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FRACTIONAL NOISES: DIFFUSIVE MODEL FOR CCD IMAGER BAND-PASS ACQUISITION CHAIN

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Abstract. Charge-Coupled Devices often deliver non standard noises, with a power spectrum proportional to a real power of 1/f over several decades (depending on biasing and temperature). On the other hand, CCD signal acquisition systems are frequently non stationary in their structure: the Wiener-Kintchin theorem is not valid for evaluating noise power and signal to noise ratio. In this paper, the chain is seen as a continuous state-variable time-varying system with a fractional-noise input, modelled from a diffusive representation of fractional integrators.

Résumé. Le bruit des composants CCD est souvent non standard, à spectre de puissance proportionnel à une puissance réelle de 1/f sur plusieurs décades. Par ailleurs, les systèmes d'acquisition de signaux CCD sont fréquemment de structure non stationnaire: le théorème de Wiener-Kintchin n'est plus applicable pour l'évaluation de la puissance de bruit ou du rapport signal/bruit. Dans cet article, la chaine d'acquisition est considérée comme un système à coefficients variables, à variable d'état continu, attaqué par un bruit fractionnaire, modélisé à partir de la représentation diffusive d'un intégrateur fractionnaire.

1. Introduction

The sensitivity of CCD imagers is limited by the noise of the on-chip MOSFET amplifier (fig1) which generates both white noise and low-frequency noise with a power spectrum proportional to $1/f^{\alpha}$, $0.8 \leq \alpha \leq 2$ (fig2), depending on biasing, temperature and the fabrication process [1].

The function of a CCD signal acquisition system is to extract the pixel information from the output signal while suppressing the noise. The information, i.e. the amount of charge collected in a pixel, is represented by the voltage difference (ΔV) between the reference level after having reset the output node and the level after the charge injection from the pixel to the floating diffusion (see fig1). This technique is known as correlated double sampling (CDS).

Several ways for evaluating ΔV have been presented in the literature [2], [3]. When the output signal of the acquisition chain is stationary, standard tools in the frequency domain may be employed and the optimal solution, with respect to the signal to noise ratio, depends on the matching between the acquisition chain transfer function and the noise power spectral density (PSD). In opposite, other CDS methods are based on commutable filters whose both output and noise signals never reach the stationary rate [3], [4]. In that case, the mathematical representation of the system necessarily involves a dynamic state representation with $1/f^{\alpha}$ input noise.

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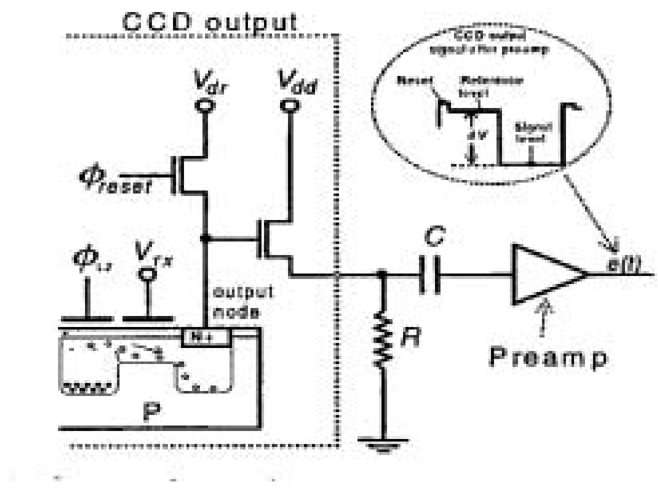

FIGURE 1. Shematic figure of a CCD output

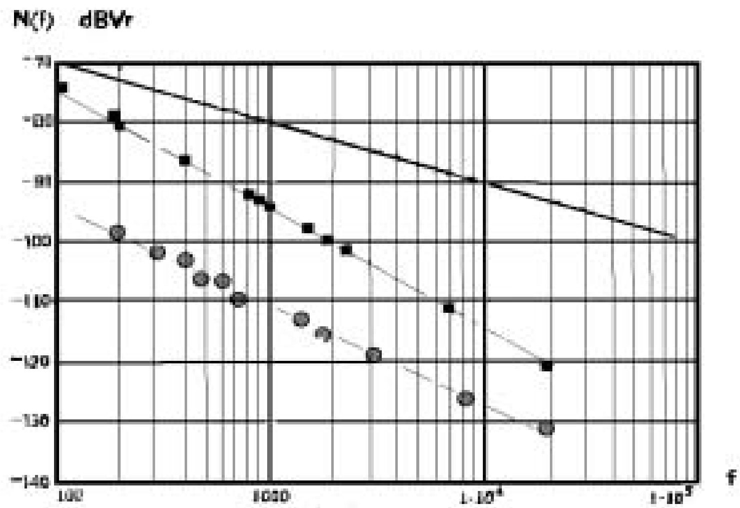

FIGURE 2. Output noise of a Thomson CCD detector type 7895M

The *fractional* $1/f^{\alpha}$ noise ($0 < \alpha < 2$), related to the fractional brownian motion [5], is difficult to manipulate under its usual formulation. Indeed, the time-convolution with a singular long-memory kernel leads to stochastic integro-differential equations. The so-described stochastic process is not markovian: from the numerical simulation point of view, its entire past is to be memorized, which leads in practice to very prohibitive memory space and computation.

