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# DISTRIBUTION PROCESSES WITH STATIONARY FRACTIONAL INCREMENTS

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Abstract. This paper continues an idea introduced by L. Bel, G. Oppenheim, L. Robbiano ans M.C. Viano. We define a linear Gaussian distribution process with stationary k-order increments and study several properties of this process, which depend on a specific family of distributions. For k=1, we study the cases in which these distributions are fractional.

Résumé. Ce travail continue une idée de L. Bel, G. Oppenheim, L. Robbiano ans M.C. Viano où nous annoncions un résultat de base démontré dans un cas particulier. On définit un processus distribution dont les accroissements d'ordre k sont stationnaires linéaires gaussiens. On étudie les propriétés de ces processus qui dépendent d'une famille de distributions spécifiques au problème. Pour k=1, on s'intéresse aux cas dans lesquels ces dernières distributions sont fractionnaires.

## 1. Introduction

We continue to explore the stochastic distribution processes we began to study in [2] and [1]. These processes form a sub-family of the Gelfand-Vilenkin's family ([5]) because the underlying continuity holds with relation to a stronger topology. Moreover they are defined through a filter $f$, the impulse response, being a convolution kernel. For this reason, we call them linear processes. We denote this family as $\mathcal{L}$. They are Gaussian and stationary processes. These processes are written:

$$\langle X;\varphi\rangle = \int \left(\check{f} * \varphi\right) dW_s.$$

We study a little different family: a family of non-stationary distribution processes, but the increments and the derivatives (with relation to the generalized functions theory) are stationary processes and belong to $\mathcal{L}$.

The understanding we can gather from the study of the process called Fractional Brownian Process, and from processes with filters whose Laplace transform is $F(s) = s^{\beta}$, give a nice feeling of several results of the present paper. Of course, the results of this paper concern more general filters.

### 1.1. The FBM

The continuous time Fractional Brownian Motion (denoted B) (Mandelbrot-Van Ness [7]) does not belong to the family $\mathcal{L}$.

It can be written as $B(t) = \int_R (f(t-s) - f(-s))dW_s$ where $f(t) = \frac{1}{\Gamma(H+\frac{1}{2})} t^{H-\frac{1}{2}} \mathbb{1}_{t>0}$ using evident notations and with $0 < H < 1$.

As we know, the increments $\Delta B_h$, process indexed by $t$, is a stationary process, because $\Delta B_h(t) = B(t+h) - B(t) = \int_R (f(t+h-s) - f(t-s))dW_s$. $\Delta B_h$ belongs



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to $\mathcal{L}$, associated with the impulse response $g(t) = f(t+h) - f(t)$. The preceeding function $f$, choosen by Mandelbrot, is a very good choice. It provides the property of self-similarity in distribution to $B$. Nevertheless this property is an advantage as well as a drawback.

- An advantage, because it permits the modelisation of "fractal" situations as for instance "electrical consumption ([3]), image analysis ([8]), synthesis of artificial mountains, Internet traffic ([11])".
- A drawback because a single parameter $H$ determines all the characteristics, the local ones as the Hölder exponents and the global ones as the speed of convergence to 0, close to infinity, of the increments covariance, which is a mesure of long dependence property. Several authors are working to loosen of this constraint. For instance Levy-Vehel ([6]) "have defined a time varying parameter Fractional Brownian Motion. The parameter $H$ usually constant through time, is taken to be a smooth function $H(t)$ of time $t$". We do not go this way further, for this moment.

It is easy to associate a distribution process to the time process $B$. The derivative process $\partial B$ (in the distribution sense) of $B$ belongs to the family $\mathcal{L}$. Its impulse response is the Inverse Laplace transform of $s^\beta = s^{-H+1/2}$.
As a consequence of these remarks, in order to include the FBM within the cases to study, we had to define, in complement to $\mathcal{L}$, a new family of non stationary distribution processes whose first order increments process belong to $\mathcal{L}$ and, are, for this reason, linear stationary Gaussian distribution processes. We are able to generalize the ideas to orders greater than 1. A member $X$ of this family is defined using expressions similar to the following one which resembles the FBM, but in which $f$ and $g$ are distributions:

$$\langle X; \varphi \rangle = \int \left( \check{f} * \varphi - \check{g} \int \varphi(t)dt \right) dW_s$$

### 1.2. Processes that can be constructed starting with Laplace transform $s^\beta$

This function belongs to a family of Laplace transforms, we have studied quite a lot. This is the family of fractional filters. See for instance ([1],[10]). The fractional basic filters $f$ are the Inverse Laplace Transform $f = \mathcal{L}^{-1}(F)$ of function $F$ such that: $F(s) = \prod_{k=1}^{K} (s - a_k)^{d_k}$.
The simplest function of this set is $F(s) = s^\beta$. We know exactly the associated $f$ when they are usual functions ($\beta < 0$) or when they are distributions ($\beta \geq 0$).
Let us have a look at the processes we are able to associate, giving greater place to temporal, even non stationary processes or, if it is impossible, to distribution processes. Let $\beta$ vary in $\mathbf{R}$, partitionned as follows:
let $H = -\beta - \frac{1}{2}$. Then

$$\mathcal{L}^{-1}(s^\beta) = \begin{cases} \delta^{(\beta)} & \text{if} \quad \beta \in \mathbb{N} \\ \dfrac{1}{\Gamma(H+\frac{1}{2})} \mathrm{vp}(t^{H-\frac{1}{2}}) & \text{if} \quad \beta \in \mathbf{R}^+ \setminus \mathbb{N} \\ \dfrac{1}{\Gamma(H+\frac{1}{2})} t^{H-\frac{1}{2}} & \text{if} \quad \beta \in \mathbf{R}^- \end{cases}$$

