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DIFFUSIVE REPRESENTATION OF PSEUDO-DIFFERENTIAL TIME-OPERATORS

GÉRARD MONTSENY

ABSTRACT. The concept of "diffusive representation" was previously introduced in the aim of transforming non standard convolutive causal operators such as fractional integrodifferential ones, into infinite-dimension dissipative classical input-output dynamic systems. The existence of an explicit dissipative semigroup makes then possible the use of classical tools of functional and numerical analysis of PDE, energy methods, control, filtering, etc., generally ill-fitted to the standard convolutive formulations, namely when long memory dynamics are present. The aim of the paper is to present in a synthetic statement, first the general frame of diffusive representations, and secondly some of the essential characteristics and properties of this new tool. Simple and concrete examples are given. More significant applications (essentially in the fractional context) will be found in the referenced papers.

RÉSUMÉ. Le concept de "représentation diffusive" a été introduit dans le but de transformer certains opérateurs convolutifs causaux non standard, tels les intégrations ou dérivations d'ordre fractionnaire, en systèmes dynamiques linéaires entrée-sortie de nature dissipative, dans un espace d'état fonctionnel. L'existence d'un semi-groupe dissipatif explicite rend alors possible l'emploi des outils classiques en analyse fonctionnelle et numérique des EDP, contrôle, filtrage, des méthodes d'énergie, etc., généralement mal adaptés aux formulations standard convolutives, en particulier dans le cas de comportements de type "mémoire longue". L'objectif de l'article est de présenter de manière synthétique, tout d'abord le cadre général des représentations diffusives, d'autre part les caractéristiques et propriétés essentielles de cet outil récemment introduit. Quelques exemples simples sont décrits. Des applications plus consistantes existent dans les articles cités en reférences.

1. INTRODUCTION

The concept of "diffusive representation" was initially introduced in the aim of transforming fractional operators into classical input-output dynamic systems (see [15], [16], [17] etc.). The corner stone of the approach lays in the observation of the fact that the heat equation (or diffusion equation) naturally generates long memory behaviors such as $\frac{1}{\sqrt{t}}$. Via convenient adaptations, it becomes possible to generate from diffusive equations (of various type) several input-output behaviors of *pseudo-differential* nature. This approach have revealed very useful in control, modelling of non standard noises, numerical approximation, etc., namely when long memory dynamics are present, for example when the problem under consideration involves fractional integrodifferential operators. Indeed, the "diffusive realization" of a pseudo-differential operator: $y := a(t, \frac{d}{dt}).u$, appears as a *Cauchy* problem, under the standard abstract form: $\frac{dX(t)}{dt} = AX(t) + Bu(t)$, $X(0) = 0$, $y(t) := C(t, X(t), u(t))$, with $X(t)$ in a Hilbert state space and A a (constant) *dissipative* operator: $(AX \mid X) < 0$, $X \neq 0$ (C independent of t when a is independent of t).

G. Montseny: LAAS du CNRS, 7, avenue du Colonel Roche, 31077 Toulouse cedex 4, France. E-mail: montseny@laas.fr.

Contrarily to the convolutive or integral expression, this equivalent representation is well-fitted to many problems, due to the dissipativity property of diffusions (see [3], [15], [16], [13], etc.).

In this paper, we present in a synthetic statement, first a general frame for diffusive representations, and second the essential properties of this tool. It is organized as follows. In section 2, we recall the definition and basic properties of pseudo-differential operators. In section 3, we introduce the concept of "basic" diffusive representation and the associated calculus rules; some fundamental examples are given in Table 1. Note that this section is self-contained. In section 4, we study the associated mathematical frame, and give some regularity and approximation results. In section 5, we present some extensions to operators of non negative order, and define the class of non constant diffusive operators and representations. Finally, in section 6, we make use of the diffusive representation to perform the analysis of pseudo-differential equations.

Some simple examples are given. More significant applications (essentially in the fractional context) will be found in the referenced papers.

2. PSEUDO-DIFFERENTIAL OPERATORS

One-dimensional pseudo-differential operators (PDO) behave like differential ones, but are generally not reducible to finite-dimensional dynamics, such as rational transfer functions. The simplest of non trivial PDO are the fractional derivators or integrators, which possess several remarkable properties. Although they are useful in many practical situations, fractional operators sometimes reveal to be insufficient, for example for time-varying behaviors[1], or simply from the fact that they do not possess a Banach structure, which may be a considerable shortcoming in various cases (identification problems [7], optimization [17], [4], etc.).

In this section, we introduce a definition of PDO. More details will be found in [2]. Note that, due to the causal nature of the operators under consideration, the Laplace transform is equivalently used in place of Fourier's one. In the sequel, we will denote $(\frac{d}{dt})^{-1}u(t) := \int_0^t u(\tau)\,d\tau$, $\mathcal{L}$ and $\mathcal{F}$ will represent respectively the Laplace and Fourier transforms, and:

$$\widehat{u} := \mathcal{L}u.$$

First remark that for any differential operator of finite order and any u sufficiently regular and causal, $\sum a_k \frac{d^k}{dt^k} u(t) = \mathcal{L}^{-1}\left(\sum a_k p^k \widehat{u}(p)\right)$; hence, this operator may be described simply by changing p in $\frac{d}{dt}$ in the "Laplace expression". This is in fact at the origin of operational calculus. Similarly, given $a(t)$ a regular function, and $h = \mathcal{L}^{-1}H$ a causal "impulse response", the operator $A : u \to a(t)\,(h \star u)\,(t)$ may be equivalently defined by: $u \to a(t)\mathcal{L}^{-1}\left(H(p)\widehat{u}(p)\right) = \mathcal{L}^{-1}\left(a(t)H(p)\widehat{u}(p)\right)$, so that we may write $A = a(t)H(\frac{d}{dt})$. Here is the basic idea of the following definition:

Definition 2.1. [2] Given a regular function $a(t,p)$ defined on $(t,p) \in \mathbf{R}_t^+ \times i\mathbf{R}$, the operator A with **symbol** $a(t,p)$ is defined by:

$$\forall u \text{ causal and regular,}$$
$$Au = a(t,\tfrac{d}{dt}).u := \mathcal{F}^{-1}\left(a(t,i\omega)\mathcal{F}u(\omega)\right) = \frac{1}{2i\pi}\int_{i\mathbf{R}} e^{pt} a(t,p)\widehat{u}(p)\,dp; \tag{2.1}$$

in the case $a(t,p) = H(p)$, the operator A is said **constant**.

An operator of symbol $a(t,p)$ is pseudo-differential if the function $a(t,i\omega)$, $(t,\omega) \in \mathbf{R}^2$, satisfies convenient regularity hypothesis which are not introduced here (see [2]). We simply consider the characteristic "pseudo local property" of PDO.

[1]Example of CCD sensors which generate fractional noise with order depending on the temperature [9].

Given $u(t)$, we define the "singular support" of u by:

$$\operatorname{sing\,supp} u = \mathbf{R} - \{t \in \mathbf{R},\ u \text{ is } C^\infty \text{ at point } t\}.$$

DEFINITION 2.2. [2] (pseudo-local property) The operator A is pseudo-differential if and only if:

$$\forall u,\ \operatorname{sing\,supp}(Au) \subseteq \operatorname{sing\,supp}(u). \tag{2.2}$$

This property means that a pseudo-differential operator may spread the support, but preserves the localization (on the time-axis) of the singularities of any function u. This is the case, for example, for any stable[2] rational function $\frac{N(p)}{D(p)}$ or any fractional operator p^α. We recall the following classical definition:

DEFINITION 2.3. A is a **causal** operator if and only if Au is causal for any u causal.

In the case of causal operators, the Laplace transform may advantageously be employed, with the property of *analyticity* of $a(t,p)$.

This defines the specific frame of the present work. We finally introduce:

DEFINITION 2.4. The **order** of a constant operator A is the number $\sigma(A) \in \overline{\mathbf{R}}$:

$$\sigma(A) = \inf\{k \in \mathbf{R},\ \frac{H(i\omega)}{(i\omega)^k} \underset{|\omega| \to +\infty}{\longrightarrow} 0\}; \tag{2.3}$$

a causal constant PDO A is said **proper** (resp. strictly proper) if $\sigma(A) < 1$ (resp. $\sigma(A) < 0$).

The interpretation of this latter definition is straightforward: it expresses the asymptotic behavior of H at high frequencies, in term of real power of $\frac{d}{dt}$.