The approach presented in this paper is introduced in [6]; it was first used in [4] for the electronic treatment of the CCD noise. It is based on a markovian input-output representation of fractional noises, elaborated from an infinite-dimension stochastic differential equation. A rigorous and general mathematical frame is presented in [7].

2. Preliminary notions

We implicitly refer to an underlying probability space and the mathematical expectation of a real random variable X is denoted by $E(X)$, the variance by $var(X)$; t denotes the time variable and f the frequency.

2.1. The mathematical model under consideration

We consider the linear (stochastic) differential system:

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)y(t), \quad x(0) = x_0, \tag{2.1}$$

with y a scalar $1/f^{\alpha}$ noise, $0 \leq \alpha \leq 2$, which means that the power spectral density $N_y(f)$ of y is expressed as:

$$N_y(f) \propto \frac{1}{|f|^{\alpha}} = \frac{1}{|f|^{2\nu}}, \quad \alpha = 2\nu;$$

y is linked to the standard white noise w from an integration of non-integer order $\nu = \alpha/2$. Note that in practice, the frequency domain of $\frac{1}{f^{\alpha}}$-noise covers several decades in CCD systems.

If $\alpha = 0$, $y(t)$ is a white noise and (2.1) is written as:

$$dx(t) = A(t)x(t)dt + B(t)d\beta(t), \tag{2.2}$$

with $y = \frac{d\beta(t)}{dt}$, $\beta(t)$ a Wiener-Levy process or standard Brownian motion[1] [8]. If $\alpha \neq 0$, $y(t)$ may be linked to a fractional brownian motion (FBM), the definition of which is recalled in section 2.3.

2.2. The fractional brownian motion

The frequency-domain definition of the fractional noise is not sufficient when the system is not time-invariant. It is then necessary to consider a convenient time-domain definition. This last is based on fractional brownian motions. Given a constant $0 < a < 1$, the fractional brownian motion (FBM), with Hurst parameter a and initial value y_0 , may be defined for $t > 0$ by :

$$y_{fbm}(t) = y_0 + \frac{(2\pi)^{a+1/2}}{\Gamma(a+\frac{1}{2})}\left\{\int_0^t \frac{d\beta(s)}{(t-s)^{\frac{1}{2}-a}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-a}} - \frac{1}{(-s)^{\frac{1}{2}-a}}\right) d\beta(s)\right\} \tag{2.3}$$

This process has remarkable properties [5]. According to the definitions of the white noise and the fractional integral (of order $\nu > 0$), denoted I^{ν} [9]:

$$I^{\nu}h(t) = \frac{1}{\Gamma(\nu)}\int_0^t \frac{h(s)}{(t-s)^{1-\nu}}ds, \tag{2.4}$$

$y_{fbm}(t)$ may be expressed under the form:

$$y_{fbm}(t) = (2\pi)^{\nu} I^{\nu} w(t) + \varsigma(t), \text{ with } \nu = a + 1/2, \ 1/2 < \nu < 3/2. \tag{2.5}$$

The quantity $\varsigma(t)$ may be seen as "a corrective term" devoted to the property of stationary increments.

Remark: It may be shown that $I^{\nu}w(t)$ asymptotically behaves as $y_{fbm}(t)$, in then sense that :

1. $I^{\nu}w(t)$ possesses *asymptoticaly* stationary increments,
2. $\Delta_{\tau} y_{fbm}(t) - \Delta_{\tau} I^{\nu} w(t) \underset{t\to+\infty}{\overset{q.m}{\rightarrow}} 0$, with Δ_{τ} the usual difference operator[2].

---

[1] This process has independant increments, with the well-known properties: $E[w(t).w(s)] = \delta(t-s)$, $E\left[(d\beta(t))^2\right] = dt$.