- We know that for $-\dfrac{3}{2} < \beta < -\dfrac{1}{2}$ there exists the FBM, a time non stationary process, characterized by the parameter $H$, $0 < H < 1$.
- Let us examine the left part of the interval $[-\dfrac{3}{2} \,;\, -\dfrac{1}{2}]$. When $-\dfrac{2k+1}{2} < \beta < -\dfrac{2k-1}{2}$ for $k = 1, 2, \ldots$, then $k - 1 < H < k$, there exists

a time process whose k-th order increments are stationary. This process is defined by

$$X(t) = \int_{\mathbf{R}} \left[f(t-s) - \sum_{0 \le j \le k-1} \frac{1}{j!} t^j f^{(j)}(-s)\right] dW_s$$

where $f^{(j)}$ is the $j$-derivative of $f$.

- Let us analyse the right of the interval $[-\frac{3}{2}\ ;\ -\frac{1}{2}]$. In this case $H > 0$. The Inverse Laplace Transform is, either a function that does not belong to $L^2$ close to 0, or is a distribution. Nevertheless, in all the cases, it is possible to define a stationary linear distribution Gaussian process. For these processes we know several properties ([1]).

### 1.3. Further extensions

Leaning on the intuitions of the FBM, we extend the ideas in several directions:

- from time processes to distribution processes,
- from order 1 to order $k$ increments,
- from the function $f$ (associated with the FBM) to couples of usual functions or distributions $f, g$ .

These extensions constitute the center of this paper.

### 1.4. Plan

After this introduction, we define and study the distribution processes $X$ whose increments $\Delta X_h$ are stationary linear processes. We present the properties of $X$ and particularly when $\Delta X_h$ is a fractional process. The last part concerns the processes whose $k$th-order increments, $\Delta^{(k)} X_h(t) = \sum_{j=0}^{k} C_k^j (-1)^{k-j} X(t+jh)$, are stationary processes.

## 2. Linear Generalized Processes with stationary increments

The time processes that can be written as

$$X_t = \int (f(t-s) - g(-s))\, dW_s$$

where $f$ and $g$ are functions belonging to $L^2(\mathbf{R}^n)$ are processes with stationary increments. Indeed

$$\Delta X_h(t) \quad = \quad X(t+h) - X(t) = \int \Delta f_h(t-s) dW_s$$

has the same distribution as $\Delta X_h(t+\delta)$ for every $\delta$.
Let $X$ be the distribution process associated with $X_t$. We get

$$\begin{aligned} \langle X; \varphi \rangle &= \int \left( \int (f(t-s) - g(-s))\, \varphi(t) dt \right) dW_s \\ &= \int \left( \check{f} * \varphi - \check{g} \int \varphi(t) dt \right) dW_s \qquad (2.1) \end{aligned}$$

The processes defined by (2.1) for distributions $f$ and $g$ such that $f * \varphi - \check{g} \int \varphi dt$ belongs to $L^2(\mathbf{R}^n)$ for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ have stationary increments. This means that the process $\Delta X_h = \tau_h X - X$ is a stationary distribution process:

$$\begin{aligned} \langle \Delta X_h; \varphi \rangle &= \langle X; \tau_{-h} \varphi \rangle = -\langle X; \varphi \rangle \\ &= \int \check{f} * (\tau_{-h} \varphi - \varphi) dW_s \qquad (2.2) \end{aligned}$$

has the same distribution as $\langle \Delta X_h ; \tau_\delta \varphi \rangle$.
The next theorem gives a necessary and sufficient condition on $f$ et $g$ for the process $X$ written in (2.1) to be well defined.
Let $H^s(\mathbf{R}^n)$ denote the $s$-index Sobolev space [9], $\langle \xi \rangle^s = (1+|\xi|^2)^{s/2}$.
We shall note $\partial_{t_j} g = \partial_j g$.

THEOREM 2.1. *Let* $f$ *and* $g \in \mathcal{D}'(\mathbf{R}^n)$, $\check{f} * \varphi - \check{g} \int \varphi(t)dt \in L^2(\mathbf{R}^n)$ *for every* $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ *if and only if*

$$i) \quad \partial_j g \in H^{-1}(\mathbf{R}^n) \ ; \ \forall j = 1 \dots, n$$
$$ii) \quad f - g \in H^{-\infty}(\mathbf{R}^n).$$

REMARK 2.2. If $g \in L^2(\mathbf{R}^n)$, the hypotheses of the theorem are satisfied for every $f \in H^{-\infty}(\mathbf{R}^n)$ and the result is a direct consequence of Theorem 6 of [2] (i.e. $\check{f} * \varphi \in L^2(\mathbf{R})$ for all $\varphi \in \mathcal{C}_0^\infty$ if and only if $f \in H^{-\infty}$). The hypothesis $\partial_j g \in H^{-1}(\mathbf{R}^n)$ is equivalent to

$$\langle \xi \rangle^{-1} |\xi_j \widehat{g}(\xi)| \in L^2(\mathbf{R}^n) \qquad j = 1, \dots, n$$

It is less stringent than the hypothesis $g \in L^2(\mathbf{R}^n)$ because it implies that $\widehat{g}(\xi)$ belongs to $L^2$ close to infinity and $\xi_j \widehat{g}$ belongs to $L^2$ close to 0.