Some examples:

- any *stable* (in the large sense) rational transfer function $\frac{N(p)}{D(p)}$ is the symbol of a causal PDO of order $\deg(N) - \deg(D)$, proper if $\deg(N) \le \deg(D)$;
- with $\alpha \in \mathbf{C}, b > 0$, p^α, $\frac{1}{p^{-\alpha}+b}$, $(p+b)^\alpha$ are the symbols of causal PDO of order $\Re(\alpha)$, proper if $\Re(\alpha) < 1$;
- $\ln(p)$ is the symbol of a causal PDO of order 0;
- $e^{-b\sqrt{p}}$, $b > 0$, is the symbol of a causal PDO of order $-\infty$;
- let A the causal PDO with symbol $\frac{1}{(t+p)^\alpha}$, $t \ge 0$, $\alpha > 0$. Denoting Y the step function, from (2.1), $AY(t) = \mathcal{L}^{-1}\left(\frac{1}{p(t+p)^\alpha}\right)(t) = \displaystyle\int_0^t \frac{\tau^{\alpha-1}e^{-\tau^2}}{\Gamma(\alpha)}\,d\tau$. This operator is not constant;
- $\sqrt{-p^2} = |p|$ is the symbol of a non causal PDO of order 1 (note the non analyticity of this function);
- the delay operator, of symbol e^{-bp}, is causal if and only if $b \ge 0$, but not a PDO if $p \ne 0$ (note that (2.2) is not verified); it is of order 0.

In the sequel, all the operators under consideration are causal.

3. THE BASIC DIFFUSIVE REPRESENTATION

In this section, we only consider *constant* pseudo-differential operators.

In its basic formulation, the concept of "Diffusive Representation" (DR) may be simply introduced as a *non standard use of the Laplace Transform* (see [15], [8]), by considering the time variable as a Laplace variable (reduced to the *positive real* axis), associated to a new induced variable $\xi \ge 0$ (in opposite to the standard use in which the classical frequency representation is obtained from reduction to the *imaginary* axis). As a consequence, both Fubini theorem and analyticity property

[2]In the unstable case, the Fourier transform may no more be defined in the sense of tempered distributions.

can be used in the aim of transforming convolutive causal time-operators into well-posed input-output ordinary differential systems in functional state spaces.

3.1. Fundamental definitions

We consider a *causal* function (or distribution) $\mu : \mathbf{R} \to \mathbf{C}$, and the "impulse response" h, defined as the LT of μ, $h = \mathcal{L}\mu$:

$$h(t) = \int_0^{+\infty} e^{-t\xi} \mu(\xi)\, d\xi. \tag{3.1}$$

Definition 3.1. The function $\mu : \mathbf{R}^+ \to \mathbf{R}$ defined by (3.1) is called the **diffusive representation** of the operator $u \to h \star u$, with symbol $H = \mathcal{L}h$.

Proposition 3.2. *The diffusive representation μ of an operator with symbol H, is defined by the equation:*

$$H(p) = \int_0^{+\infty} \frac{\mu(\xi)}{p+\xi}\, d\xi, \tag{3.2}$$

with $p \in \mathcal{D}_\mu \subset \mathbf{C}$, the convergence domain of the integral.

Proof. From Laplace transform of h and Fubini theorem. ∎

Remark 3.3.
1. For $p = i\omega$, (3.2) links the frequency response and the diffusive representation.
2. For $p > 0$, $H(p) = \left(-\mathrm{pv}\frac{1}{\xi} \star \mu\right)(-p)$,[3] thus H may be obtained from analytic extension of the Hilbert transform of μ.

Note that, analogously to the "ordinary" frequency domain, $\int_0^{+\infty} e^{-t\xi}\mu(\xi)\, d\xi$ may be called the *diffusive synthesis* of h. We may remark that the variable ξ is a *frequency* variable, but μ is not the frequency response associated to h; ξ must in fact be seen as a *cut-off* frequency. Note also that the "diffusive transform" may be defined as $\mathcal{L}^{-1} \circ \mathcal{A}$, with $\mathcal{A}f$ the analytic extension of a function f defined on $\mathbf{R}_t^+$. We may conclude that, in opposite to the frequency representation, many classical impulse responses (for example the delay $\delta(t-a)$) do not admit an equivalent diffusive representation.

Definition 3.4. An operator H_μ defined by:

$$H_\mu : u \to h_\mu \star u, \quad h_\mu \,\text{given by (3.1)},$$

is called of *diffusive type.*

Definition 3.5. For any $\mu, \nu,\ h = \mathcal{L}\mu,\ k = \mathcal{L}\nu$, the **diffusive product** is defined when it exists, by:

$$\mathcal{L}(\mu \# \nu) := \mathcal{L}(\mu) \star \mathcal{L}(\nu). \tag{3.3}$$

So, from Laplace transform, the diffusive product is changed into convolution. Note that, with Fourier transform in place of Laplace, # would be the ordinary product, which would not present any interest.

Theorem 3.6. *When this expression is well-defined, the diffusive product takes the explicit form:*

$$\mu \# \nu = -\mu.(\nu \star\ \mathrm{pv}\frac{1}{\xi}) - \nu.(\mu \star\ \mathrm{pv}\frac{1}{\xi}). \tag{3.4}$$

[3]The "principal value" of $\frac{1}{\xi}$ is the distribution defined by: $\int_{\mathbf{R}} \mathrm{pv}\,\frac{1}{\xi} f(\xi)\, d\xi = \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}-[-\varepsilon,+\varepsilon]} \frac{f(\xi)}{\xi}\, d\xi$.

Proof. With sufficiently regular μ, ν and the integrals understood in the principal value sense,

$$\begin{aligned}
\mathcal{L}\mu \star \mathcal{L}\nu &= \int_0^t \int_0^{+\infty} e^{-(t-\tau)\xi}\mu(\xi)\,d\xi \int_0^{+\infty} e^{-\tau\zeta}\nu(\zeta)d\zeta d\tau \\
&= \int_0^{+\infty}\int_0^{+\infty}\left(\int_0^t e^{-t\xi+\tau\xi-\tau\zeta}d\tau\right)\mu(\xi)\nu(\zeta)d\zeta\,d\xi \\
&= \int_0^{+\infty}\nu(\zeta)\left(\int_0^{+\infty}\frac{\mu(\xi)}{\xi-\zeta}\,d\xi\right)e^{-\zeta t}d\zeta + \int_0^{+\infty}\mu(\xi)\left(\int_0^{+\infty}\frac{\nu(\zeta)}{\zeta-\xi}d\zeta\right)e^{-\xi t}\,d\xi \\
&= \mathcal{L}\left(-\nu\int_0^{+\infty}\frac{\mu(\xi)}{.-\xi}\,d\xi - \mu\int_0^{+\infty}\frac{\nu(\xi)}{.-\xi}\,d\xi\right).
\end{aligned}$$

∎

Some examples (we denote $\mu_\alpha(\xi) := \frac{\sin(\pi\alpha)}{\pi}\frac{1}{\xi^\alpha}$):

1. $a>0$, $(\mu_\alpha \# \delta_a)(\xi) = \mu_\alpha(\xi)\,\mathrm{pv}\,\frac{1}{a-\xi} + k\delta_a(\xi)$, $k = \int_0^{+\infty}\mu_\alpha(\xi)\,\mathrm{pv}\,\frac{1}{a-\xi}\,d\xi$ (from (3.4));
2. $a,b>0$, $a\neq b$, $\delta_a\#\delta_b = \frac{1}{b-a}\delta_a + \frac{1}{a-b}\delta_b$ (from (3.4));
3. $a\geq 0$, $\delta_a\#\delta_a = \mathcal{L}^{-1}\left(e^{-at}\star e^{-at}\right)(\xi) = \mathcal{L}^{-1}\left(te^{-at}\right)(\xi) = \delta_a'(\xi)$;
4. $a>0$, $(\mu_\alpha\#\delta)(\xi) = \mathcal{L}^{-1}\left(t.\frac{t^{\alpha-1}}{\alpha\Gamma(\alpha)}\right) = \frac{1}{\alpha}\frac{d}{d\xi}\mu_\alpha(\xi) = -\frac{\sin(\pi\alpha)}{\pi}\,\mathrm{fp}\,\frac{1}{\xi^{1+\alpha}}$.