[2] This operator is defined by: $\Delta_{\tau}\varphi(t) := \varphi(t) - \varphi(t-\tau)$

The FBM is the stationarized (*in the increment sense*) ν -integration of a white noise process, with $1/2 < \nu < 3/2$. As usual in physics, we will call such a process, the "fractional noise of order ν", $1/2 < \nu < 3/2$. For $0 < \nu < 1/2$, the fractional noise of order ν is the derivative of the FBM of parameter $a = \nu + 1/2$. Such a process is then stationary if $\nu < 1/2$, and stationary-increment if $\nu > 1/2$; it behaves asymptotically as $I^{\nu} w(t)$. In the sequel, we only consider the case $\nu \geq 1/2$. The case $\nu < 1/2$ may be tackled in a similar way, by considering, as in [6], the derivative of a FBM of parameter $a < 1/2$.

3. Diffusive markovian representation of the fractional noise

This representation have its origin in the diffusive representations of fractional integrators introduced in [10] and successfully used in [11], [12]. It permits to translate the functional mapping : $z := I^{\nu} u$ into the input-output diffusion equation with null initial condition:

$$\left\{ \begin{array}{l} \frac{\partial Y(r,t)}{\partial t} = \frac{\partial^2 Y(r,t)}{\partial r^2} + \delta(r).u(t), \qquad Y(0,t) = 0 \\ z(t) := \displaystyle\int_{-\infty}^{+\infty} m_{\nu}(r) Y(r,t) dr, \end{array} \right. \tag{3.1}$$

with $m_{\nu}(r)$ a convenient distribution. From Fourier transform with respect to r, and introducing a non null initial condition in order to warrant the stationarity of the process, we consider in the same way:

Definition 3.1. The **Diffusive stationary input-output representation** *is defined as:*

$$\left\{ \begin{array}{l} dy_{\xi}(t) = -\xi y_{\xi}(t) dt + d\beta(t), \quad y_{\xi}(0) = y_{\xi 0}, \ \xi > 0, \\ y(t) := \displaystyle\int_{0}^{\infty} y_{\xi}(t) \mu_{\nu}(\xi) d\xi \end{array} \right. \tag{3.2}$$

The fractional noise y will be elaborated from (3.2), with $\mu_{\nu}(\xi)$ a real measure defined on $\mathbf{R}^{*+}$ to be specified. $y_{\xi 0}$ is a centered gaussian random variable chosen to impose the stationarity on the process.

Theorem 3.2. *For* $\mu_{\nu}(\xi) = \frac{\sin(\pi\nu)}{\pi} \left(\frac{2\pi}{\xi}\right)^{\nu}$, $1/2 < \nu < 1$, *if* $y_{\xi 0}$ *verifies* $E[y_{\xi 0}] = 0$ *and* $E[{y_{\xi 0}}^2] = \frac{1}{2\xi}$, *then* $y(t)$ *is a* $1/f^{\alpha}-$*noise,* $\alpha = 2\nu$ *.*

The proof is a direct consequence of lemma (4,5,6,7) in section 6.

4. Application to fractional stochastic differential linear model.

The aim of the sequel is to use the markovian representation of the fractional noise in the study of noise performances of a CCD chain acquisition. The introduction of the markovian model in (2.1), transforms this stochastic *fractional* differential equation into an ordynary one of infinite dimension. Classical stochastic computation may then be used. For convenience, we denote $(f|g)_{\xi} = \int_0^{\infty} fg d\xi$.

4.1. A convenient representation

Let us consider the linear equation (2.1), with y a fractional noise of order ν , $1/2 < \nu < 1$. From the above, this model may be rewritten as :

$$\left\{ \begin{array}{l} dy_{\xi}(t) = -\xi y_{\xi}(t) dt + d\beta(t), \ \ y_{\xi}(0) = y_{\xi 0}\,, \, \xi > 0 \\ dx(t) = A(t) x(t) dt + B(t). \left(\mu_{\nu} | y_{\xi}(t)\right) \,, \ \ x(0) = x_0, \end{array} \right. \tag{4.1}$$

with $\mu_{\nu}(\xi) = \frac{\sin(\pi\nu)}{\pi} \left(\frac{2\pi}{\xi}\right)^{\nu}$, $0 < \nu < 1$ and $E[{y_{\xi 0}}^2] = \frac{1}{2\xi}$. The following representation (convenient for the computation of moments) may also be considered, due to the linearity property:

$$\left\{\begin{array}{l} dy_\xi(t) = -\xi y_\xi(t)dt + d\beta(t), \;\; y_\xi(0) = y_{\xi 0}\,,\, \xi > 0 \\ dx_\xi(t) = A(t)x_\xi(t)dt + B(t)y_\xi(t)\,, \;\; x_\xi(0) = 0\,,\, \xi > 0 \\ d\overline{x}(t) = A(t)\overline{x}(t)dt\,,\, x(0) = x_0 \\ x(t) := (\mu_\nu | x_\xi(t))_{\xi \in \mathbf{R}^{*+}} + \overline{x}(t). \end{array}\right. \tag{4.2}$$