*Proof.* We shall need the following lemma:

LEMMA 2.3. *Let* $g \in \mathcal{D}'(\mathbf{R}^n)$. *If* $\forall j = 1 \dots, n$ ; $\partial_j g \in H^{-1}(\mathbf{R}^n)$ *then* $\forall \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ *we have*

$$\check{g} * \varphi - \check{g} \int \varphi(t)dt \in L^2(\mathbf{R}^n)$$

Suppose the lemma has been proved.
Let

$$H(\varphi) = \check{f} * \varphi - \check{g} \int \varphi(t)dt$$

Suppose that for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, $H(\varphi) \in L^2(\mathbf{R}^n)$. Then

$$H(\partial_j \varphi) = \partial_j \check{f} * \varphi \in L^2(\mathbf{R}^n)$$

and from the Theorem 6 of [2], there exists $\rho$ such that $\partial_j \check{f} \in H^\rho(\mathbf{R}^n)$.
Moreover, if $H(\varphi) \in L^2(\mathbf{R}^n)$ then $\partial_j(H(\varphi)) \in H^{-1}(\mathbf{R}^n)$. As

$$\partial_j(H(\varphi)) = \partial_j(\check{f}) * \varphi - \partial_j(\check{g}) \int \varphi(t)dt$$

we have $\partial_j(\check{g}) \int \varphi(t)dt \in H^{-1}(\mathbf{R}^n)$.
Let $\varphi$ such that $\int \varphi(t)dt = 1$ we get

$$\partial_j(\check{g}) \in H^{-1}(\mathbf{R}^n) \ \ \forall j = 1 \dots, n.$$

Furthermore

$$H(\varphi) = (\check{f} - \check{g}) * \varphi - (\check{g} * \varphi - \check{g} \int \varphi(t)dt)$$

the second term belongs to $L^2(\mathbf{R}^n)$, thanks to the lemma we get

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n) \qquad (\check{f} - \check{g}) * \varphi \in L^2(\mathbf{R}^n)$$

and $\check{f} - \check{g} \in H^{-\infty}(\mathbf{R}^n)$ from Theorem 6 of [2].
Conversely suppose that the hypotheses i) and ii) of the Theorem are satisfied,

$$H(\varphi) = (\check{f} - \check{g}) * \varphi + (\check{g} * \varphi - \check{g} \int \varphi(t)dt)$$

The first term belongs to $L^2(\mathbf{R}^n)$ from the Theorem 6 of [2] and the second term belongs to $L^2(\mathbf{R}^n)$ thanks to the lemma. In consequence $H(\varphi) \in L^2(\mathbf{R}^n)$ for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$. □

*Proof of lemma 2.3.* If $\forall j = 1 \dots, n$ $\partial_j g \in H^{-1}(\mathbf{R}^n)$, then $g \in \mathcal{S}'(\mathbf{R}^n)$ and

$$\begin{aligned} \|\breve{g} * \varphi - \breve{g} \int \varphi(t) dt\|^2_{L^2(\mathbf{R}^n)} &= \int \mid \widehat{g}(-\xi) \mid^2 \mid \widehat{\varphi}(\xi) - \widehat{\varphi}(0) \mid^2 d\xi \\ &\leq C \int \langle \xi \rangle^{-2} \mid \xi \mid^2 \mid \widehat{g}(-\xi) \mid^2 d\xi \\ &\leq C \sum_{j=1}^n \int \langle \xi \rangle^{-2} \mid \partial_j \widehat{g(\xi) \mid^2} d\xi \\ &\leq C \sum_{j=1}^n \|\partial_j g\|^2_{H^{-1}(\mathbf{R}^n)} < +\infty \end{aligned}$$

so $\breve{g} * \varphi - \breve{g} \int \varphi(t) dt \in L^2(\mathbf{R}^n)$. □

## 2.1. Properties of the derivative

Let $X$ be a process defined by (2.1), the process $\partial^j X$ is a stationary linear distribution process and the results of [2] permits to settle easily its properties.
We recall that, for the distribution process, the regularity spaces $C^s(\mathbf{R}^n, L^2(\Omega))$, $s \in \mathbf{R}$ are defined by ([2]):

$$C^s(\mathbf{R}^n, L^2(\Omega)) = \{X \in \mathcal{S}'(\mathbf{R}^n, L^2(\Omega));\ \forall k \geq -1, \left\| \|\Delta_k X\|_{L^2(\Omega)} \right\|_{L^\infty(\mathbf{R}^n)} \leq C 2^{-ks}\}.$$

the $\Delta_k$ being the operators entering in the Littlewood-Paley decomposition.