In the class of diffusive operators, we thus have the three equivalent representations:

$$\begin{array}{ccccc}
\text{diffusive rep.} & \xrightarrow{\mathcal{L}} & \text{convolutive rep.} & \xrightarrow{\mathcal{L}} & \text{symbol} \\
\mu(\xi) & & h(t) & & H(p) \\
\mu\#\nu & & h\star k & & H(p).K(p)
\end{array} \tag{3.5}$$

The major interest of the DR lies in the following definition, which enables to transform convolutive operators of diffusive type into standard memoryless (*a priori* infinite-dimensional) input-output state representations.

THEOREM 3.7. (**Definition**) *The "basic diffusive realization" of the operator* : $u\to y = H(\frac{d}{dt}).u$, *is the dynamic input-output system defined by the following diffusion equation:*

$$\mathrm{D}_\mu \equiv \left\{\begin{array}{l}
\partial_t\psi(\xi,t) = -\xi\psi(\xi,t) + u(t), \quad \xi\in\mathbf{R}^+, \quad \psi(\xi,0)=0 \\
y(t) := \displaystyle\int_0^{+\infty}\mu(\xi)\psi(\xi,t)\,d\xi,
\end{array}\right. \tag{3.6}$$

with μ *defined by (3.5).*

Proof. From Fubini theorem, $h\star u = \displaystyle\int_0^{+\infty}(e^{-t\xi}\star u).\mu(\xi)\,d\xi$. ∎

The state representation (3.6) is in fact an equivalent (time-) representation of $y = h\star u$, based on an "internal" state variable ψ, convenient (from the dissipative nature of (3.6)) for analysis, approximation, control, etc. of models which involve non standard operators of diffusive type (see [3], [5], [14], [16], [21], etc.). Indeed, this internal variable appears in some sense as an "analytical" representation of the operator, in opposite to $h\star(.)$ which is in fact "synthetic". This is a direct consequence of the use of Fubini theorem which plays a fundamental role in the diffusive approach. So, natural tools such as energy functionals may be introduced. For example, with $E(\psi) = \frac{1}{2}\|\psi\|^2_{L^2(\mathbf{R}^+)}$, we have the following dissipativity result: if ψ is solution of (3.6), then $\frac{dE(\psi(.,t))}{dt}\leq 0$ if $u(t)=0$. This is naturally related to Lyapunov methods; analogously, finite-dimensional numerical approximations are easily elaborated from hilbertian techniques.

Note that in opposite to the frequency representation, the analysis of a dynamic response in the ξ-variable: $\widetilde{y}(\xi) = (\mu\#\widetilde{u})(\xi)$, would be rather complex, due to the expression of the diffusive product $\#$ and the fact that generally, the input u is not a priori of the form $u = \mathcal{L}\theta$.

Corollary 3.8. *The following diffusive realizations are equivalent to* (3.6)*:*

$$\begin{cases} \partial_t \Psi = -4\pi^2\zeta^2\Psi + u, \quad \zeta \in \mathbf{R}, \quad \Psi(\zeta,0) = 0 \\ y := \displaystyle\int_{-\infty}^{+\infty} \widehat{m}.\Psi d\zeta, \end{cases} \tag{3.7}$$

$$\begin{cases} \partial_t \Phi = \partial_x^2\Phi + \delta \otimes u, \quad x \in \mathbf{R}, \quad \Phi(x,0) = 0 \\ y := \displaystyle\int_{-\infty}^{+\infty} m.\Phi dx. \end{cases} \tag{3.8}$$

Proof. From changes : $\xi = 4\pi^2\zeta^2$ and $m(\zeta) := 4\pi^2\zeta\mu(4\pi^2\zeta^2)$, (3.7)$\Leftrightarrow$(3.6). From Fourier transform with respect to the ζ-variable: $\Psi = \mathcal{F}\Phi$, it is easily shown that (3.8)$\Leftrightarrow$(3.7). ∎

The realization (3.8) is well-fitted to physical interpretations and exhibits specific properties of the heat equation (namely the maximum principle), which may be useful for analysis (see for example [3]). The realization (3.7) presents the simplicity of (3.6) and is linked to (3.8) from an *isometric* L^2-transformation (Fourier-Plancherel). The use of one or another is suggested by the specificities of the problem under consideration. Note the fundamental property that *any* diffusive operator is obtained from the *same* state equation: $\partial_t\psi(\xi,t) = -\xi\psi(\xi,t) + u(t)$, $\xi \in \mathbf{R}^+$, by conveniently choosing the output operator (the distribution μ) only, which allows to tackle optimization problems with respect to μ [4], [7], [17].

3.2. Some examples of DR

In Table 1, various examples of DR of PDO are given[4,5]. They are computed from standard functions and use of (3.3). All these operators are strictly proper.

Remark 3.9.
- $\operatorname{sing\,supp}(\frac{e^{-a^2/4t}}{\sqrt{\pi t}}) = \emptyset$: the corresponding PDO is of order $-\infty$; under heat equation form (3.8), the diffusive realization is given by: $y(t) = \int \delta_a(x)\Phi(x,t)dx = \Phi(a,t)$; such operator is used in [10] in the closed-loop stabilization of the heat equation.
- the operator with symbol $(p^\alpha + a)^{-1}$ is associated to the fractional differential equation: $\frac{d^\alpha}{dt^\alpha}X = -aX + u$, which may thus be rewritten: $X = (p^\alpha + a)^{-1}u$, and obtained as the solution of: $\partial_t\psi = -\xi\psi + u$, $X := \int \mu(\xi)\psi(\xi,t)\,d\xi$, with $\mu > 0$; in this expression, X appears as an *output* of a linear constant dynamic system with input u.

[4] $\mathrm{Ei}(t) = \int_t^{+\infty} \frac{e^{-u}}{u}\,du$. $\mathcal{E}_\alpha(\lambda,t) = \Sigma_{k=0}^{+\infty}\lambda^k\frac{t^{(1+k)\alpha-1}}{\Gamma(\alpha k+\alpha)}$ (as defined in [12]).

[5] We define the finite part in the sense of distributions [20].

restrictions	**symbol** $(p = \frac{d}{dt})$	**diffusive rep.** (ξ)	*impulse resp.* (t)
$\mathcal{L}\mu \in L^1_{\text{loc}}$	$H = \mathcal{L}h$	$\mu = \mathcal{L}^{-1}h$	h
$\mathcal{L}(\xi\mu) \in L^1_{\text{loc}}$	$pH(p)$	$-\xi\mu$	$\frac{d}{dt}h$
$0 < \Re(\alpha) < 1$	$p^{-\alpha}$	$\mu_\alpha(\xi) := \frac{\sin(\pi\alpha)}{\pi}\frac{1}{\xi^\alpha}$	$\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$
$\left.\begin{matrix} 0 < \Re(\alpha) < 1 \\ n \in \mathbf{N} \end{matrix}\right\}$	$p^{-(\alpha+n)}$	$(-1)^n \frac{\sin(\pi\alpha)}{\pi}\text{fp}\frac{1}{\xi^{\alpha+n}}$	$\frac{1}{\Gamma(\alpha+n)}t^{\alpha+n-1}$
$0 < \Re(\alpha) < 1$	$p^{-\alpha}H(p)$	$\mu_\alpha \# \mu$	$I^\alpha h$
$a > 0$	$H(p+a)$	$\mu(\xi - a)$	$e^{-at}h(t)$
$-$	p^{-1}	δ	1
$-$	$p^{-1}H(p)$	$\delta \# \mu$	$\int_0^t h(\tau)d\tau$
$a, b \geq 0$	$(p+a)^{-1}(p+b)^{-1}$	$\frac{1}{b-a}\delta_a + \frac{1}{a-b}\delta_b$	$e^{-at} \star e^{-bt}$
$a \geq 0,\, n \in \mathbf{N}$	$(p+a)^{-(n+1)}$	$\delta_a^{\#(n+1)} = \delta_a^{(n)}$	$\frac{t^n}{n!}e^{-at}$
$\left.\begin{matrix} 0 < \Re(\alpha) < 1, \\ a > 0 \\ k = \int \text{pv}\frac{\mu_\alpha\, d\xi}{a-\xi} \end{matrix}\right\}$	$p^{-\alpha}(p+a)^{-1}$	$\mu_\alpha(\xi)\text{pv}\frac{1}{(a-\xi)} + k\delta_a$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \star e^{-at}$
$\left.\begin{matrix} 0 < \Re(\alpha) < 1 \\ a > 0, \end{matrix}\right\}$	$p^{1-\alpha}(p+a)^{-1}$	$-\mu_\alpha(\xi)\text{pv}\frac{\xi}{(a-\xi)} - ka\delta_a$	$\frac{d}{dt}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \star e^{-at}\right)$
	$-p^{-1}(\gamma + \ln(p))$	$-\text{fp}\frac{1}{\xi}$	$\ln(t)$
$\left.\begin{matrix} 0 < \Re(\alpha) < 1 \\ a > 0, \end{matrix}\right\}$	$(p^\alpha + a)^{-1}$	$\frac{\sin(\pi\alpha)\xi^\alpha/\pi}{\xi^{2\alpha}+2a\cos(\alpha\pi)\xi^\alpha+a^2}$	$\mathcal{E}_\alpha(-a, t)$
$a > 0$	$e^{ap}\,\text{Ei}(ap)$	$e^{-a\xi}$	$\frac{1}{t+a}$
$a > 0$	$\frac{e^{-a\sqrt{p}}}{\sqrt{p}}$	$\frac{\cos(a\sqrt{\xi})}{\pi\sqrt{\xi}}$	$\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$

