overline{u},\overline{\varphi},\overline{\lambda},\overline{z}^*)(u-\overline{u}) \geq 0 \quad \forall u \in \mathcal{U}^{ad} \tag{3.6}$$

$$\sum_{j=m_1+1}^{m} \overline{\lambda}_j F_j(\overline{y}) = 0. \tag{3.7}$$

In order to simplify our notation, derivatives taken at $(\overline{y},\overline{u},\overline{\varphi},\overline{\lambda})$ will be indicated by a bar. For instance, $\overline{\mathcal{L}}_y y$, $\overline{\mathcal{L}}_u(u-\overline{u})$ would stand for the derivatives in (3.5) and (3.6), respectively.

Next we state some useful results on certain linearized versions of the state equation. Regard first the linear system

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega \\ \partial_\nu y + \beta y = g & \text{on } \Gamma, \end{cases} \tag{3.8}$$

where $\beta \in L^\infty(\Gamma)$ is nonnegative. Owing to Casas [2], this system admits for each pair $(f,g) \in L^1(\Omega) \times L^1(\Gamma)$ a unique solution $y \in W^{1,\sigma}(\Omega)$ with $\sigma < n/(n-1)$. (Note that a function of L^1 can be considered as a Borel measure.) On the other hand, we have that the solution y of (3.8) belongs to $H^1(\Omega) \cap C(\overline{\Omega})$, if

$(f,g) \in L^q(\Omega) \times L^p(\Gamma)$. This regularity result is well known for domains with C^1–boundary. Nevertheless, it holds also true for domains with Lipschitz boundary in the sense of Nečas [7] (cf. Stampacchia [8] and Murthy and Stampacchia [6]). In view of these results we see that the mapping $T : (f,g) \mapsto (y, y|_\Gamma)$ is a mapping from $L^1(\Omega) \times L^1(\Gamma)$ into $L^s(\Omega) \times L^t(\Gamma)$ with $s < n/(n-2)$ and $t < (n-1)/(n-2)$ by embedding theorems for $W^{1,\sigma}(\Omega)$ (cf. again Nečas [7]) and from $L^q(\Omega) \times L^p(\Gamma)$ into $L^\infty(\Omega) \times L^\infty(\Gamma)$. In both cases, the mapping is linear and continuous. Interpolation theory applies to show the following results for T viewed as a mapping defined on $L^2(\Omega) \times L^2(\Gamma)$:

$$y \in \begin{cases} C(\overline{\Omega}), & n = 2 \\ L^r(\Omega)\ \forall r < \infty, & n = 3 \\ L^r(\Omega)\ \forall r < \dfrac{2n}{n-3} & n \geq 4 \end{cases} \qquad y|_\Gamma \in \begin{cases} C(\Gamma) & n = 2 \\ L^s(\Gamma)\ \forall s < \infty & n = 3 \\ L^s(\Gamma)\ \forall s < \dfrac{2(n-1)}{n-3} & n \geq 4. \end{cases} \tag{3.9}$$

4 Regularity condition and linearization theorem

Recall that we consider a fixed reference pair $(\overline{y}, \overline{u})$ satisfying together with $(\overline{\varphi}, \overline{\lambda})$ the first order necessary conditions (3.5) – (3.7). The *linearized cone* of $\mathcal{U}^{ad}$ at $\overline{u}$ is the set $\mathcal{C}(\overline{u}) = \{v \in L^\infty(\Gamma) \,|\, v = \varrho(u - \overline{u}),\ \varrho \geq 0,\ u \in \mathcal{U}^{ad}\}$. Let us introduce the sets $I = \{1, ..., m\}$, $I_o = \{i \in I \,|\, F_i(\overline{y}) = 0\}$ and $I_+ = \{i \in I \,|\, i > m_1 \text{ and } \overline{\lambda}_i > 0\}$. The set of all pairs (y, u) satisfying all constraints of our control problem is denoted by $\mathcal{M}$. Let $F = F(y)$ denote the mapping $y \mapsto (F_1(y), \ldots, F_m(y))^T$ from Y to $\mathbb{R}^m$.