PROPOSITION 2.4. *$\partial^j X$ is a linear distribution process associated with the filter $\partial^j f$,*

- *the spectral density is* $\mid \xi_j \widehat{f}(\xi) \mid^2$,
- *the covariance function is* $\sigma = \mathcal{F}^{-1}(\mid \xi_j \widehat{f}(\xi) \mid^2)$,
- *$\partial_j X$ belongs to $C^s(\mathbf{R}^n, L^2(\Omega))$ if and only if $\partial_j f$ belongs to $B^s_{2,\infty}(\mathbf{R}^n)$.*

*Proof.* Let $X$ be a distribution process defined by (2.1) then

$$\langle \partial_j X; \varphi \rangle = -\langle X; \partial_j \varphi \rangle = \int \partial_j \check{f} * \varphi dW_s.$$

If $f$ and $g$ satisfy the hypotheses of Theorem 2.1, $\partial_j f$ belongs to $H^{-\infty}(\mathbf{R}^n)$ and $\partial_j X$ is a linear stationary distribution process with a spectral density $\mid \widehat{\partial_j f(\xi)} \mid^2 = \mid \xi_j \widehat{f}(\xi) \mid^2$ its covariance is given by $\sigma = \mathcal{F}^{-1}(\mid \xi_j \widehat{f}(\xi) \mid^2)$ and it belongs to $C^s(\mathbf{R}^n, L^2(\Omega))$ if and only if $\partial_j f$ belongs to $B^s_{2,\infty}(\mathbf{R}^n)$ ([2]). □

PROPOSITION 2.5. *If $\partial_j f$ belongs to $B^{s-1}_{2,+\infty}(\mathbf{R}^n)$, the process defined by (2.1) belongs to $C^s_{loc}(\mathbf{R}^n)$.*

*Proof.* If for every $j$, $\partial_j f$ belongs to $B^{s-1}_{2,+\infty}(\mathbf{R}^n)$ the process $\partial_j X$ belongs to $C^{s-1}(\mathbf{R}^n)$ and $X$ belongs to $C^s_{loc}(\mathbf{R}^n, L^2(\Omega))$. □

### 2.2. Properties of the increments

Let $X$ be a process (2.1), the process $\Delta X_h$ is a linear stationary distribution process and in this case we can also settle easily its properties.

Proposition 2.6. *$\Delta X_h$ is a linear distribution process associated with the filter $\Delta f_h$:*

- *the spectral density is* $4|\widehat{f}(\xi)|^2 \sin^2(\frac{h\xi}{2})$,
- *the covariance function is* $\sigma_h = \mathcal{F}^{-1}\left(4|\widehat{f}(\xi)|^2 \sin^2(\frac{h\xi}{2})\right)$,
- *$X$ belongs to $C^{s+1}(\mathbf{R}^n, L^2(\Omega))$ if $\partial_j f$ belongs to $B^s_{2,\infty}(\mathbf{R}^n)$ for every $j$.*

*Proof.*

$$\begin{aligned}\langle \Delta X_h;\varphi\rangle &= \int \check{f} * (\tau_{-h}\varphi) dW_s \\ &= \int (\tau_h \check{f} - \check{f}) * \varphi dW_s\end{aligned}$$

So $\Delta f_h$ is the filter of the linear distribution process $\Delta X_h$. In consequence its density is:

$$\begin{aligned}|\mathcal{F}(\Delta f_h)|^2 &= |\widehat{f}(\xi)|^2 (e^{ih\xi}-1)(e^{-ih\xi}-1) \\ &= 4|\widehat{f}(\xi)|^2 \sin^2(\frac{h\xi}{2})\end{aligned}$$

and its covariance is: $\sigma_h = \mathcal{F}^{-1}\left(4|\widehat{f}(\xi)|^2 \sin^2(\frac{h\xi}{2})\right)$ [5].

Suppose now that for every $j$, $\partial_j f \in B^s_{2,\infty}(\mathbf{R}^n)$, $\chi$ is the truncature function entering in the Littlewood-Paley decomposition of [9] for $k \geq 0$.

$$\begin{aligned}||\Delta_k \Delta f_h||^2_{L^2(\mathbf{R}^n)} &= ||\chi(2^{-k}\xi)\widehat{\Delta f_h}||^2_{L^2(\mathbf{R}^n)} \\ &= 4\int_{2^{k+1}<|\xi|<2^{k+2}} \chi^2(2^{-k}\xi)|\widehat{f}(\xi)|^2 \sin(\frac{h\xi}{2}) d\xi \\ &\leq C2^{-k}\int_{2^{k+1}<|\xi|<2^{k+2}} \left(\frac{\chi(2^{-k}\xi)}{|2^{-k}\xi|}\right)^2 |\xi\widehat{f}(\xi)|^2 d\xi \\ &\leq C2^{-k(s+1)}\end{aligned}$$

for $k=-1$

$$\begin{aligned}||\Delta_{-1}\Delta f_h||^2_{L^2(\mathbf{R}^n)} &= ||\psi(\xi)\widehat{\Delta f_h}||^2_{L^2(\mathbf{R}^n)} \\ &= 4\int_{|\xi|<2} \psi(\xi)^2 |\widehat{f}(\xi)|^2 \sin(\frac{h\xi}{2}) d\xi \\ &\leq C\int_{|\xi|<2} \psi(\xi)|\xi|^2 \widehat{f}(\xi)|^2 d\xi \\ &\leq C\end{aligned}$$

In consequence $\Delta f_h$ belongs to $B^{s+1}_{2,\infty}(\mathbf{R}^n)$ and $\Delta X_h$ belongs to $C^{s+1}(\mathbf{R}^n, L^2(\Omega))$. □

### 2.3. Processes with stationary fractional increments

We have defined in [1] the distribution fractional ARMA processes by

$$\langle X;\varphi\rangle = \int \check{f} * \varphi dW_s$$

with

$$f = \mathcal{L}^{-1}(F) \qquad F = \prod_{k=1}^{K}(s-a_k)^{d_k}$$

the parameters $a_k \in \mathbb{C}$ and $d_k \in \mathbb{C}$ satisfying the following conditions:

- $\Re e(a_k) \leq 0$ if $a_k$ singular
- if $\Re e(a_k) = 0$ and $a_k$ singular, then $\Re e(d_k) > -\frac{1}{2}$.