Table 1

4. Mathematical analysis

The previously introduced notions were only formal. This is sufficient in many situations, but from a general point of view, it is necessary to define the associated mathematical framework correctly. It is not unique and must be chosen according to the problem under consideration. For example, convenient Hilbert spaces will be searched for approximations are to be performed. In opposite, the largest possible class (of distributions μ) will be considered if we want to extend the field of interest of the DR, with the aim of unification. An interesting question is the determination of a sufficiently large space of diffusive operators, such that the diffusive product be internal. Natural extensions of this question concern the existence of a unity and inversibility. The latter two questions need convenient extensions of the diffusive representations (see section 5).

In this paper, we only give some partial answers. A more detailed analysis will be presented in a forthcoming work. In this section, the operators under consideration are *constant.*

4.1. The topological framework

Let $\mathcal{S}'_+ \subset \mathcal{S}'(\mathbf{R})$ be the space of *causal* tempered distributions [20] on $\xi \in \mathbf{R}$.

Definition 4.1. The set of **regular** diffusive representations is defined by:

$$\Delta_0 = \mathcal{S}(\mathbf{R})/\ker(\mathcal{L}). \tag{4.1}$$

The Laplace transform is then defined on Δ_0 and injective. Obviously, $\mathcal{L}(\Delta_0)\mid_{t\in\mathbf{R}^+} \subset L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$.

THEOREM 4.2. *There exists a norm $\|.\|$ such that $(\Delta_0, \|.\|)$ is a separable normed space.*

Proof. Due to the *analyticity*[6] of $\mathcal{L}\mu$, for $0 < T < +\infty$, the seminorm on $L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$ defined by $q = \|.\|_{L^1(0,T)}$, is a norm on $\mathcal{L}(\Delta_0)\mid_{t\in\mathbf{R}^+}$(due to the uniqueness of the analytic extension). From $\mathcal{L}^{-1}$, this norm induces a norm on Δ_0 and Δ_0 is isometric to a subspace of $L^1(0,T)$ which is separable. ∎

Note that, for the q-topology, $\mathcal{L}(\Delta_0)$ is a Hausdorff (due to the analyticity of $\mathcal{L}\mu$) subspace of $L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$. Furthermore, the product $\#$ is obviously internal in Δ_0. For the q-typology, the convolution $\star : L^1_{\mathrm{loc}}(\mathbf{R}^+_t) \times L^1_{\mathrm{loc}}(\mathbf{R}^+_t) \to L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$ is continuous, with: $q(h \star k) \le q(h).q(k)$; so is it for the diffusive product $\#$ and so, $(\Delta_0, +, ., \#)$ is a (commutative) normed algebra (isometric to the sub-algebra $\mathcal{L}(\Delta_0) \subset L^1_{\mathrm{loc}}(\mathbf{R}^+_t))$.

DEFINITION 4.3. We denote $\overline{\Delta} := \overline{\Delta}_0$, the completion[7] of Δ_0, and $\mathcal{L}\left(\overline{\Delta}\right) := \overline{\mathcal{L}(\Delta_0)}^q$.

So, $\overline{\Delta}$ is a (real or complex) normed and complete algebra, isometric to $\mathcal{L}\left(\overline{\Delta}\right) \subset \left(L^1_{\mathrm{loc}}(\mathbf{R}^+_t), q\right)$. However, the unity for $\#$ does not exist in $\overline{\Delta}$ because $\delta \notin L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$. Note that the extension of $L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$ to the Radon measures would not be a convenient frame, because, for instance, $\delta \notin \mathcal{L}(\mathcal{S}')$. The construction of extended frames of DR with unity (Banach algebras) is presented in section 5.

In practice, the initial topology on Δ_0 induced by q is not useful (namely for approximations) because it is implicit. Furthermore, it is probable that $\overline{\Delta} \not\subset \mathcal{S}'_+$: the frontier of Δ_0 is not suitable for the representations (3.1), (3.6). We consider the canonical injection $\Delta_0 \subset \mathcal{S}'_+$ and given $\Omega \subset \mathbf{R}^n$, we denote $H^k(\Omega) \supset \mathcal{S}(\Omega)$, $k \ge 0$, the usual Sobolev space defined on Ω [1], with topological dual $H^k(\Omega)'$. A sufficient condition for $\mu \in \overline{\Delta}$ is given by:

THEOREM 4.4. *If $\mu \in \mathcal{S}'_+$ and $\exists X > 0$, such that $\frac{\mu(\xi)}{\xi} \in L^1(X, +\infty)$, then $\mu \in \overline{\Delta}$.*

Proof. There exists $k > 0$ such that μ is continuous on $\mathcal{S}(\mathbf{R})$ for the norm of $H^k(0,X)$. Identifying $L^2(0,X)$ with its topological dual, we consider the duality triplet $H^k(0,X) \subset L^2(0,X) \subset H^k(0,X)'$. We first prove that $\mathcal{L}\mu \in L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$, i.e. $\int_0^T |\mathcal{L}\mu|\, dt < +\infty$[8]:

$$\begin{aligned} \int_0^T |\mathcal{L}\mu|\, dt &= \int_0^T \left| < \mu, e^{-\xi t} >_{\mathcal{S}',\mathcal{S}} \right| dt \\ \le \int_0^T \left| \int_0^X \mu(\xi) e^{-\xi t}\, d\xi \right| dt &+ \int_0^T \left| \int_X^{+\infty} \mu(\xi) e^{-\xi t}\, d\xi \right| dt \\ &:= \int_0^T A(t) dt + \int_0^T B(t) dt; \end{aligned}$$

without loss of generality, we may suppose that μ is not singular at $\xi = X$. The map: $t \in [0,T] \to e^{-\xi t} \in H^k(0,X)$ is continuous, then

$$t \to A(t) = < \mu, e^{-\xi t} >_{H^k(0,X)', H^k(0,X)}$$

[6] This property is fundamental from the point of view of the topological framework for diffusive representations.

[7] The distribution μ is causal: rigorously speaking, $\mu \in \overline{\Delta} \Leftrightarrow \mu \in \widetilde{\mu} \in \overline{\Delta}_0$, $\operatorname{supp}\mu \subset \mathbf{R}^+$.

[8] Rigorously speaking: $e^{-t\xi} = \varphi(\xi, t)$, $\xi \ge 0$, $\varphi(., t) \in \mathcal{S}(\mathbf{R})$.

is continuous on $[0,T]$ and $\int_0^T A\,dt$ is finite. Then,

$$\int_0^T B\,dt \le \int_X^{+\infty} |\mu(\xi)| \int_0^T e^{-\xi t}\,dt\,d\xi = \int_X^{+\infty} \left|\frac{\mu(\xi)}{\xi}\right| \left(1-e^{-\xi T}\right)\,dt\,d\xi < +\infty.$$

Secondly, we consider a sequence $\mu_n \in \Delta_0$, $\mu_n \to \mu$ in $H^k(0,X)'$, and $\frac{\mu_n}{\xi} \to \frac{\mu}{\xi}$ in $L^1(X,+\infty)$. This is possible because Δ_0 is dense in $H^k(0,X)'$ and in $L^1(X,+\infty)$. We have:

$$\begin{aligned}\int_0^T \left| \int_0^X (\mu_n(\xi)-\mu(\xi))\,e^{-\xi t}\,d\xi \right| dt &= \int_0^T \left| < \mu_n-\mu, e^{-\xi t} >_{H^{k\prime},H^k} \right| dt \\ &\le \|\mu_n-\mu\|_{H^{k\prime}} \int_0^T \left\|e^{-\xi t}\right\|_{H^k}\,dt \to 0.\end{aligned} \tag{4.2}$$