Following Maurer and Zowe [5], the *linearized cone* $L(\mathcal{M}, \overline{w})$ at $\overline{w} = (\overline{y}, \overline{u})$ is defined by

$$L(\mathcal{M}, \overline{w}) = \{w = (y, u) \,|\, u \in \mathcal{C}(\overline{u}) \text{ and } (y, u) \text{ satisfies (4.1) –(4.3)}\},$$

where

$$\begin{cases} -\Delta y + y = 0 & \text{in } \Omega \\ \partial_\nu y = b_y(\cdot, \overline{y}, \overline{u}) y + b_u(\cdot, \overline{y}, \overline{u}) u & \text{on } \Gamma \end{cases} \tag{4.1}$$

$$F_i'(\overline{y}) y = 0, \quad \forall\, i \in I_o,\ i \leq m_1, \tag{4.2}$$

$$F_i'(\overline{y}) y \leq 0, \quad \forall\, i \in I_o,\ i > m_1. \tag{4.3}$$

The following *regularity assumption* **(R)** is essential for our analysis:

(R) For arbitrary real numbers z_j, $j \in I_o$, the system $F_j'(\overline{y}) G'(\overline{u}) u = z_j$, $j \in I_o$, has at least one solution $u \in \mathcal{C}(\overline{u})$.

This condition is equivalent to a very general regularity condition introduced by Zowe and Kurcyusz [11], which is sufficient for the existence of nondegenerate Lagrange multipliers. The following condition was introduced by Casas and Tröltzsch [3] to derive second order necessary optimality conditions: There is an $\varepsilon > 0$ such that $\forall i \in I_o\, \exists h_i \in L^\infty(\Gamma)$ with $supp\, h_i \in \Gamma(\varepsilon) = \{x \in \Gamma \,|\, u_a(x) + \varepsilon \leq \overline{u}(x) \leq u_b(x) - \varepsilon\}$ such that $F_j'(\overline{y}) G'(\overline{u}) h_i = \delta_{ij}$, $j \in I_o$. It is easy to see that this condition implies **(R)** (take $u = \rho(\overline{u} + \sum_{i \in I_o} \lambda_i h_i - \overline{u})$ for ρ large). We should mention that $h_i \leq 0$ and $h_i \geq 0$, respectively, could be allowed outside of $\Gamma(\varepsilon)$ for this purpose. The strong requirement $h_i = 0$ on $\Gamma \setminus \Gamma(\varepsilon)$ was needed for second order *necessary* conditions only.

Theorem 1 *Suppose that* **(R)** *is satisfied. Then for all pairs* $(\hat{y}, \hat{u}) \in \mathcal{M}$ *there is a pair* $(y, u) \in L(\mathcal{M}, \overline{w})$ *such that the difference* $r = (r^y, r^u) = (\hat{y}, \hat{u}) - (\overline{y}, \overline{u}) - (y, u)$ *fulfils the following estimates:*

$$\|r\|_{Y \times L^\infty(\Gamma)} \leq C_{L,p} \|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \|\hat{u} - \overline{u}\|_{L^p(\Gamma)} \quad \forall p > n - 1 \tag{4.4}$$

$$\|r\| \leq C_{L,2} \|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \|\hat{u} - \overline{u}\|_{L^2(\Gamma)}, \tag{4.5}$$

where $\|r\| = \|r^y\|_2 + \|r^u\|_{L^2(\Gamma)}$. *If* $b(x, y, u) = b_1(x, y) + b_2(x)u$, *then*

$$\|r\|_{Y \times L^\infty(\Gamma)} \leq C_{L,p} \|\hat{u} - \overline{u}\|^2_{L^p(\Gamma)} \quad \forall p > n - 1. \tag{4.6}$$

This theorem can be shown by the following idea: Take $v = \hat{u} - \overline{u}$ and define $\tilde{y}$ as the solution of the linear system (4.1) associated to $u := v$. Then $\tilde{y}$ will "almost" belong to the linearized cone, as $F_i'(\overline{y})\tilde{y} = e_i$, $1 \leq i \leq m_1$, and $F_i'(\overline{y})\tilde{y} \leq e_i$, $m_1 < i \leq m_1$, where $|e_i| = o(\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)})$. The order of the error e is given in Lemma 1. Invoking **(R)**, we find a certain correction r^u having the same order and removing the error e. Then $u = r^u + \hat{u} - \overline{u}$ belongs together with the associated solution y of (4.1) to the linearized cone.

We conclude this section with another important assumption and some useful estimates for $\mathcal{L}''$ and certain remainder terms. First, we derive the expression for $\overline{\mathcal{L}}''[(y_1, u_1), (y_2, u_2)] = \mathcal{L}''(\overline{y}, \overline{u}, \overline{\varphi}, \overline{\lambda})[(y_1, u_1), (y_2, u_2)]$, where $\mathcal{L}''$ denotes the second order derivative of $\mathcal{L}$ with respect to (y, u). We have

$$\begin{aligned}\overline{\mathcal{L}}''[(y_1, u_1), (y_2, u_2)] = \int_\Omega f_{yy}(\cdot, \overline{y}) y_1 y_2 \, dx + \int_\Gamma (y_1, u_1) g''(\cdot, \overline{y}, \overline{u})(y_2, u_2)^T \, dS \\ + \int_\Gamma \overline{\varphi} \cdot, (y_1, u_1) b''(\cdot, \overline{y}, \overline{u})(y_2, u_2)^T \, dS + \sum_{i=1}^m \overline{\lambda}_i F_i''(\overline{y})[y_1, y_2].\end{aligned} \tag{4.7}$$