Denoting:

$$\mathbf{X} \;:\; = \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} := (X_\xi)_{\xi \in \mathbf{R}^{*+}} := \begin{pmatrix} y_\xi \\ x_\xi \end{pmatrix}_{\xi \in \mathbf{R}^{*+}},\; \mathbf{X}_0 = \begin{pmatrix} y_{\xi 0} \\ 0 \end{pmatrix}_{\xi \in \mathbf{R}^{*+}}$$

and

$$\begin{aligned} \mathbf{A} \;&:\; = (A_\xi)_{\xi \in \mathbf{R}^{*+}} := \begin{bmatrix} -\xi & 0 \\ B & A \end{bmatrix}_{\xi \in \mathbf{R}^{*+}}, \quad \mathbf{B} := (B)_{\xi \in \mathbf{R}^{*+}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\xi \in \mathbf{R}^{*+}}, \\ \mathbf{C} \;&:\; = \begin{bmatrix} \mathbf{0} & M_\nu \end{bmatrix}, \;\text{ with }\; M_\nu := (\mu_\nu | \,.\,)_{\xi \in \mathbf{R}^{*+}}\,, \end{aligned} \tag{4.3}$$

the system may be rewritten under the tensor form:

$$\left\{\begin{array}{l} d\mathbf{X} = \mathbf{A}\mathbf{X}dt + \mathbf{B}dt \\ d\overline{x}(t) = A(t)\;\overline{x}(t)\;dt,\; x(0) = x_0 \\ x := \mathbf{C}\mathbf{X} + \overline{x}(t). \end{array}\right. \tag{4.4}$$

4.2. The correlation function of the process x

We suppose, without loss of generality, that $E\left[\mathbf{X}_0\right] = 0$ and $x_0 = 0$, which implies that $E\left[\mathbf{X}(t)\right] = 0$. The correlation function of x : $R_{xx}(t,s) := E\left[x(t).x^T(s)\right]$, is obtained from the $\mathbf{X}$ one, defined by :

$$R_{x_\xi x_\eta}(t,s) := E\left[x_\xi(t).x_\eta^T(s)\right],\; \xi,\eta,t,s > 0.$$

The correlation tensor $\left\{R_{x_\xi x_\eta}(t,s)\right\}$ is the solution of the set of equations [6] :

$$\left\{\begin{array}{l} \frac{d}{ds} R_{x_\xi x_\eta}(s,s) = A_\xi(s).R_{x_\xi x_\eta}(s,s) + R_{x_\xi x_\eta}(s,s).A_\eta^T(s) + B.B^T \\ R(0,0) = \begin{bmatrix} \frac{1}{\xi+\eta} & 0 \\ 0 & 0 \end{bmatrix} \\ \frac{\partial}{\partial t} R_{x_\xi x_\eta}(t,s) = A_\xi(t).R_{x_\xi x_\eta}(t,s), \end{array}\right. \tag{4.5}$$

and the correlation function $R_{xx}(t,s)$ is given by :

$$R_{xx}(t,s) = (\mu_\nu \otimes \mu_\nu | R_{x_\xi x_\eta}(t,s)). \tag{4.6}$$

The properties (4.5) and (4.6) are obvious (see ([6]) for the general case) and will be sued in the next section for CCD application. Due to the linearity of the system (4.2), and using (8.10), (4.6) may be rewritten under the more convenient form if only the computation of the moments of the output signal is considered :

$$R_{xx}(t,s) = (\varphi_\nu(\xi) | R_{x_\xi x_\xi}(t,s))\,, \quad \varphi_\nu(\xi) = 4(\frac{2\pi}{\xi})^{2\nu-1}\sin(\pi\nu). \tag{4.7}$$

Details are given in the particular case of the CCD acquisition chain in the next section.

5. Application to commutable pass-band filter for CCD signal acquisition

The acquisition chain is a commutable band-pass filter whose function is the removal of both low-frequency excess noise (or fractional noise) and high-frequency noise in order to maximize the signal to noise ratio. When the output signal is the reference level (fig 1) the time constants are made very small for resetting the system and memorize the reference level. In the second part of the lecture sequence, the time constants are chosen in order to obtain the maximum output level at a desired date.