Furthermore,

$$\begin{aligned}\int_0^T \left| \int_X^{+\infty} (\mu_n(\xi)-\mu(\xi))\,e^{-\xi t}\,d\xi \right| dt &\le \int_0^T \int_X^{+\infty} |(\mu_n(\xi)-\mu(\xi))|\,e^{-\xi t}\,d\xi\,dt \\ = \int_X^{+\infty} |(\mu_n(\xi)-\mu(\xi))| \frac{(1-e^{-\xi t})}{\xi}\,d\xi &\le \left\| \frac{\mu_n(\xi)}{\xi} - \frac{\mu(\xi)}{\xi} \right\|_{L^1(X,+\infty)} \to 0,\end{aligned}$$

which implies with (4.2), that $\|\mathcal{L}\mu_n - \mathcal{L}\mu\|_{L^1(0,T)} \to 0$ and so, $\mu \in \overline{\Delta}$. ∎

DEFINITION 4.5. We call:

$$\Delta := \left\{ \mu \in \overline{\Delta} \cap \mathcal{S}'_+,\ \exists X > 0,\ \frac{\mu(\xi)}{\xi} \in L^1(X,+\infty) \right\} \tag{4.3}$$

the **reference** space of diffusive representations.

This space is normed, but not complete, and the product # is probably not internal. Note that the two conditions expressed in (4.3) in fact express respectively the high and low frequency behaviors of the operator defined by $(\mathcal{L}.\mathcal{L}.\mu)(\frac{d}{dt})$. Such conditions are suitable for analysis and numerical approximations. Concrete separable Hilbert spaces included in Δ may be introduced (for example Sobolev spaces of tempered distributions). Similarly, convenient subspaces of $L^1_{\text{loc}}(\mathbf{R}_t^+)$ with Hilbert structure are to be considered. This will be examined in section 4.3.

DEFINITION 4.6. Given $\mathcal{M} \subset \Delta$, we denote $\mathcal{O}(\mathcal{M})$ the space of (symbols of) diffusive operators generated by $\mathcal{M}$:

$$\mathcal{O}(\mathcal{M}) := \mathcal{L}\left(\mathcal{L}(\mathcal{M})|_{\mathbf{R}_t^+}\right); \tag{4.4}$$

with D_μ defined by (3.6), we denote:

$$\mathrm{DR}(\mathcal{M}) = \{\mathrm{D}_\mu,\ \mu \in \mathcal{M}\}. \tag{4.5}$$

With respect to the induced operations and topologies, the spaces $\mathcal{O}(\mathcal{M})$, $\mathcal{L}(\mathcal{M})$ and $\mathrm{DR}(\mathcal{M})$ (of operators, impulses responses and diffusive realizations respectively) are isomorphic to $(\mathcal{M},+,.,\#)$ (the space of diffusive representations).

REMARK 4.7. The diffusive representations and realizations can be extended to the vector frame, by considering, as usual: $\mathcal{U},\mathcal{Y}$ two Banach spaces, and H opering in $C^\infty(\mathbf{R}_t^+;\mathcal{U})$ with value in $C^\infty(\mathbf{R}_t^+;\mathcal{Y})$ (see [17]). Example: $\mathcal{U}=\mathbf{R}^n$, $\mathcal{Y}=\mathbf{R}^m$, $\mu \in (\mathcal{S}'_+)^{m\times n}$, $\psi(\xi,t) \in \mathbf{R}^n$.

4.2. REGULARITY RESULTS

The proof of the following results is easily obtained, from classical analysis. We suppose $\mu \in \Delta$ and we denote: $h = \mathcal{L}\mu =< \mu, e^{-\xi t} >_{\mathcal{S}',\mathcal{S}}$, $H = \mathcal{L}h$.

THEOREM 4.8. 1. $h \in C^\infty(]0,+\infty[)$.

2. *$H(i\omega) \underset{|\omega|\to+\infty}{\to} 0$ and equivalently, $\int_0^t h(\tau)d\tau \in C^0([0,+\infty[)$.*
3. *If $\mu \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$, then $i\omega H(i\omega) \underset{|\omega|\to 0}{\to} 0$ and equivalently, $h(t) \underset{t\to+\infty}{\to} 0$.*
4. *$H(p)$ has no pole with positive real part.*

Proof. 1. For any $t>0$, $e^{-\xi t} \in \mathcal{S}/\ker(\mathcal{L})$, then $\frac{d^n}{dt^n}h = <\mu, (-1)^n \frac{\partial^n}{\partial t^n} e^{-\xi t}>$.

2. $H(i\omega) = \int_0^X \frac{\mu(\xi)}{i\omega+\xi}\,d\xi + \int_X^{+\infty} \frac{\mu(\xi)}{i\omega+\xi}\,d\xi = A(\omega) + B(\omega)$.
   $A(\omega) = <\mu,\varphi>$ with $\operatorname{supp}\varphi \supset [0,X]$ and $\varphi\,|_{[0,X]} = \frac{1}{i\omega+\xi} \Rightarrow \varphi \underset{|\omega|\to+\infty}{\to} 0$ in $\mathcal{S}$; on $[X,+\infty[$, $\left|\frac{\mu(\xi)}{i\omega+\xi}\right| \le \left|\frac{\mu(\xi)}{\xi}\right| \in L^1(X,+\infty)$ and we conclude from Lebesgue dominated convergence theorem.
3. First, from Lebesgue theorem, $\lim_{t\to+\infty} \left|\int_0^X \mu(\xi)e^{-\xi t}\,d\xi\right| = 0$.
   secondly, $\int_X^{+\infty} \mu(\xi)e^{-\xi t}\,d\xi = <\mu,\varphi>$ with $\operatorname{supp}\varphi \supset [X,+\infty[$ and $\varphi\,|_{[X,+\infty[} = e^{-\xi t} \Rightarrow \varphi \underset{t\to+\infty}{\to} 0$ in $\mathcal{S}$.
4. It is sufficient to prove that $h(t)$ is tempered. First, $\int_X^{+\infty} \mu(\xi)e^{-\xi t}\,d\xi \underset{t\to+\infty}{\to} 0$.
   secondly, $\left|\int_0^X \mu(\xi)e^{-\xi t}\,d\xi\right| = \left|<\mu, e^{-\xi t}>_{H^k(0,X)',H^k(0,X)}\right| \le \|\mu\|_{H^{k\prime}} \cdot \left\|e^{-\xi t}\right\|_{H^k}$
   $\le \|\mu\|_{H^{k\prime}} .P(t) \int_0^{+\infty} e^{-x}\,dx$, with P a polynomial of order $k-1$.

∎

Corollary 4.9. *The operator with symbol $H(p)$ is pseudo-differential.*

Proof. This operator is constant, hence, from theorem 4.8-1, $\operatorname{sing\,supp} h \subset \{0\}$; hence:

$$\begin{aligned}\operatorname{sing\,supp} Hu = \operatorname{sing\,supp} h \star u &\subset \operatorname{sing\,supp} h + \operatorname{sing\,supp} u \\ &\subset \operatorname{sing\,supp} u + \{0\} = \operatorname{sing\,supp} u.\end{aligned}$$

∎

4.3. Optimal diffusive representations

We consider here $\mathcal{M} \subset \Delta$ a separable Hilbert space (of distributions), and $\mathcal{H} \subset L^1_{\mathrm{loc}}(\mathbf{R}^+_t)$ a Hilbert space of functions defined on $t \in [0,T]$. Given $h \in \mathcal{H}$, a fundamental problem is to find $\mu \in \mathcal{M}$ such that $\mathcal{L}\mu$ is the closest possible to the impulse response h. In this context, the following result is obvious (due to the injectivity of $\mathcal{L}$):

Theorem 4.10. *If the linear map $\mathcal{L} : \mathcal{M} \to \mathcal{H}$ is continuous with range closed in $\mathcal{H}$, then the problem:*

$$\min_{\mu\in\mathcal{M}} \|\mathcal{L}\mu - h\|^2_{\mathcal{H}} \tag{4.6}$$

has a unique solution: $\mu^* = \mathcal{L}^\dagger h$, $h^* = \mathcal{L}\mu^*$, with $\mathcal{L}^\dagger$ the pseudo-inverse[9] of $\mathcal{L}$.