It is the term connected with $\overline{\varphi}$, which causes troubles, more precisely,

$$I = \int_\Gamma \overline{\varphi} \, (b_{yy}(\cdot, \overline{y}, \overline{u}) y_1 y_2 + b_{yu}(\cdot, \overline{y}, \overline{u})(y_1 u_2 + y_2 u_1) + b_{uu}(\cdot, \overline{y}, \overline{u}) u_1 u_2) \, dS. \tag{4.8}$$

We shall need an estimate of I with respect to the norm $\|y\|_2 + \|u\|_{L^2(\Gamma)}$ (cf. (4.13)). This would require at least $\overline{\varphi} \in L^2(\Gamma)$ in the second item and $\overline{\varphi} \in L^\infty(\Gamma)$ in the third one. Without additional assumption only $\overline{\varphi} \in L^r(\Gamma)$ with $r < (n-1)/(n-2)$ follows from $\overline{\varphi} \in W^{1,\sigma}(\Omega)$, cf. Nečas [7], p. 84. This means in particular $\overline{\varphi} \in L^r(\Gamma)$ for all $r < \infty$, if $n = 2$, $\overline{\varphi} \in L^r(\Gamma)$, for $r < 2$, if $n = 3$. Therefore the following additional assumption is crucial for our analysis:
(A4) Let one of the following statements be true:

(i) $\overline{\varphi} \in L^\infty(\Gamma)$.

(ii) $b_{uu}(x, y, u) = 0$ on $\Gamma \times \mathbb{R}^2$ and, if $n \geq 3$, then $\overline{\varphi} \in L^r(\Gamma)$ for some $r > n-1$.

(iii) $b_{uu}(x, y, u) = b_{yu}(x, y, u) = 0$ on $\Gamma \times \mathbb{R}^2$ and, if $n \geq 4$, then $\overline{\varphi} \in L^r(\Gamma)$ for some $r > (n-1)/2$.

(iv) $b''(\cdot, y, u) = 0$.

b with respect to u, i. e. $b(x, y, u) = b_0(x, y) + b_1(x, y)u$.
As a consequence of **(A3)** and **(A4)**, *pointwise state–constraints* on the whole set $\overline{\Omega}$ can only be handled by our theory, if u appears linearly in the boundary condition and $n = 2$. In the considerations below we denote by r_i^T the remainder terms of ith order of the Taylor expansion of a mapping T.

According to our assumptions on b' and b'', the estimates

$$|r_1^b| \leq C_M(|y - \overline{y}|^2 + |u - \overline{u}|^2) \tag{4.9}$$
$$|r_2^b| \leq C_M\,\eta(|y - \overline{y}| + |u - \overline{u}|)(|y - \overline{y}|^2 + |u - \overline{u}|^2) \tag{4.10}$$

are valid for all $|y|, |\overline{y}|, |u|, |\overline{u}| \leq M$. Moreover, we are able to derive the following estimates for the Lagrange function:

$$|r_1^{\mathcal{L}}| \leq C_{\mathcal{L}}(\|y - \overline{y}\|_2^2 + \|u - \overline{u}\|_{L^2(\Gamma)}^2) \tag{4.11}$$
$$|r_2^{\mathcal{L}}| \leq C_{\mathcal{L}}\,\eta(\|y - \overline{y}\|_{C(\overline{\Omega})} + \|u - \overline{u}\|_{L^\infty(\Gamma)})\cdot(\|y - \overline{y}\|_2^2 + \|u - \overline{u}\|_{L^2(\Gamma)}^2) \tag{4.12}$$

and

$$|\overline{\mathcal{L}}''[(y_1, u_1), (y_2, u_2)]| \leq C_{\mathcal{L}}(\|y_1\|_2 + \|u_1\|_{L^2(\Gamma)})(\|y_2\|_2 + \|u_2\|_{L^2(\Gamma)}) \tag{4.13}$$

with some $C_{\mathcal{L}} > 0$, which depends in particular on $\overline{\varphi}$. These decisive estimates are explained in section 6.2.

5 Second order sufficient optimality condition

We aim to establish sufficient optimality conditions, which are close to the necessary ones derived in Casas and Tröltzsch [3]. Therefore, we have to take into account *first order sufficient optimality conditions*. We combine an approach going back to Zowe and Maurer [5] with a splitting technique, which was known for the optimal control of ordinary differential equations and has been extended to the case of elliptic equations without state–constraints by the authors in [4].