5.1. The model

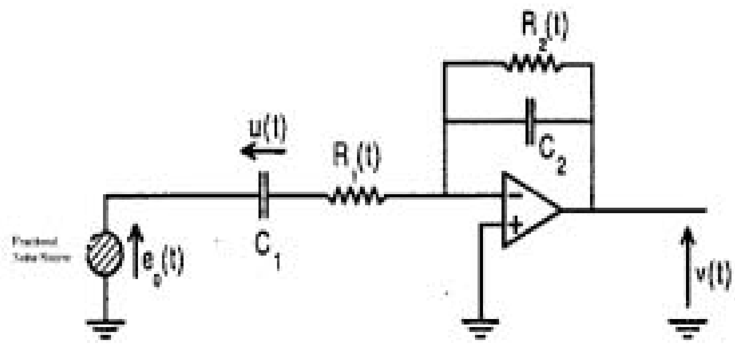

Figure 3. Variable band-pass filter

We consider, as in [4], the following state-representation of the commuted band-pass filter described in figure 3, with fractional noise $y(t)$ input and output $s(t)$.

$$\left\{\begin{array}{r} du = -a(t)u(t)dt + a(t)y(t)dt, \quad u(0) = 0 \\ dv = -b(t)v(t)dt + c(t)u(t)dt - c(t)y(t)dt, \;\; v(0) = 0, \end{array}\right. \tag{5.1}$$

$a(t) := (R1.C1)^{-1}$, $b(t) := (R2.C2)^{-1}$, $c(t) := (R1.C2)^{-1}$.

R_1 and R_2 are time-dependant resistors (using in practice JFETS as parallel switches). The function of this circuit is to reduce the noise bandwidth while measuring ΔV. It allows to reset C_1 and C_2 at every period of the read sequence. Only the variance of $v(t)$ is of interest in this application. From (4.2), (5.1) is written :

$$\left\{\begin{array}{l} dy_\xi(t) = -\xi y_\xi(t)dt + d\beta(t) \\ du_\xi = -au_\xi dt + ay_\xi dt \\ dv_\xi = -bv_\xi dt + cu_\xi dt - cy_\xi dt \\ u(t) = \displaystyle\int_0^\infty u_\xi(t)\mu_\nu(\xi)d\xi \\ v(t) = \displaystyle\int_0^\infty v_\xi(t)\mu_\nu(\xi)d\xi. \end{array}\right. \tag{5.2}$$

5.2. Analysis

After convenient reorganization of (5.2), we obtain:

$$d\begin{pmatrix} y_\xi \\ u_\xi \\ v_\xi \end{pmatrix} = \begin{bmatrix} -\xi & 0 & 0 \\ a & -a & 0 \\ -c & c & -b \end{bmatrix}.\begin{pmatrix} y_\xi \\ u_\xi \\ v_\xi \end{pmatrix} dt + \begin{bmatrix} d\beta(t) \\ 0 \\ 0 \end{bmatrix}, \tag{5.3}$$

rewritten as:

$$dx_\xi = A_\xi x_\xi dt + [d\beta], \tag{5.4}$$

where: $x_\xi := \begin{pmatrix} y_\xi \\ u_\xi \\ v_\xi \end{pmatrix}$, $A_\xi := \begin{bmatrix} -\xi & 0 & 0 \\ a & -a & 0 \\ -c & c & -b \end{bmatrix}$, $[d\beta] := \begin{bmatrix} d\beta(t) \\ 0 \\ 0 \end{bmatrix}$

It is obvious that $E(v(t)) = 0$, and we denote $R_{w,z}(t,t') := E(v(t).z^T(t'))$, so:

$$\begin{cases} R_{vv}(t,t') := E(v(t)v(t')) = \int_0^\infty \varphi_\nu(\xi) R_{v_\xi v_\xi}(t,v) d\xi \\ \text{with } \varphi_\nu(\xi) := 4(\frac{2\pi}{\xi})^{2\nu-1}\sin(\pi\nu). \end{cases} \tag{5.5}$$

Using (5.4), and $t > t'$,

$$dR_{x_\xi,x_\xi}(t,t') = E\left(d_t\left(x_\xi(t)x_\xi(t')^T\right)\right) = E\left(A_\xi x_\xi(t)x_\xi(t')^T dt + [d\beta(t)]\, x_\xi(t')^T\right) \tag{5.6}$$

where $E\left([d\beta(t)]\,.x_\eta(t')^T\right) = 0$ because $\beta(t)$ is an independent increment process [14]. From linearity of (5.4), we then obtain:

$$\frac{\partial}{\partial t} R_{x_\xi x_\xi}(t,t') = A_\xi R_{x_\xi x_\xi}(t,t'). \tag{5.7}$$

The evaluation of the tensor $\mathbf{R}_{x_\xi x_\xi}(t,t)$ is quite identical:

$$dR_{x_\xi,x_\xi}(t,t') = E\left(d_t x_\xi(t)x_\xi(t)^T + x_\xi(t)d_t x_\xi(t)^T + d_t x_\xi(t) d_t x_\xi(t)^T\right), \tag{5.8}$$

with (due to the classical rules of stochastic calculus) :

$$E\left(d_t x_\xi(t) d_t x_\xi(t)^T\right) = E\left([d\beta(t)]\,.\,[d\beta(t)]^T\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} dt.$$