We introduce now the distribution processes with fractional increments in the following way: they are processes that satisfy (2.1) with

$$f = \mathcal{L}^{-1}(F+G) \qquad g = \mathcal{L}^{-1}(G) \tag{2.3}$$

with

$$F(s) = C\prod_{k=1}^{K}(s-a_k)^{d_k} \qquad G(s) = s^d\prod_{j=1}^{J}(s-b_j)^{\delta_j} \tag{2.4}$$

Proposition 2.7. *If*

- $\Delta = d + \sum_{j=1}^{J}\delta_j < -\frac{1}{2}$,
- $\Re e(a_k) < 0$ *if* $a_k$ *singular or* $\Re e(d_k) > -\frac{1}{2}$ *if* $\Re e(a_k) = 0$ *and* $a_k$ *singular*
- $\Re e(b_j) < 0$ *if* $b_j$ *singular or* $\Re e(\delta_j) > -\frac{1}{2}$ *if* $\Re e(b_j) = 0$ *and* $b_j$ *singular,*
- $\Re e(d) > -\frac{3}{2}$

*then the process* $X$ *defined by (2.1) with (2.3) and (2.4) is a distribution process with stationary increments.*

*Proof.* With these conditions on $a_k$ and $d_k$, from [1], we have $\mathcal{L}^{-1}(F) \in H^{-\infty}(\mathbf{R})$ thus $f - g \in H^{-\infty}(\mathbf{R})$. The condition $\Delta = d + \sum_{j=1}^{J}\delta_j < -\frac{1}{2}$ implies that $G(s)$ is $L^2$ close to infinity, while the conditions $d > -\frac{3}{2}$ and $\delta_j > -\frac{1}{2}$ if $\Re e(b_j) = 0$ assure that $\langle\xi\rangle^{-1}\mathcal{F}(\partial g) = \langle\xi\rangle^{-1}G(i\xi)$ could be integrated at 0, which implies that $\partial g \in H^{-1}(\mathbf{R})$. □

Property 2.8. *The process with fractional increments defined by (2.1) with (2.3) and (2.4) belongs to* $C_{loc}^{-\sup(\Delta,D)-\frac{1}{2}}(\mathbf{R}, L^2(\Omega))$.

*Proof.* It was proved in [1] that $\partial f \in B_{2,\infty}^{-D-\frac{3}{2}}(\mathbf{R})$, $\partial g \in B_{2,\infty}^{-D-\frac{3}{2}}(\mathbf{R})$ so the sum belongs to $B_{2,\infty}^{-\sup(\Delta,D)-\frac{1}{2}}(\mathbf{R})$ and, applying proposition 2.5, we get the result. □

Proposition 2.9. *The increments* $\Delta X_h$ *of the fractional process defined by (2.1) with (2.3) and (2.4) posses the following properties:*

- *their spectral density is* $4(|(F+G)(i\xi)|)^2\sin^2(\frac{h\xi}{2})$,
- *their covariance function is* $\sigma_h = \mathcal{F}^{-1}\left(4|(F+G)(\xi)|^2\sin^2(\frac{h\xi}{2})\right)$,
- *they belong to* $C^{-\sup(D,\Delta)-\frac{3}{2}}(\mathbf{R}, L^2(\Omega))$,
- *they are mixing if and only if for singular* $a_k$, $\Re e(a_k) < 0$.

*Proof.* The three first properties are direct consequences of proposition (2.6). Let us prove the fourth.

Let $X^{(f)}$ be the stationary process associated with the filter $f$.

$$\langle \Delta X_f ; \varphi \rangle = \langle X^{(f)} ; \Delta \varphi_{-h} \rangle$$

Let $\varphi \in \mathcal{C}_0^\infty(]-\infty;0])$ and $\psi \in \mathcal{C}_0^\infty([T,+\infty[)$.

$$E(X^{(f)}(\Delta\varphi_{-h}).X^{(f)}(\Delta\psi_{-h})) = E(X^{(f)}(\tau_h \Delta\varphi_{-h}).X^{(f)}(\tau_h\Delta\psi_{-h}))$$

$\text{supp}(\tau_h\Delta\varphi_h) \subset ]-\infty;0]$ and $\text{supp}(\tau_h\Delta\psi_h) \subset [T-h;\infty[$ thus if $\Re e(a_k) < 0$ for all singular $a_k$, from the Theorem 9 of [1]

$$\text{cor}(\Delta X_h(\phi);\Delta X_h(\psi)) \leq Ce^{-b(T-h)} \leq C_h e^{-T}$$

and $\Delta X_h$ is a mixing process.
If $\Re e(a_k) = 0$, for some singular $a_k$ then $\Delta X_h$ is not a mixing process, from Theorem 11 of [1]. □