This result is useful to compute diffusive representations concretely, from a given impulse response (namely by means of convergent numerical approximations). From (3.2), it may be equivalently extended to the frequency representation $H(i\omega)$, $\omega \in \mathbf{R}$ (via Fourier transform on Sobolev spaces $\mathcal{H}$). Of course, the solution will generally be "bad" (i.e. $\|\mathcal{L}\mu - h\|^2_{\mathcal{H}} \gg 0$) if $\mathcal{M}$ and $\mathcal{H}$ are not well-fitted. Note

[9] defined by: $\mathcal{L}^\dagger = (\mathcal{L}^*\mathcal{L})^{-1}\mathcal{L}^*$.

that the choice of $\mathcal{M}$ and $\mathcal{H}$ is concretely determined from additional informations on the problem.

The continuity of the product $\star$: $\|h \star k\|_{\mathcal{H}} \leq k\,\|h\|_{\mathcal{H}} \cdot \|k\|_{\mathcal{H}}$ is a too strong property, in regard to the classical Sobolev spaces (this product is generally not internal on $\mathcal{H}$). So is it about the product $\#$. Nevertheless, we have the following results, which allows to tackle approximation problems efficiently (by combining the pseudo-inversion and the products):

THEOREM 4.11. *If the canonical injection: $\mathcal{H} \subset \left(L^1_{\mathrm{loc}}(\mathbf{R}^+_t), q\right)$ is continuous, then:*

$$\left.\begin{array}{r} h_n \to h \ \text{ in } \mathcal{H} \\ k_n \to k \ \text{ in } \mathcal{H} \\ \|h_n \star k_n\|_{\mathcal{H}} \ \text{ bounded} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} h \star k \in \mathcal{H} \\ h_n \star k_n \to h \star k \ \text{ in } (\Delta, q) \\ h_n \star k_n \rightharpoonup h \star k \ \text{ weakly in } \mathcal{H}. \end{array}\right. \tag{4.7}$$

Proof. From extraction of weakly convergent subsequences in $\mathcal{H}$ and uniqueness of the limit. ∎

COROLLARY 4.12. *Under the hypothesis of theorem 4.10 and theorem 4.11 and if the injection $\mathcal{M} \subset \Delta$ is continuous:*

$$\left\{\begin{array}{l} \mu_n^* \# v_n^* \to \mu^* \# \nu^* \ \text{ in } (\Delta, q) \\ \mathcal{L}^\dagger(h_n \star k_n) \rightharpoonup \mathcal{L}^\dagger(h \star k) \ \text{ weakly in } \mathcal{H}; \end{array}\right. \tag{4.8}$$

with the additional hypothesis: $\mathcal{L}\mathcal{L}^\dagger(h \star k) \in \mathcal{L}\mathcal{L}^\dagger(\mathcal{H})$, we have:

$$\left\{\begin{array}{l} (h_n \star k_n)^* - h_n^* \star k_n^* \to 0 \ \text{ in } (\Delta, q) \\ (h_n \star k_n)^* - h_n^* \star k_n^* \rightharpoonup 0 \ \text{ weakly in } \mathcal{H} \\ \mathcal{L}^\dagger(h_n \star k_n) - \mathcal{L}^\dagger(h_n) \# \mathcal{L}^\dagger(k_n) \to 0 \ \text{ in } (\Delta, q) \\ \mathcal{L}^\dagger(h_n \star k_n) - \mathcal{L}^\dagger(h_n) \# \mathcal{L}^\dagger(k_n) \rightharpoonup 0 \ \text{ weakly in } \mathcal{H}. \end{array}\right. \tag{4.9}$$

Proof. Similar to theorem 4.11, with continuity of $\mathcal{L}^\dagger$ and $\mathcal{L}\mathcal{L}^\dagger$. ∎

REMARK 4.13. All these results are equivalently expressed in the frequency domain, from Fourier transform on Sobolev spaces.

4.4. FINITE-DIMENSIONAL REALIZATIONS

We only give some indications. Precise examples may be found in references. Given a diffusive PDO with symbol $H \in \mathcal{O}(\mathcal{M})$, we consider its diffusive representation $\mu \in \mathcal{M} \subset \Delta$, and the diffusive realization $\mathrm{D}_\mu \in \mathrm{DR}(\mathcal{M})$, defined by: $y = \int \mu \psi \, d\xi$, with ψ solution of (3.6).

4.4.1. APPROXIMATION OF THE DIFFUSIVE REPRESENTATION. Given $\mathcal{M}_n$ a sequence of n-dimensional subspaces of $\mathcal{M}$, with $\overline{\bigcup_n \mathcal{M}_n} = \mathcal{M}$, we may consider an approximation $\mu_n \in \mathcal{M}_n$ of μ in the sense: $\mu_n \overset{\mathcal{M}}{\to} \mu$. We then obtain a natural approximation of H from $H_n = \mathcal{L}\mathcal{L}\mu_n$. As an example, we consider $\mathcal{M}_n$ the space of atomic measures with support in $\mathcal{N}_n = \{\xi_1, \xi_2, ..., \xi_n\}$, such that the network $\mathcal{N}_n$ "fills up" $\mathbf{R}^+_\xi$ when $n \to +\infty$: $\overline{\bigcup_n \mathcal{N}_n} = \mathbf{R}^+$. Obviously, $\bigcup_n \mathcal{M}_n$ is dense in the Sobolev space $H^1(\mathbf{R}^+)'$ and the associated diffusive representations and realizations are given by:

$$\mu_n(\xi) = \sum_{i=1}^{n} \mu(\xi_i) \delta_{\xi_i}(\xi) \tag{4.10}$$

$$\mathrm{D}_{\mu_n} : \; y_n = \sum_{i=1}^{n} \mu(\xi_i) \psi(\xi_i, t), \tag{4.11}$$

so that the diffusive realization becomes finite-dimensional (only a finite number of $\psi(\xi, t)$ are necessary).

Nevertheless, the property of finite dimension of D_{μ_n} is ruined if μ_n is not atomic. Note also that in general, the diffusive product is defined nowhere in $\mathcal{M}_n$ (namely in the atomic case, due to the property: $\delta_a \# \delta_a = \delta'_a$).

4.4.2. Approximation of the diffusive realization. finite-dimensional realizations may advantageously be constructed from approximations of the function ψ. Given $\mathcal{K} \ni \psi(.,t)$ a Hilbert state space, and $\mathcal{K}_n$ a sequence of n-dimensional subspaces of $\mathcal{K}$, with $\overline{\bigcup_n \mathcal{K}_n} = \mathcal{K}$, many approximation methods of D_μ may be considered. A simple and efficient one is the following.

Given $\mathcal{N}_n$ defined as above, we consider the finite-dimensional functional approximation of ψ defined by:

$$\widetilde{\psi}_n(\xi,t) = \sum_{i=1}^{n} \psi(\xi_i,t)\Lambda_i(\xi), \tag{4.12}$$

with Λ_i a finite element basis of $\mathcal{K}$. Given u sufficiently regular, we have:

$$\left\| \widetilde{\psi}_n - \psi \right\|_{\mathcal{K}} \underset{n \to +\infty}{\to} 0;$$

then, under weak hypothesis and with respect to convenient topologies, we obtain a convergent approximation $\widetilde{y}_n$ of y as:

$$\begin{aligned} \widetilde{y}_n(t) &= \sum_{i=1}^{n} \lambda_i \psi(\xi_i,t), \\ \lambda_i &= \int_0^{+\infty} \mu(\xi)\Lambda_i(\xi)\, d\xi, \end{aligned} \tag{4.13}$$

which defines an *approximated diffusive realization* of dimension n, under the form:

$$\widetilde{\mathrm{D}}^n_\mu \equiv \left\{ \begin{array}{l} \frac{d}{dt}\psi_i = -\xi_i \psi_i + u, \ \psi_i(0) = 0, \ i = 1, n \\ \widetilde{y}_n(t) := \sum_{i=1}^{n} \lambda_i \psi_i(t). \end{array} \right. \tag{4.14}$$

Concrete examples of approximated fractional operators may be found in ([8], [15], [16], etc.).

Remark 4.14. In practice, small n (about 10-30) are sufficient for good approximations, namely when specific Λ_i are chosen. This latter point is essential, namely for long memory operators whose standard approximations based on the convolutive expression generally lead to algorithms of great complexity [11].

5. Extended diffusive representations

The unity for # is associated to the identity operator (of order 0). Furthermore, it may be of great interest to consider pseudo-differential operators of order $\sigma \in]0,1[$ (see for example [16]). Such orders are not accessible from the basic diffusive representation space Δ, but may easily be obtained from convenient extensions as introduced in sections 5.1, 5.2.