In [5], Maurer and Zowe introduce first order sufficient optimality conditions taking into account a general constraint $g(w) \leq 0$. Aiming to apply this approach to our problem in its full generality, we observed that this type of first order sufficient optimality conditions considerably complicates the presentation and the assumptions. Therefore, we introduce in a first step the first order sufficient optimality condition only for the constraints on the control. Later, we deal in the same way with the state–constraints. Define for fixed $\tau > 0$ (arbitrarily small)

$$\Gamma_\tau = \{x \in \Gamma \,|\, |g_u(x, \overline{y}(x), \overline{u}(x)) + \overline{\varphi}(x) b_u(x, \overline{y}(x), \overline{u}(x))| \geq \tau\}.$$

Γ_τ is a subset of "strongly active" control constraints (cf. (3.4)).

Let $\mathcal{P}_\tau : L^\infty(\Gamma) \to L^\infty(\Gamma)$ denote the projection operator $u \mapsto \chi_{\Gamma \setminus \Gamma_\tau} u = \mathcal{P}_\tau u$. In other words, $(\mathcal{P}_\tau u)(x) = u(x)$ on $\Gamma \setminus \Gamma_\tau$ and $(\mathcal{P}_\tau u)(x) = 0$ on Γ_τ. We start with the following slightly too strong *second order sufficient optimality condition.*

(SSC) There exist positive numbers τ and δ such that

$$\mathcal{L}''(\overline{y}, \overline{u}, \overline{\varphi}, \overline{\lambda})[w_2, w_2] \geq \delta \|u_2\|^2_{L^2(\Gamma)} \tag{5.1}$$

holds for all $w_2 = (y_2, u_2)$ obtained in the following way: For every $w = (y, u) \in L(\mathcal{M}, \overline{w})$ we split up the control part u by $u_1 = (u - \mathcal{P}_\tau u)$ and $u_2 = \mathcal{P}_\tau u$. Finally, we denote by y_i the linearized state associated to u_i, i. e.

$$\begin{cases} -\Delta y_i + y_i = 0 & \text{in } \Omega \\ \partial_\nu y_i = b_y(\cdot, \overline{y}, \overline{u}) y_i + b_u(\cdot, \overline{y}, \overline{u}) u_i & \text{on } \Gamma. \end{cases} \tag{5.2}$$

According to this, we get the splitting $w = w_1 + w_2 = (y_1, u_1) + (y_2, u_2)$.

Theorem 2 *Let the feasible pair $\overline{w} = (\overline{y}, \overline{u})$ satisfy the regularity condition* **(R)***, the first order necessary optimality conditions* (3.5)–(3.7) *and the second order sufficient optimality condition* **(SSC)***. Suppose further that the general assumptions* **(A1)**–**(A4)** *are satisfied. Then there are constants $\varrho > 0$ and $\delta' > 0$ such that*

$$F_0(\hat{y}, \hat{u}) \geq F_0(\overline{y}, \overline{u}) + \delta' \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} \tag{5.3}$$

holds for all feasible pairs $\hat{w} = (\hat{y}, \hat{u})$ such that $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} < \varrho$.

proof: Let $\hat{w} = (\hat{y}, \hat{u})$ be a given feasible pair. We use for convenience the notation $\overline{l} = (\overline{\varphi}, \overline{\lambda})$ for the Lagrange multipliers appearing in the first order necessary optimality conditions. Obviously, we have

$$F_0(\hat{w}) - F_0(\overline{w}) = \mathcal{L}(\hat{w}, \overline{l}) - \mathcal{L}(\overline{w}, \overline{l}) - \sum_{i \in I_+} \overline{\lambda}_i (F_i(\hat{y}) - F_i(\overline{y})). \tag{5.4}$$

It holds $-\sum_{i \in I_+} \overline{\lambda}_i (F_i(\hat{y}) - F_i(\overline{y})) \geq 0$, hence we may avoid this term, and a second order Taylor expansion yields with $\mathcal{L}''(\overline{w}, \overline{l})[\hat{w} - \overline{w}]^2 = \mathcal{L}''(\overline{w}, \overline{l})[(\hat{w} - \overline{w}), (\hat{w} - \overline{w})]$

$$\begin{aligned} F_0(\hat{w}) - F_0(\overline{w}) &\geq \mathcal{L}(\overline{w}, \overline{l}) - \mathcal{L}(\overline{w}, \overline{l}) \\ &\geq \int_\Gamma l_u (\hat{u} - \overline{u}) \, dS + \frac{1}{2} \mathcal{L}''(\overline{w}, \overline{l})[\hat{w} - \overline{w}]^2 + r_2^{\mathcal{L}}(\overline{w}, \hat{w} - \overline{w}) \end{aligned}$$

where $l_u(x) = g_u(x, \overline{y}(x), \overline{u}(x)) + \overline{\varphi}(x) b_u(x, \overline{y}(x), \overline{u}(x))$. Hence

$$F_0(\hat{w}) - F(\overline{w}) \geq \tau \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS + \frac{1}{2} \mathcal{L}''(\overline{w}, \bar{l})[\hat{w} - \overline{w}]^2 + r_2^{\mathcal{L}}(\overline{w}, \hat{w} - \overline{w}). \quad (5.5)$$