According to (8.2) and the stationarity of y_ξ, $E(y_\xi(t)y_\xi(t)) = \frac{1}{2\xi}$; from the above and after convenient reorganization we obtain from (5.8), the infinite set of ordinary differential system (for each $\xi \in \mathbf{R}^{*+}$):

$$\frac{d}{dt}\begin{bmatrix} R_{y_\xi u_\xi}(t,t) \\ R_{y_\xi v_\xi}(t,t) \\ R_{u_\xi u_\xi}(t,t) \\ R_{u_\xi v_\xi}(t,t) \\ R_{v_\xi v_\xi}(t,t) \end{bmatrix} = \begin{bmatrix} -\xi-a & 0 & 0 & 0 & 0 \\ c & -\xi-b & 0 & 0 & 0 \\ 2a & 0 & -2c & 0 & 0 \\ -c & a & c & 0 & 0 \\ 0 & -2c & 0 & 2c & -2b \end{bmatrix}.\begin{bmatrix} R_{y_\xi u_\xi}(t,t) \\ R_{y_\xi v_\xi}(t,t) \\ R_{u_\xi u_\xi}(t,t) \\ R_{u_\xi v_\xi}(t,t) \\ R_{v_\xi v_{n\xi}}(t,t) \end{bmatrix} + \begin{bmatrix} \frac{a}{2\xi} \\ \frac{-c}{2\xi} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \xi > 0, \text{ with null initial condition.} \tag{5.9}$$

The variance of the output signal is finally:

$$var(v) := R_{vv}(t,t) = 4\int_0^\infty (2\pi/\xi)^{2\nu-1}\sin(\pi\nu)R_{v_\xi v_\xi(t)}d\eta = \int_0^\infty \varphi_\nu(\xi)R_{v_\xi v_\xi(t)}d\xi \quad (5.10)$$

with $\varphi_\nu(\xi)$ defined in (5.5).

We may obtain numerically a convergent evaluation of $var(v)$ versus time by discretizing ξ. The proof of the convergence may be found in [7]. If we choose a finite set of positive numbers:

$$0 \leq \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N,$$

the reduction of (5.9) to this set leads to the following approximation: $var(v) \simeq \sum_{i=0}^{N} \varphi_\nu(\xi_i)R_{v_{\xi_i}v_{\xi_i}(t)}\Delta\xi_i$. In practice, N may be low (typically 10) which allows an implementation with very ordinary means.

This methodology has been implemented on a Thomson CCD detector (type 7895M) whose output noise spectral density was proportional to $1/f^2$ at room temperature (typically 295K) and $1/f^{1.5}$ at $T = 233$K (-40^oC), in the aim to compare experimental data with the theoretical ones obtained from (5.10).

6. Experimental versus theoretical results.

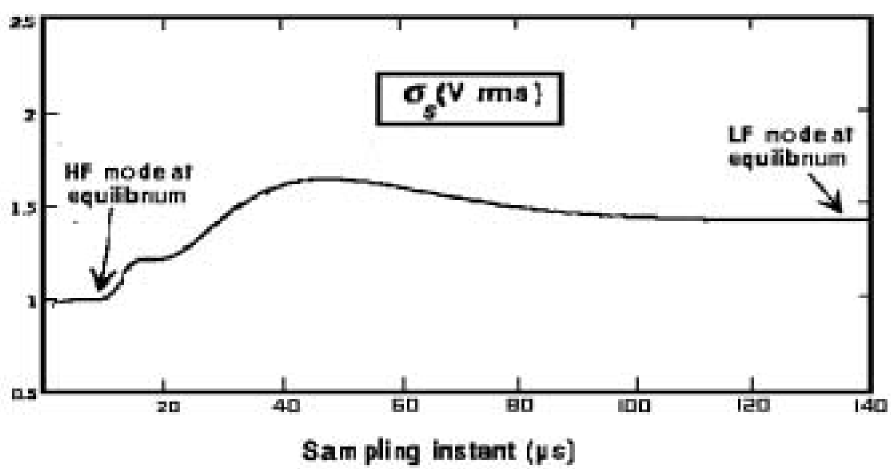

Figure 4. RMS output noise versus time with $1/f$ input noise, ($\nu = 0.5$)

Fig 4 shows the theoretical behavior of the square root σ of the output variance for an input with $1/f$ PSD. In fig 5 , both experimental points (dots) and theoretical (line) are included for $1/f^{1.5}$input noise. The experimental values of σ are obtained by a direct measure on the output noise. The theoretical values come from the numerical resolution of the deterministic model (5.9) and 5.10). A good overall agreement is observed. The small discrepancies are due to our simplified model of the switches and operational amplifier during the transition. More technical details may be found in [13] and [4]