5.1. Diffusive representation with unity

The most simple extension of DR consists in completing $\mathcal{M}$ with the direction of unity denoted: ϵ, associated to the impulse response $\delta(t)$ (with symbol 1):

$$\mathcal{M}_e := \mathcal{M} \oplus \{\lambda\epsilon\}_{\lambda \in \mathbf{C}}. \tag{5.1}$$

The associated class of PDO and diffusive realizations are then defined in a natural way, respectively:

$$\mathcal{O}(\mathcal{M}_e) = \mathcal{O}(\mathcal{M}) \oplus \{\lambda.1\}_{\lambda \in \mathbf{C}},$$

$$\mathrm{D}_{\mu+\lambda\epsilon} \equiv \begin{cases} \partial_t \psi = -\xi\psi + u \\ y(t) := \displaystyle\int_0^{+\infty} \mu(\xi)\psi(\xi,t)\,d\xi + \lambda u(t). \end{cases} \tag{5.2}$$

The diffusive product is extended following the calculus rule:

$$(\mu + \lambda\epsilon)\#(\mu' + \lambda'\epsilon) = \mu\#\mu' + \lambda'\mu + \lambda\mu' + \lambda\lambda'\epsilon. \tag{5.3}$$

The completed space $(\overline{\Delta}_e, +, ., \#)$ is then a *Banach algebra* of operators. Note that for the diffusive product, the set of fractional operators of non positive order (fractional *integrators*) has the structure of semigroup with one (complex) parameter, isomorphic to $(\mathbf{R}^- + i\mathbf{R}, +)$.

5.2. Extension by derivation

Given a DR $\mu \in \mathcal{M}$, recall that the diffusive realization D_μ is defined by: $\partial_t\psi = -\xi\psi + u$, $y(t) := \int_0^{+\infty} \mu\psi\,d\xi$.

Definition 5.1. The **extended** diffusive realization is defined, for any $\mu \in \Delta$, by:

$$\mathrm{ED}_\mu \equiv \begin{cases} \partial_t \psi = -\xi\psi + u,\ \xi \in \mathbf{R}^+,\ \psi(\xi,0) = 0 \\ y(t) := \frac{d}{dt}\displaystyle\int_0^{+\infty} \mu(\xi)\psi(\xi,t)\,d\xi = \int_0^{+\infty} \left(-\xi\mu(\xi)\psi(\xi,t) + \mu(\xi)u(t)\right)\,d\xi. \end{cases} \tag{5.4}$$

We have the following:

Theorem 5.2. *For any* $\mathrm{D}_\mu \in \mathrm{DR}(\Delta)$, $\mathrm{D}_\mu = \mathrm{ED}_{\delta\#\mu}$.

Proof. From Table 1, $\delta\#\mu$ is the DR of the operator with symbol: $(\frac{d}{dt})^{-1}H(\frac{d}{dt}) \in \mathcal{O}(\Delta)$; then, $\int_0^{+\infty} \mu\psi\,d\xi = \frac{d}{dt}\int_0^{+\infty}(\delta\#\mu)\psi\,d\xi = \int_0^{+\infty}\left(-\xi(\delta\#\mu)\psi + (\delta\#\mu)u\right)\,d\xi$.

From Table 1, $-\xi(\delta\#\mu) = \mu$ for any regular μ, and necessarily: $\int_0^{+\infty} \delta\#\mu\,d\xi = 0$. The result is extended to any $\mu \in \Delta$ by continuity-density. ∎

Corollary 5.3. *Denoting:*

$$\mathrm{EDR}(\Delta) := \{\mathrm{ED}_\mu,\ \mu \in \Delta\} \ \textit{and}\ \mathrm{E}\mathcal{O}(\Delta) := \{pH(p),\ H(p) \in \mathcal{O}(\Delta)\},$$

the canonical injections: $\mathrm{DR}(\Delta) \subset \mathrm{EDR}(\Delta)$ *and* $\mathcal{O}(\Delta) \subset \mathrm{E}\mathcal{O}(\Delta)$ *are defined by the linear map:* $\mu \in \Delta \to \delta\#\mu \in \Delta$.

The map $\mu \to \delta\#\mu$ is of course not inversible (for example for $\alpha \in [0,1[$, $p^\alpha \in \mathrm{E}\mathcal{O}(\Delta)$, $p^\alpha \notin \mathcal{O}(\Delta)$). Note that the identity operator is in $\mathrm{E}\mathcal{O}(\Delta)$: it corresponds to the DR δ (from $\xi\delta(\xi) = 0$ and $\int \delta\,d\xi = 1$). We may deduce:

Corollary 5.4. $\mathcal{O}(\Delta_e) \subset \mathrm{E}\mathcal{O}(\Delta)$. *Furthermore,* $\mathrm{D}_{\mu+\lambda\epsilon} = \mathrm{ED}_{\delta\#\mu+\lambda\delta}$.

Proof. $\forall H(p) \in \mathcal{O}(\Delta_e)$, with diffusive realization $H(\frac{d}{dt}).u := \int_0^{+\infty} \mu\psi\,d\xi + \lambda u$, $p^{-1}H(p) \in \mathcal{O}(\Delta)$, with DR: $\delta\#\mu + \lambda\delta$. ∎

Hence, Δ may be equipped with a new product (not internal, with unity δ, induced by the map $\mu \to \delta\#\mu$):

Definition 5.5. Given $pH(p)$, $pQ(p) \in \mathrm{E}\mathcal{O}(\Delta)$, with DR respectively μ, ν (i.e. $\frac{d}{dt}H(\frac{d}{dt})u := \frac{d}{dt}\displaystyle\int_0^{+\infty} \mu\psi\,d\xi$, $\frac{d}{dt}Q(\frac{d}{dt})u := \frac{d}{dt}\displaystyle\int_0^{+\infty} \nu\psi\,d\xi$), the DR of $p^2H(p)Q(p)$ (if in $\mathrm{E}\mathcal{O}(\Delta)$) is denoted $\mu \bullet \nu$, with $p^2H(p)Q(p)u := \frac{d}{dt}\displaystyle\int_0^{+\infty} (\mu\bullet\nu)\psi\,d\xi$.

THEOREM 5.6. *The* **extended** *diffusive product* $\bullet$ *is given (when it is in* Δ*) by:* $\mu \bullet \nu = -\xi.(\mu \# \nu)$.

Proof. From Table 1, $p^2 H(p) Q(p) u = \left(\frac{d}{dt}\right)^2 \int (\mu \# \nu) \psi \, d\xi = \frac{d}{dt} \int -\xi (\mu \# \nu) \psi \, d\xi$. ∎

The extended diffusive realizations allow to express under the standard form: $\frac{d}{dt} X = AX + Bu, \ y := C(X, u)$, a wide variety of pseudo-differential operators, namely any fractional integrator of any order and any fractional derivator of order less than 1. From simple algebraic manipulations, it is possible to determine the associated diffusive representation μ. As an example, we may explicitly compute the diffusive realization of the extended diffusive operator $(p+a)^{1-\alpha}$, $0 < \alpha < 1$, as follows:

$$\begin{aligned} (p+a)^{1-\alpha} = (p+a)\,(p+a)^{-\alpha} = p \left(\tfrac{1}{(p+a)^\alpha} + \tfrac{a}{p(p+a)^\alpha} \right) \\ \Rightarrow y = \tfrac{d}{dt} \int (\mu + a\delta \# \mu) \psi \, d\xi, \end{aligned} \tag{5.5}$$

with $\mu(\xi) = \mu_\alpha(\xi - a)$, $\delta \# \mu(\xi) = k\delta(\xi) + \frac{\mu_\alpha(\xi - a)}{\xi}$, $k = \int_a^{+\infty} -\frac{\mu_\alpha(\zeta - a)}{\zeta} d\zeta$.

It may be noticed that operators of order $\sigma \in]0,1[$ are proper, in the sense that the extended diffusive realization is a well-posed equation without any derivative of u (for example the fractional derivators of order $\sigma \in]0,1[$). Such operators are used in [16] and [4] (this last work is based on an extension of CRONE control [19] into the general diffusive frame [17]). Note that when the order is strictly positive, $\mu \notin L^1(X, +\infty)$ for any $X > 0$ (but it may be proved that $(-\xi\mu\psi + \mu u) \in L^1(X, +\infty)$).