Let us examine the case where $F = 0$, that is when $f = g = \mathcal{L}^{-1}(G)$. With the hypothesis on $g$, $g(t-s) - g(-s)$ belongs to $L^2$ and the process is a continuous time usual process. If one writes

$$G(s) = s^\delta \prod_{j=1}^{J} (s-b_j)^{\delta_j} \text{ with } b_j \neq 0,$$

we can see that the defined processes are a simple extension of the Brownian Fractional Processes, which correspond to the case where $G(s) = s^{-H-\frac{1}{2}}$ and $g(t) = \frac{1}{\Gamma(H+\frac{1}{2})} t^{H-\frac{1}{2}} \mathbb{I}_{t>0}$. If $0 < H < 1$, we have $-H - \dfrac{1}{2} > -\dfrac{3}{2}$. The only processes of the family that are self-similar are the FBM. Nevertheless, in this family we have a greater variety of sample regularity, since, deduced from property 2.8, the regularity index is $-\sup(\Delta, D) - \frac{1}{2}$.
The derivative of the FBM is a stationary distribution process, the fractional Gaussian white noise, which density is given by ([4]) :

$$\mid \Gamma(H+\frac{1}{2})\xi\widehat{g}(i\xi) \mid^2 = \mid \xi(i\xi)^{-H-\frac{1}{2}} \mid^2 = \mid \xi \mid^{1-2H} .$$

## 3. Distribution Processes with stationary $k$th-order increments

The time processes that may be written as

$$X(t) = \int f(t-s) - \sum_{|\alpha| \leq k-1} t^\alpha g_\alpha(-s)\, dW_s$$

with $f$, $g$ in $L^2(\mathbf{R}^n)$, have stationary $k$th-order increments. Indeed

$$\begin{aligned} \Delta^{(k)} X_h(t) &= \sum_{j=0}^{k} C_k^j (-1)^{k-j} X(t+jh) \\ &= \int \Delta_h^{(k)} f(t-s) dW_s \end{aligned}$$

are stationary processes. We may define, similarly, distribution processes with stationary $k$th-order increments by:

$$\langle X;\varphi\rangle = \int \left( \check{f} * \varphi(s) - \sum_{|\alpha|\leq k-1} g_\alpha(-s) \int \varphi(t) t^\alpha \, dt \right) dW_s$$

Then we have a result that generalises Theorem 2.1

THEOREM 3.1. *Let $f$ and $g_\alpha$, $|\alpha| \leq k-1 \in \mathcal{D}'(\mathbf{R}^n)$,*
$\check{f} * \varphi(s) - \sum_{|\alpha|\leq k-1} g_\alpha(-s) \int \varphi(t)t^\alpha dt \in L^2(\mathbf{R}^n)$ *for every* $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ *if and only if*

*i) for every* $\beta, \alpha \in \mathbb{N}^n$, $1 \leq |\beta| \leq k$, $|\alpha| + |\beta| = k$; $\partial^\beta g_\alpha \in H^{-|\beta|}(\mathbf{R}^n)$,
*ii)* $\beta \in \mathbb{N}^n$, $1 \leq |\beta| \leq k-1$, $\partial^\beta g_0 - \beta! g_\beta \in H^{-|\beta|}(\mathbf{R}^n)$,
*iii)* $f - g_0 \in H^{-\infty}(\mathbf{R}^n)$.

*Proof.* In order to simplify, we prove the result in $\mathbf{R}$. We note $f = \check{f}$ and $\check{g}_j = g_j$. We prove the theorem by induction. It has been proved for $k=1$. Suppose it is true for $k-1$. Let

$$H_k^{f,g}(\varphi) = f * \varphi - \sum_{j=0}^{k-1} g_j \int t^j \varphi(t) dt$$

By induction hypothesis, $H_{k-1}^{f,g}(\varphi) \in L^2(\mathbf{R})$ for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$ if and only if

$$\begin{array}{ll} i) & \partial^j g_{k-1-j} \in H^{-j}(\mathbf{R}) \qquad j = 1, \ldots, k-1 \\ ii) & \partial^j g_0 - (-1)^j j! g_j \in H^{-j}(\mathbf{R}) \qquad j = 1, \ldots, k-2 \\ iii) & f - g_0 \in H^{-\infty}(\mathbf{R}) \end{array}$$

Suppose that $H_k^{f,g}(\varphi) \in L^2(\mathbf{R})$ for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$.

$$\begin{aligned} H_k^{f,g}(\partial\varphi) &= \partial f * \varphi + \sum_{j=1}^{k-1} g_j \int j t^{j-1} \varphi(t) dt \\ &= \partial f * \varphi + \sum_{j=0}^{k-2} g_{j+1} \int (j+1) t^j \varphi(t) dt \\ &= H_{k-1}^{\widetilde{f},\widetilde{g}}(\varphi) \end{aligned}$$

with $\widetilde{f} = \partial f$ and $\widetilde{g}_j = (j+1) g_{j+1}$. $H_{k-1}^{\widetilde{f},\widetilde{g}}(\varphi) \in L^2(\mathbf{R})$ for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$, thus, because of hypothesis i) of the induction.

$$\partial^j \widetilde{g}_{k-1-j} = (k-j) \partial^j g_{k-j} \in H^{-j}(\mathbf{R}) \qquad \text{for } j = 1, \ldots, k-1. \tag{3.1}$$