7. Conclusion

We have used a new markovian model of fractional noises. This approach allowed us to compare experimental and theoretical data of a band-pass filter used

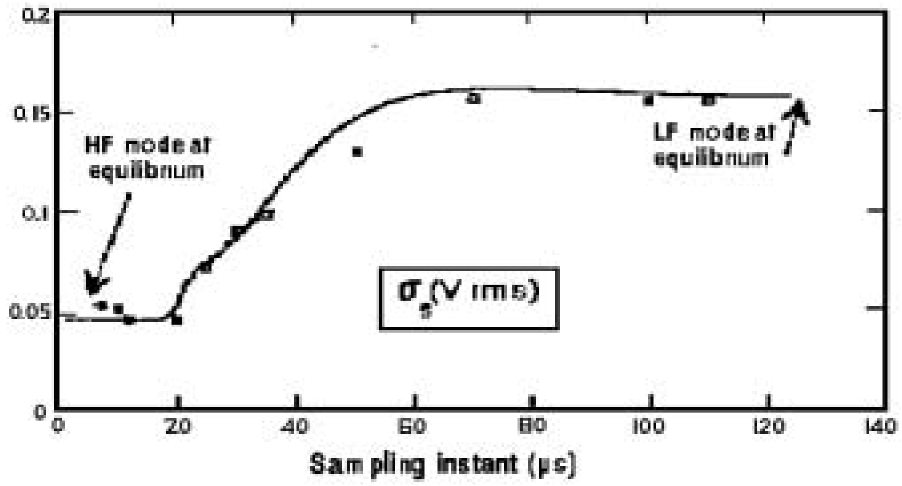

FIGURE 5. RMS output noise versus time with $1/f^{1.5}$ input noise, ($\nu = 0.75$)

in a CCD camera with a good agreement that confirms the validity of above approach. The model may also be used in Monte-Carlo simulations, for instance for the optimization of the signal to noise ratio. Remark that, in the hereditary model of the fractional noise, the integro-differential system is no more usable because of the computation and memory volume which is prohibitive (memorization of the past).

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8. Appendix

LEMMA 8.1. ***(stationarity of the process)*** *The vector process $\{y_\xi(t)\}_{\xi>0}$is centered, gaussian and WSS if and only if $\{y_\xi(t)\}_{\xi>0}$is a centered gaussian random vector with covariance:*

$$E(y_{\xi 0}y_{\eta 0}) = \frac{1}{\xi+\eta} \tag{8.1}$$

Proof. According to independant-increment property of β , the correlation function of y_ξ is given, for any $s, t > 0$, by:

$$R_{\xi\xi}(t,s) := E[y_\xi(t)\, y_\xi(s)] = e^{-\xi(t+s)}E[y_{\xi 0}^2] + \frac{1}{2\xi}\left(e^{-\xi|t-s|} - e^{-\xi(t+s)}\right)$$

Thus a first necessary condition for $\{y_\xi\}_{\xi>0}$ to be stationary is:

$$E[{y_{\xi 0}}^2] = \frac{1}{2\xi} \tag{8.2}$$

which gives:

$$R_{\xi\xi}(t,s) = \frac{1}{2\xi}e^{-\xi|t-s|} \tag{8.3}$$

Similarly, the asymptotic intercorrelation function of $\{y_\xi(t)\}_{\xi>0}$ is in the same way and after simplification given by:

$$R_{\xi\eta}(t,s) := E[y_\xi(t), y_\eta(s)] = \begin{cases} \frac{e^{-\eta(s-t)}}{\xi+\eta} & \text{if } s>t \\ \frac{e^{-\xi(t-s)}}{\xi+\eta} & \text{if } s<t \\ \frac{1}{\xi+\eta} & \text{if } s=t \end{cases}$$

we then obtain the stationary condition:

$$R_{\xi\eta}(0,0) := E[y_{\xi 0}, y_{\eta 0}] = \frac{1}{\xi+\eta}. \tag{8.4}$$

∎

LEMMA 8.2. ***(Determination of $\mu_\nu(\xi)$)*** *The process y defined by (3.2) is a fractional stationary noise of order ν $(1/2 < \nu < 1)$ if and only if:*

$$\mu_\nu(\xi) = \frac{\sin(\pi\nu)}{\pi}\left(\frac{2\pi}{\xi}\right)^\nu \tag{8.5}$$

Proof. From stochastic integration, we obtain from (3.2):

$$y_\xi(t) = y_{\xi 0}e^{-\xi t} + \int_0^t e^{-\xi(t-s)}d\beta(s).$$

Then:

$$\begin{aligned} y(t) &= \int_0^\infty y_\xi(t)\mu_\nu(\xi)d\xi && (8.6)\\ &= \int_0^\infty y_{\xi 0}e^{-\xi t}\mu_\nu(\xi)d\xi + \int_0^t \left(\int_0^\infty e^{-\xi(t-s)}\mu_\nu(\xi)d\xi\right)d\beta(s). && (8.7)\end{aligned}$$