We introduce for convenience the bilinear form $B = \mathcal{L}''(\overline{w}, \bar{l})$ and approximate $\hat{w} - \overline{w}$ by $w = (y, u) \in L(\mathcal{M}, \overline{w})$, according to Theorem 1. In this way we get $r = (r^y, r^u)$ such that $\hat{w} - \overline{w} = w + r$ and

$$\|r\| \leq C_L \|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \|\hat{u} - \overline{u}\|_{L^2(\Gamma)}. \quad (5.6)$$

Then $B[\hat{w} - \overline{w}]^2 = B[w]^2 + 2B[r, w] + B[r]^2$. We have $w \in L(\mathcal{M}, \overline{w})$, hence **(SSC)** applies to $B[w]^2$. Splitting up $w = w_1 + w_2$ as in **(SSC)**,

$$\begin{aligned} B[w]^2 &= B[w_2]^2 + 2B[w_1, w_2] + B[w_1]^2 \\ &\geq \delta \|u_2\|^2_{L^2(\Gamma)} - C_{\mathcal{L}} \{ 2(\|y_1\|_2 + \|u_1\|_{L^2(\Gamma)})(\|y_2\|_2 + \|u_2\|_{L^2(\Gamma)}) \\ &\quad + (\|y_1\|_2 + \|u_1\|_{L^2(\Gamma)})^2 \} \end{aligned}$$

by (**SSC**) and (4.13). Let $\varrho < 1$ and $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} < \varrho$. In the following, c denotes a generic constant. By $\|y_i\|_2 \leq c \|u_i\|_{L^2(\Gamma)}$ and Young's inequality,

$$\begin{aligned} B[w]^2 &\geq \delta \|u_2\|^2_{L^2(\Gamma)} - \frac{\delta}{2} \|u_2\|^2_{L^2(\Gamma)} - c \|u_1\|^2_{L^2(\Gamma)} \geq \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} u^2 \, dS - c \int\limits_{\Gamma_\tau} u^2 \, dS \\ &\geq \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}|^2 \, dS - c \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}| \, |r^u| \, dS - c \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}|^2 \, dS \\ &\quad - c \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, |r^u| \, dS - c \int\limits_{\Gamma_\tau} |r^u|^2 \, dS. \end{aligned}$$

The third integrand is estimated by $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} |\hat{u} - \overline{u}|$, in the other integrals (excepting the first) we insert the estimate (5.6). This leads to

$$B[w]^2 \geq \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}|^2 \, dS - c\varrho \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS - c\varrho \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}. \quad (5.7)$$

The estimation of $B[r, w]$ and $B[r]^2$ is simpler.

$$|B[r, w]| \leq c \|r\| \|u\|_{L^2(\Gamma)} = c \|r\| \|\hat{u} - \overline{u} + r^u\|_{L^2(\Gamma)} \leq c\varrho \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}.$$

The same estimate applies to $B[r]^2$. Altogether,

$$B[\hat{w} - \overline{w}]^2 \geq \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}|^2 \, dS - c\varrho \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS - c\varrho \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} \quad (5.8)$$

is obtained. Inserting (5.8) in (5.5), we get

$$F_0(\hat{w}) - F_0(\overline{w}) \geq (\tau - c\varrho) \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}|\, dS + \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}|^2\, dS - \\ - c\varrho \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} - |r_2^{\mathcal{L}}| \\ \geq \frac{\tau}{2} \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}|\, dS + \frac{\delta}{2} \int\limits_{\Gamma \setminus \Gamma_\tau} |\hat{u} - \overline{u}|^2\, dS - c\varrho \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} - |r_2^{\mathcal{L}}|.$$

Because of $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \leq 1$, we have $|\hat{u} - \overline{u}| \geq |\hat{u} - \overline{u}|^2$ almost everywhere. Using this in the first integral, setting $\delta' = \min\{\tau/2, \delta/2\}$, and inserting the estimate (4.12) for $r_2^{\mathcal{L}}$, we arrive at

$$F_0(\hat{w}) - F_0(\overline{w}) \geq \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} (\delta' - c\varrho - \eta(c\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)})) \geq \frac{\delta'}{2} \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}$$

for sufficiently small $\varrho > 0$.