Finally, it may be conjectured that any extended diffusive operator of order $\sigma \in]-1,1[$ is invertible in $\mathrm{E}\mathcal{O}(\Delta)$, with diffusive representation μ^{-1}, such that $\mu \bullet \mu^{-1} = \delta$, which confers to $(\mathrm{E}\mathcal{O}(\Delta), \bullet)$ a structure of local group. This is obviously the case for fractional operators.

6. APPLICATION TO PSEUDO-DIFFERENTIAL EQUATIONS

In this section, we consider possibly time-varying distributions $\mu(t, \xi)$, which allows to represent *non constant* PDO[10]. Details will be given in a forthcoming wortk.

Given a diffusive pseudo-differential operator $\mathcal{D}^{-1}$ with diffusive representation $\mu(t, \xi)$, we consider the functional equation:

$$\mathcal{D} X = f(X, u, t). \tag{6.1}$$

Equation (6.1) is obviously rewritten as the following semilinear diffusive equation (Cauchy problem):

$$\left\{ \begin{array}{l} \partial_t \psi = -\xi \psi + f\left(\displaystyle\int \mu \psi \, d\xi, u, t \right), \ \psi(\xi, 0) = 0 \\ X := \displaystyle\int \mu \psi \, d\xi. \end{array} \right. \tag{6.2}$$

We may note that in (6.2), X appears as an output. This simple example highlights the efficiency of the diffusive representations in the field of pseudo-differential equations: they allow to transform non standard models involving diffusive operators (such as fractional) into ordinary differential systems (of infinite dimension) for which classical powerful tools may be used: Lyapunov and Galerkin methods, LaSalle's invariance principle, finite-element approximations, etc. Various examples of fractional differential equations have been successfully treated following this approach (see [3], [18], [9], [16], [13]).

[10]For example, $\Sigma_i \, a_i(t)(d/dt)^{-\alpha_i}$ is associated to the diffusive representation: $\mu(t, \xi) = \Sigma_i \, a_i(t) \mu_{\alpha_i}(\xi)$.

Vector expressions are perfectly compatible with (6.1),(6.2), which makes this approach quite generic. As an example, with A a positive self-adjoint operator on a Hilbert space E, we consider the problem in E : $\mathcal{D}\frac{d}{dt}X = -AX + v \Leftrightarrow \frac{d}{dt}X = -A\mathcal{D}^{-1}X + u,\ X(0) = X_0 \in E$, or equivalently in the state space $E \times E$:

$$\left\{ \begin{array}{l} \mathcal{D}Y = X \\ \frac{d}{dt}X = -AY + Bu,\ X(0) = X_0; \end{array} \right. \tag{6.3}$$

by use of the diffusive realization of $\mathcal{D}^{-1}$, this problem is equivalently transformed into the "diffusive form" in $L^2(\mathbf{R}^+_\xi; E) \times E$:

$$\left\{ \begin{array}{l} \partial_t \psi = -\xi\psi + \sqrt{A}X,\ \psi(\xi,0) = 0 \\ \frac{d}{dt}X = -\sqrt{A}\displaystyle\int \mu(t,\xi)\psi\, d\xi + Bu,\ X(0) = X_0 \end{array} \right. \tag{6.4}$$

REMARK 6.1. Here, $\mu(t,\xi)$ is a linear operator on E.

Under the hypothesis $\mu(t,\xi) \geq 0$, and self-adjoint, we may introduce the "global energy" functional:

$$\Xi(X,\psi) := \frac{1}{2}\left(\|X\|_E^2 + \int \left\| \sqrt{\mu(t,\xi)}\psi \right\|_E^2 d\xi \right). \tag{6.5}$$

THEOREM 6.2. *If $\partial_t\mu(t,\xi) \geq 0$, then the system (6.4) is dissipative in $\mathcal{E}$.*

Proof. If $u(t) = 0$:

$$\begin{aligned} \frac{d\Xi}{dt} &= -(X \mid \sqrt{A}\int \mu\psi\, d\xi)_E + \int (\mu\psi \mid \sqrt{A}X)_E\, d\xi \\ &\quad -\tfrac{1}{2}\int ((\partial_t\mu)\psi \mid \psi)_E\, d\xi - \int \left\|\sqrt{\xi}\psi\right\|_E^2 d\xi \\ &= -\tfrac{1}{2}\int \left\|\sqrt{\partial_t\mu}\psi\right\|_E^2 d\xi - \int \left\|\sqrt{\xi}\psi\right\|_E^2 d\xi \leq 0. \end{aligned}$$

∎

Under some convenient additional hypothesis, we may deduce existence and uniqueness of a weak solution in the induced energy space, and, from LaSalle invariance principle, the asymptotic stability for the autonomous system:

$$\Xi(X,\psi) \underset{t\to+\infty}{\rightarrow} 0 \Rightarrow X \underset{t\to+\infty}{\overset{E}{\rightarrow}} 0. \tag{6.6}$$

Note that such results would be very difficult to obtain under the initial pseudo-differential or convolutive form. In fact, $\Xi(X,\psi)$ appears as a Lyapunov functional for (6.4); in opposite, Lyapunov candidates for (6.3) are generally based on the positiveness of $\int_0^t u.(h \star u)d\tau$, which is much less accurate[11]. Furthermore, (6.4) is also well-fitted for numerical approximations and simulations as well as control problems, due to the Hilbert underlying structures and the highly dissipative nature of diffusion equations.

Examples:

- $\alpha \in]0,2[,\ \frac{d^\alpha}{dt^\alpha}x = \lambda x + v(t)$ is stable if $\lambda \leq 0$.
- $\alpha \in]0,1[,\ \partial_t^{1+\alpha}\theta(x,t) = \Delta\theta(x,t) + u(x,t)$ is dissipative (in the sense of the global energy (6.5)) and $\|\theta(x,t)\|^2_{L^2(\Omega)} \underset{t\to+\infty}{\rightarrow} 0$ if $u(t) = 0,\ t \geq T$. Such equation, intermediate between heat and wave equations, has been studied namely in [6]). Note that $\psi = \psi(x,\xi,t),\ x \in \Omega \subset \mathbf{R}^n$.

[11] This appears clearly when studying this property from the diffusive realization of h. Endeed, it is easily shown for example that $\int_0^t u(\tau)I^\alpha u(\tau)d\tau = \frac{1}{2}\int_0^{+\infty}\mu_\alpha(\xi)\psi^2(\xi,t)d\xi + \int_0^t\int_0^{+\infty}\xi\mu_\alpha(\xi)\psi^2(\xi,t)d\xi d\tau$.

- $\alpha, \beta \in]-1,0[$, $\partial_t \theta(x,t) = \left(a(t)\partial_t^\alpha + b(t)\partial_t^\beta\right) \Delta\theta(x,t)$ is dissipative if $a, b \geq 0$, $\dot{a}\ \mu_{-\alpha} + \dot{b}\ \mu_{-\beta} \geq 0$. This is verified for example if $b = 1 - a$, $0 \leq a \leq 1$, $0 \leq \dot{a} \leq 1$.

REMARK 6.3. Similar results are obtained in the frame of *extended* diffusive realizations (see for example [16]).

7. Conclusion

The diffusive representation of pseudo-differential operators allow to tackle a wide class of problems involving non-standard operators, such as fractional ones, or more generally, long memory non-oscillating operators, by transforming them into input-output well-posed differential equations with Markov property. Diffusive PDO may be viewed as a convenient extension to the infinite-dimensional frame, of classical transfer functions with stable real poles. In this sense, these representations may be called of "first kind". Many questions are to be studied, concerning approximation, control, etc., of problems involving such operators expressed under the diffusive form, which allows to introduce in a natural way powerful tools such as global energy functionals, taking into account the whole past of the system simply through a diffusive state-variable ψ.

More complex pseudo-differential operators such as $\left(\frac{d^2}{dt^2} + a^2\right)^\alpha$, with long-memory *oscillating* behavior, have no diffusive representation in the sense introduced in this paper. Such operators are of great interest, for example in absorbing feedback problems for propagative 2D-systems, in which natural matched impedances are of type $1/\sqrt{\partial_t^2 - \partial_y^2}$. The concept of diffusive representation and realization for " second kind" operators, as extension to the infinite-dimensional frame of classical transfer functions with stable complex poles, is currently under study.

Finally, in the time-varying case (touched on in section 6), diffusive representations remain useful, in the sense that a large class of (time-varying) diffusive realizations are obtained in a natural way, with the same properties as in the constant case. Details will be given in a forthcoming work.

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