Moreover

$$\partial^k (H_k^{f,g}(\varphi)) = (\partial^k f) * \varphi - \sum_{j=0}^{k-1} (\partial^k g_j) \int t^j \varphi(t) dt \in H^{-k}(\mathbf{R})$$

We have $\partial^k g_j = \partial^j \partial^{k-j} g_j \in H^{-k}(\mathbf{R})$ for $j = 1, \ldots, k-1$ by (3.1) and

$$H_k^{f,g}(\partial^k \varphi)) = (\partial^k f) * \varphi \in L^2(\mathbf{R}) \subset H^{-k}(\mathbf{R})$$

thus $\partial^k g_0 \int t^k \varphi(t) dt \in H^{-k}(\mathbf{R})$ and together with (3.1)

$$\partial^j g_{k-j} \in H^{-j}(\mathbf{R}) \qquad \text{for } j = 1, \ldots, k.$$

Thus we obtain condition i).
Let $1 \leq l \leq k$,

$$\begin{aligned} H_k^{f,g}(\partial^l \varphi) &= \partial^l f * \varphi - (-1)^l \sum_{j=l}^{k-1} g_j \int j(j-1)\ldots(j-l+1) t^{j-l} \varphi(t) dt \\ &\in L^2(\mathbf{R}) \end{aligned} \tag{3.2}$$

Moreover

$$\begin{aligned}\partial^l(H_k^{f,g}(\varphi)) &= \partial^l f*\varphi-\sum_{j=0}^{k-1}(\partial^l g_j)\int t^j\varphi(t)dt\\ &= \partial^l f*\varphi-\sum_{j=0}^{k-l-1}\partial^l g_j\int t^j\varphi(t)dt-\sum_{j=k-l}^{k-1}\partial^l g_j\int t^j\varphi(t)dt\\ &\in H^{-l}(\mathbf{R})\end{aligned}$$

For $l\geq k-j$, $\partial^l g_j=\partial^{l-(k-j)}\partial^{k-j}g_j\in H^{-l}(\mathbf{R})$, by condition i), and in consequence

$$\partial^l f*\varphi-\sum_{j=0}^{k-l-1}\partial^l g_j\int t^j\varphi(t)dt\in H^{-l}(\mathbf{R})$$

but

$$\begin{aligned}\partial^l f*\varphi \; - & \sum_{j=0}^{k-l-1}\partial^l g_j\int t^j\varphi(t)dt=\\ & \partial^l f*\varphi-(-1)^l\sum_{j=l}^{k-1}g_j\int j(j-1)...(j-l+1)t^{j-l}\varphi(t)dt\\ & +(-1)^l\sum_{j=l}^{k-1}g_j\int j(j-1)...(j-l+1)t^{j-l}\varphi(t)dt\\ & -\sum_{j=0}^{k-l-1}\partial^l g_j\int t^j\varphi(t)dt\end{aligned}$$

The first term belongs to $L^2(\mathbf{R})$ from (3.2) thus to $H^{-l}(\mathbf{R})$ and

$$(-1)^l\sum_{j=l}^{k-1}g_j\int j(j-1)...(j-l+1)t^{j-l}\varphi(t)dt-\sum_{j=0}^{k-l-1}\partial^l g_j\int t^j\varphi(t)dt=$$
$$\sum_{j=0}^{k-l-1}\left[(-1)^l(j+l)(j+l-1)...(j+1)g_{j+l}-\partial^l g_j\right]\int t^j\varphi(t)dt\in H^{-l}(\mathbf{R})$$

This relation being true for every $\varphi\in\mathcal{C}_0^\infty(\mathbf{R})$, we obtain

$$(-1)^l(j+l)(j+l-1)...(j+1)g_{j+l}-\partial^l g_j\in H^{-l}(\mathbf{R})\text{ for }j=0,\dots,k-1-l$$

For $j=0$, we get $(-1)^l l!g_l-\partial^l g_0\in H^{-l}(\mathbf{R})$ for $l=1,\dots,k-1$ and we obtain condition ii).

LEMMA 3.2. *If $\partial^j g\in H^{-j}(\mathbf{R})$ for $j=1,\dots,k$, then for every $\varphi\in\mathcal{C}_0^\infty(\mathbf{R})$,*

$$g\int\varphi(t)dt-\sum_{h=0}^{j-1}\frac{1}{h!}\partial^h g*t^h\varphi\in L^2(\mathbf{R})$$

Let us suppose the lemma has been proved.

$$\begin{aligned} H_k^{f,g}(\varphi) &= f*\varphi - \sum_{j=0}^{k-1}\left\{ g_j \int t^j\varphi(t)dt - \sum_{h=0}^{k-1-j} \frac{\partial^h g_j}{h!} * t^{j+h}\varphi \right\} \\ &\quad - \sum_{l=0}^{k-1}\left\{ \sum_{h=0}^{l} \frac{\partial^h g_{l-h}}{h!} * t^l\varphi \right\} \\ &= (f-g_0)*\varphi - \sum_{j=0}^{k-1}\left\{ g_j \int t^j\varphi(t)dt - \sum_{h=0}^{k-1-j} \frac{\partial^h g_j}{h!} * t^{j+h}\varphi \right\} \\ &\quad - \sum_{l=1}^{k-1}\left\{ \sum_{h=0}^{l} \frac{\partial^h g_{l-h}}{h!} * t^l\varphi \right\} \end{aligned} \tag{3.3}$$