The study of the paper [5] reveals that first order sufficient optimality conditions can be extended also to state–constraints. However, this leads to a quite involved construction and more restrictive assumptions. We have to suppose that the function b is linear with respect to the control u and $n = 2$. The associated theorem is stated below. For fixed $\beta > 0$ and $\tau > 0$ we define the following subset of $L(\mathcal{M}, \overline{w})$:

$$L_{\beta,\tau}(\mathcal{M}, \overline{w}) = \{w = (y, u) \,|\, w \in L(\mathcal{M}, \overline{w}) \text{ and satisfies (5.9) below}\}.$$

The decisive inequality characterising $L_{\beta,\tau}$ is

$$\sum_{i \in I_+} F_i'(\overline{y}) y \geq -\beta \int_{\Gamma \setminus \Gamma_\tau} |u(x)|\, dS(x). \tag{5.9}$$

$L_{\beta,\tau}(\mathcal{M}, \overline{w})$ is the subset of $L(\mathcal{M}, \overline{w})$, where first order sufficient optimality conditions are not very supported by the term $\sum\limits_{i \in I_+} F_i(\overline{y}) y$. It is only this set, where we have to require second order conditions, namely

(SSC') There exist positive numbers β, τ, and δ such that

$$\mathcal{L}''(\overline{y}, \overline{u}, \overline{\varphi}, \overline{\lambda})[w_2, w_2] \geq \delta \|u_2\|^2_{L^2(\Gamma)} \tag{5.10}$$

holds for all $w_2 = (y_2, u_2)$ obtained in the same way as in **(SSC)** by elements w taken from the smaller set $L_{\beta,\tau}(\mathcal{M}, \overline{w})$.

Using this condition we formulate

Theorem 3 *Let the feasible pair $\overline{w} = (\overline{y}, \overline{u})$ satisfy the regularity condition* **(R)**, *the first order necessary optimality conditions* (3.5)–(3.7) *and the second order sufficient optimality condition* **(SSC')**. *Suppose further that* *the general assumptions* **(A1)**–**(A4)** *are satisfied. Moreover, assume that* $n = 2$ *and* $b(x, y, u) = b_1(x, y) + b_2(x)u$. *Then there are constants* $\varrho > 0$ *and* $\delta' > 0$ *such that*

$$F_0(\hat{y}, \hat{u}) \geq F_0(\overline{y}, \overline{u}) + \delta' \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} \tag{5.11}$$

holds for all feasible pairs $\hat{w} = (\hat{y}, \hat{u})$ *such that* $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} < \varrho$.

proof: We start exactly in the same way we have shown Theorem 2 and arrive at

$$F_0(\hat{w}) - F_0(\overline{w}) = \mathcal{L}(\hat{w}, \bar{l}) - \mathcal{L}(\overline{w}, \bar{l}) - \sum_{i \in I_+} \overline{\lambda}_i (F_i(\hat{y}) - F_i(\overline{y})). \tag{5.12}$$

Once again, $\hat{w} - \overline{w} = w + r$. Now we distinct between two cases.

Case I: First order sufficiency yields (5.3):

In this case

$$- \sum_{i \in I_+} F_i(\overline{y}) y > \beta \int\limits_{\Gamma \setminus \Gamma_\tau} |u(x)| \, dS(x), \tag{5.13}$$

i. e. $w = (y, u) \in L(\mathcal{M}, \overline{w}) \setminus L_{\beta,\tau}(\mathcal{M}, \overline{w})$. Here we handle (5.12) as follows

$$\begin{aligned} F_0(\hat{w}) - F_0(\overline{w}) &= \mathcal{L}'(\overline{w}, \bar{l})(\hat{w} - \overline{w}) + r_1^{\mathcal{L}}(\overline{w}, \hat{w} - \overline{w}) - \sum_{i \in I_+} \overline{\lambda}_i (F_i(\hat{y}) - F_i(\overline{y})) \\ &= \mathcal{L}_y(\overline{w}, \bar{l})(\hat{y} - \overline{y}) + \mathcal{L}_u(\overline{w}, \bar{l})(\hat{u} - \overline{u}) - \sum_{i \in I_+} \overline{\lambda}(F_i'(\hat{y} - \overline{y})) \\ &\quad + r_1^{\mathcal{L}}(\overline{w}, \hat{w} - \overline{w}) - \langle \overline{\lambda}, r_1^F(\overline{y}, \hat{y} - \overline{y}) \rangle \\ &= 0 + \int\limits_{\Gamma} l_u(x)(\hat{u}(x) - \overline{u}(x)) \, dS(x) - \sum_{i \in I_+} \overline{\lambda}_i F_i'(\overline{y}) y \\ &\quad + r_1^{\mathcal{L}}(\overline{w}, \hat{w} - \overline{w}) - \langle \overline{\lambda}, F'(\overline{y}) r^y + r_1^F(\overline{y}, \hat{y} - \overline{y}) \rangle, \end{aligned} \tag{5.14}$$

where $l_u(x) = g_u(x, \overline{y}(x), \overline{u}(x)) + \overline{\varphi}(x) b_u(x, \overline{y}(x), \overline{u}(x))$.