According to the well-known formulas, when $\tau > 0$:

$$0 < \nu < 1, \quad \int_0^\infty \xi^{-\nu} \exp(-\xi\tau) d\xi = \frac{\Gamma(1-\nu)}{\tau^{1-\nu}} \text{ and } \frac{1}{\Gamma(\nu).\Gamma(1-\nu)} = \frac{\sin(\pi\nu)}{\pi} \quad (8.8)$$

it is verified that, with $\mu_\nu(\xi) = \frac{\sin(\pi\nu)}{\pi}\left(\frac{2\pi}{\xi}\right)^\nu$ and $t > s$:

$$\int_0^\infty e^{-\xi(t-s)} \mu_\nu(\xi) d\xi = \frac{(2\pi)^\nu}{\Gamma(\nu)} \frac{1}{(t-s)^{1-\nu}}$$

and

$$\int_0^t \left(\int_0^\infty e^{-\xi(t-s)} \mu_\nu(\xi).d\xi \right) d\beta(s) = \frac{(2\pi)^\nu}{\Gamma(\nu)} \int_0^t \frac{d\beta(s)}{(t-s)^{1-\nu}}.$$

∎

So, for $1/2 < \nu < 1$, the noise generated by (3.2) and the fractional brownian motion (2.3) asymptoticaly behaves identically in the increment sense (for the proof, see [6] and [7]).

Lemma 8.3. *(**Covariance function of** $y(t)$). The covariance function of the* $1/f^{2\nu}$*-noise* $y(t)$ *is given by:*

$$R_y(t,s) = \int_0^\infty \varphi_\nu(\xi) R_{\xi\xi}(t,s) d\xi, \qquad \varphi_\nu(\eta) = 4(\frac{2\pi}{\xi})^{2\nu-1} \sin(\pi\nu)$$

Proof. For $s > t \in \mathbf{R}^+$,

$$E(y(t)y(s)) = R_y(t,s) = \int_0^\infty \int_0^\infty E(y_\xi(t) y_\eta(s)) \mu_\nu(\xi) \mu_\nu(\eta) d\xi d\eta$$

$$R_y(t,s) = (2\pi)^\nu \frac{\sin^2(\pi\nu)}{\pi^2} \int_0^\infty \frac{\exp(-\eta(s-t))}{\eta^\nu} \left(\int_0^\infty \frac{1}{(\eta+\xi).\xi^\nu} d\xi \right) d\eta$$

With the well-known formula $\int_0^\infty \frac{1}{(1+x)x^\nu} dx = \frac{\pi}{\sin(\pi\nu)}$, $0 < \nu < 1$,
it may be verified that:

$$R_y(t,s) = 4 \int_0^\infty (2\pi/\eta)^{2\nu-1} \sin(\pi\nu) \frac{\exp(-\eta(s-t))}{2\eta} d\eta \quad (8.9)$$

The covariance function of $y(t)$, $R_y(t,s)$,may be expressed with the covariance function $R_{\eta\eta}(t,s)$ of the unique output y_η only, which will give an important shortening in computations.

$$R_y(t,s) = \int_0^\infty \varphi_\nu(\eta) R_{\eta\eta}(t,s) d\eta, \qquad \varphi_\nu(\eta) = 4(\frac{2\pi}{\eta})^{2\nu-1} \sin(\pi\nu) \quad (8.10)$$

∎

Lemma 8.4. *(**evaluation of the power spectral density** $N_y(f)$ **of the fractional noise**). The PSD of the fractional noise* $y(t)$ *defined by (3.2) is:*

$$N_y(f) = \frac{1}{|f|^{2\nu}}$$

Proof. : $y(t)$ and $y_\xi(t)$ are a stationary random noise. In the sequel, we denote $t - s = \tau$. $y_\xi(t)$ is the output signal of a low-pass filter whose transfert function $H_\xi(f)$ is obtained by from Fourier transform with respect to t of equation (3.2). It is obvious that :

$$H_\xi(f) = \frac{1}{\xi + i2\pi f}.$$

Due to the linearity (3.2), the global transfert function of the system with input signal $w(t)$ and output signal $y(t)$ is given by :

$$H(f) = \int_0^\infty H_\xi(f)\mu_\nu(\xi)d\xi = \frac{1}{(i2\pi f)^\nu}. \tag{8.11}$$

Then the spectral density (or PSD) of $y(t)$ is:

$$N_y(f) = |H(f)|^2 N_w(f) = |H(f)|^2 = \frac{1}{|f|^{2\nu}} \tag{8.12}$$

$\blacksquare$