By the lemma, the first sum belongs to $L^2(\mathbf{R})$. We can write

$$\begin{aligned} \sum_{h=0}^{l} \frac{\partial^h g_{l-h}}{h!} &= \frac{1}{l!}\left\{ \sum_{h=0}^{l} C_l^h(-\partial^l g_0 + (-1)^{l-h}(l-h)!\partial^h g_{l-h}) \right. \\ &\quad \left. + \sum_{h=0}^{l}(-1)^{l-h} C_l^h \partial^l g_0 \right\} \end{aligned}$$

The second term is equal to 0 and by condition ii)

$$-\partial^l g_0 + (-1)^{l-h}(l-h)!\partial^h g_{l-h} = \partial^h(-\partial^{l-h} g_0 + (l-h)! g_{l-h}) \in H^{-l}(\mathbf{R}),$$

thus the terms $\displaystyle\sum_{h=0}^{l} \frac{\partial^h g_j}{h!} * t^l\varphi$ belong to $L^2(\mathbf{R})$ and $(f-g_0)*\varphi \in L^2(\mathbf{R})$; in consequence $f-g_0$ belongs to $H^{-\infty}(\mathbf{R})$ and we obtain the condition iii).

*Proof of lemma 3.2.* Let $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$,

$$\widehat{\varphi}(0) = \widehat{\varphi}(\xi) - \xi\partial\widehat{\varphi}(\xi) - \ldots - \xi^{j-1}\partial^{j-1}\widehat{\varphi}(\xi) + O(\xi^j)$$

thus

$$|\widehat{\varphi}(0) - \widehat{\varphi}(\xi) - \xi\partial\widehat{\varphi}(\xi) - \ldots - \xi^{j-1}\partial^{j-1}\widehat{\varphi}(\xi)| \leq C\frac{|\xi|^j}{\langle\xi\rangle^j}$$

If $\partial^j g \in H^{-j}(\mathbf{R})$, $|\xi|^j\widehat{g}\langle\xi\rangle^{-j} \in L^2(\mathbf{R})$ and

$$\widehat{g}(\xi)\left(\widehat{\varphi}(0) - \widehat{\varphi}(\xi) - \xi\partial\widehat{\varphi}(\xi) - \ldots - \xi^{j-1}\partial^{j-1}\widehat{\varphi}(\xi)\right) \in L^2(\mathbf{R}),$$

now $\xi^h\widehat{g}(\xi)\partial^h\widehat{\varphi}(\xi) = \mathcal{F}(\partial^h g * t^h\varphi)$, thus

$$g\int\varphi(t)dt - g*\varphi - \partial g*t\varphi - \ldots - \partial^{j-1}g*t^{j-1}\varphi \in L^2(\mathbf{R}).$$

□

Conversely, we suppose

*i)* $\partial^j g_{k-j} \in H^{-j}(\mathbf{R}) \qquad j = 1,\ldots,k$

*ii)* $\partial^j g_0 - (-1)^j j! g_j \in H^{-j}(\mathbf{R}) \qquad j = 1,\ldots,k-1$

*iii)* $f - g_0 \in H^{-\infty}(\mathbf{R})$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$,

$$\begin{aligned} H_k^{f,g}(\varphi) &= (f-g_0)*\varphi - \sum_{j=0}^{k-1}\left\{ g_j \int t^j \varphi(t)dt - \sum_{h=0}^{k-1-j}(-1)^h \frac{\partial^h g_j}{h!} * t^{j+h}\varphi \right\} \\ &\quad - \sum_{l=1}^{k-1}\left\{ \sum_{h=0}^{l}(-1)^h \frac{\partial^h g_{l-h}}{h!} * t^l \varphi \right\} \end{aligned}$$

Thanks to condition iii) the first term belongs to $L^2(\mathbf{R})$, thanks to condition i) and the lemma, the second term belongs to $L^2(\mathbf{R})$ and thanks to condition ii) the third term belongs to $L^2(\mathbf{R})$. Thus for every $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$, $H_k^{f,g}(\varphi) \in L^2(\mathbf{R})$. □

Let $k-1 < H < k$, $f(t) = t^{H-\frac{1}{2}}\mathbb{1}_{t>0}$ and $g_j(t) = \dfrac{1}{j!}f^{(j)}(t)$. Then, it is obvious that,

$$\begin{aligned} &\partial^j \check{g}_0 - j!\check{g}_j = 0 \in H^{-j}(\mathbf{R}) \qquad j = 1, \dots, k-1 \\ &\check{f} - \check{g}_0 = 0 \in H^{-\infty}(\mathbf{R}) \end{aligned}$$

$\partial^j \check{g}_{k-j} = \partial^k \check{f}^{(k)}$ and $\langle\xi\rangle^{-1}|\partial^k \widehat{f}| = C\langle\xi\rangle^{-1}|\xi|^{-H-\frac{1}{2}+k} \in L^2(\mathbf{R})$, so $\partial^k \check{f} \in H^{-1}(\mathbf{R}) \subset H^{-j}(\mathbf{R})$ for $j = 1, \dots k$. Thus the process

$$X_t = \int_{\mathbf{R}} f(t-s) - \sum_{j=0}^{k-1}\frac{t^j}{j!}f^{(j)}(-s)dW_s$$

is well defined; its $k$ order increments $k$ form a stationary process and its $k-1$th-order derivative is a Fractional Brownian process with a parameter value equal to $H-k+1$.

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