Owing to $n = 2$ and $b(x, y, u) = b_1(x, y) + b_2(x)u$, we are able to apply the strong estimate (4.6) for $p = 2$. That is

$$\|r\|_{Y \times L^\infty(\Gamma)} \leq C_{L,2} \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}. \tag{5.15}$$

By Theorem 1, (5.15), (4.11), and **(A3), (ii)** we have

$$\max\{\|r^y\|_2, |r_1^{\mathcal{L}}|, \|r_1^F\|_Z\} \leq c(\|\hat{y} - \overline{y}\|_2^2 + \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}).$$

Thus the Lipschitz property of $u \mapsto y(u) = G(u)$ from $L^2(\Gamma)$ into $C(\overline{\Omega})$ (note that $n = 2$) implies that the last three items of (5.14) can be estimated by $c \, \|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}$. To the second one we apply (5.13), while the first one is treated by Γ_τ:

We know that $l_u(x)(\hat{u}(x) - \overline{u}(x)) \geq 0$ a. e. on Γ, hence

$$\int\limits_{\Gamma} l_u(\hat{u} - \overline{u}) \, dS \geq \int\limits_{\Gamma_\tau} l_u(\hat{u} - \overline{u}) \, dS = \int\limits_{\Gamma_\tau} |l_u| \, |\hat{u} - \overline{u}| \, dS \geq \tau \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS.$$

Now (5.14) can be continued by

$$\begin{aligned} F_0(\hat{w}) - F_0(\overline{w}) &\geq \tau \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS + \beta \int\limits_{\Gamma \backslash \Gamma_\tau} |u| \, dS - c\|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} \\ &\geq \tau \int\limits_{\Gamma_\tau} |\hat{u} - \overline{u}| \, dS + \beta \int\limits_{\Gamma \backslash \Gamma_\tau} |\hat{u} - \overline{u}| \, dS - c\|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)} \end{aligned}$$

as $\|r^u\|_{L^\infty(\Gamma)} \leq c\|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}$. Proceeding with the estimation, we have

$$F_0(\hat{w}) - F_0(\overline{w}) \geq \min\{\beta, \tau\}\|\hat{u} - \overline{u}\|_{L^1(\Gamma)} - c\varrho\|\hat{u} - \overline{u}\|_{L^1(\Gamma)} \geq \beta'\|\hat{u} - \overline{u}\|_{L^1(\Gamma)}$$

with some $\beta' > 0$, provided that $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \leq \varrho \leq \varrho_1$, where ϱ_1 is sufficiently small. Assume additionally that $\varrho_1 \leq 1$. Then $|\hat{u} - \overline{u}|^2 \leq |\hat{u} - \overline{u}|$, hence $F_0(\hat{w}) - F_0(\overline{w}) \geq \beta'\|\hat{u} - \overline{u}\|^2_{L^2(\Gamma)}$ for $\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)} \leq \varrho_1$.

Case II: Partial use of first order sufficient optimality conditions

Here, we avoid the term $\sum\limits_{i \in I_+} \overline{\lambda}_i(F_i(\hat{y}) - F_i(\overline{y}))$ and proceed word by word as in the proof of Theorem 2, using $L_{\beta,\tau}$ instead of L.

6 Appendix

6.1 Linearization theorem

To show Theorem 1 we have used the following auxiliary result:

Lemma 1 *Let $\overline{u}, \hat{u} \in \mathcal{U}^{ad}$ be given with associated states $\overline{y}, \hat{y}$ defined by* (2.2). *Define $y \in Y$ as the solution of the linearized state equation*

$$\begin{cases} -\Delta y + y = 0 & \text{in } \Omega \\ \partial_\nu y = b_y(\cdot, \overline{y}, \overline{u})y + b_u(\cdot, \overline{y}, \overline{u})(\hat{u} - \overline{u}) & \text{on } \Gamma. \end{cases} \tag{6.1}$$

Then there are constants C_p, C_2 such that

$$\|\hat{y} - \overline{y} - y\|_Y \leq C_p\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)}\|\hat{u} - \overline{u}\|_{L^p(\Gamma)} \quad \forall p > n - 1 \tag{6.2}$$

$$\|\hat{y} - \overline{y} - y\|_2 \leq C_2\|\hat{u} - \overline{u}\|_{L^\infty(\Gamma)}\|\hat{u} - \overline{u}\|_{L^2(\Gamma)}. \tag{6.3}$$

In the case, where $b_u(x, y, u)$ does not depend on y and u, it holds

$$\|\hat{y} - \overline{y} - y\|_Y \leq C_p\|\hat{u} - \overline{u}\|^2_{L^p(\Gamma)} \quad \forall p > n - 1. \tag{6.4}$$

proof: We use the first order expansion of b at $(x, \hat{y}, \hat{u})$ and $(x, \overline{y}, \overline{u})$ and obtain from (2.2), (6.1) that

$$\begin{aligned} -\Delta(\hat{y} - \overline{y} - y) + (\hat{y} - \overline{y} - y) &= 0 \quad \text{in } \Omega \\ \partial_\nu(\hat{y} - \overline{y} - y) - b_y(\cdot, \overline{y}, \overline{u})(\hat{y} - \overline{y} - y) &= r_1^b \quad \text{on } \Gamma, \end{aligned}$$

where the first order remainder term r_1^b satisfies

$$|r_1^b(x)| \le C_M(|\hat{y}(x) - \overline{y}(x)|^2 + |\hat{u}(x) - \overline{u}(x)|^2)$$

and M depends on $\mathcal{U}^{ad}$ (note that the boundedness of $\mathcal{U}^{ad}$ implies a uniform bound on all admissible states). We continue as in [4], Lemma 3.

6.2 Estimates of the Lagrange function

In this subsection we explain the estimates (4.11)–(4.13) for $r_1^{\mathcal{L}}$, $r_2^{\mathcal{L}}$, and $\mathcal{L}''$. They depend mainly on the estimation of I defined in (4.8), which is performed by the discussion of the following three integrals,

$$\int_\Gamma |\overline{\varphi}|\, u^2 \, dS \le c\|u\|^2_{L^2(\Gamma)} \tag{6.5}$$

provided that assumption **(A4), (i)** is fulfilled, and

$$\begin{aligned}\int_\Gamma |\overline{\varphi}|\,|y|\,|u|\,dS &\le c\|\overline{\varphi} y\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)} \le c\|\overline{\varphi}^2\|^{1/2}_{L^{(s/2)'}(\Gamma)}\|y^2\|^{1/2}_{L^{s/2}(\Gamma)}\|u\|_{L^2(\Gamma)} \\ &\le c\|\overline{\varphi}\|_{L^{2s/(s-2)}(\Gamma)}\|y\|_{L^s(\Gamma)}\|u\|_{L^2(\Gamma)}. \end{aligned} \tag{6.6}$$

These estimates are justified by **(A4), (ii)**: For $n = 2$ we know $y \in C(\Gamma)$ and $\varphi \in L^r(\Gamma)$ $\forall r < \infty$. If $n \ge 3$, then $y \in L^s(\Gamma)$ for all $s < 2(n-1)/(n-3)$ (including $s < \infty$ for $n = 3$). The function $2s/(s-2) = 2/(1-1/s)$ is monotone decreasing. Therefore, $s \uparrow 2(n-1)/(n-3)$ implies $2s/(s-2) \downarrow n-1$, so that $\overline{\varphi} \in L^r(\Gamma)$ for some $r > n-1$ is sufficient to justify (6.6) with a sufficiently large s. Finally,

$$\int_\Gamma |\overline{\varphi}| y^2 \, dS \le \|\overline{\varphi}\|_{L^{(s/2)'}(\Gamma)}\|y^2\|_{L^{s/2}(\Gamma)} = \|\overline{\varphi}\|_{L^{s/(s-2)}(\Gamma)}\|y\|^2_{L^s(\Gamma)} \tag{6.7}$$

is estimated by **(A4), (iii)**: In the case $n = 2$ we can take $s = \infty$, as $y \in C(\Gamma)$ and $\varphi \in L^1(\Gamma)$ is true without any additional assumption. For $n = 3$ we know $y \in L^s(\Gamma)$ for all $s < \infty$. If $s \uparrow \infty$, then $s/(s-2) \downarrow 1 < n/(n-1)$. Since $\varphi \in L^r(\Gamma)$ for all $r < n/(n-1)$, (6.7) is true for sufficiently large s. Now it is very easy to derive the estimates (4.11)–(4.13) for $\mathcal{L}''$, $r_1^{\mathcal{L}}$, and $r_2^{\mathcal{L}}$. We leave the details to the reader.

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