ESAIM: PROCEEDINGS
FRACTIONAL DIFFERENTIAL SYSTEMS:
MODELS, METHODS AND APPLICATIONS
URL: HTTP://WWW.EMATH.FR/PROC/VOL.5/

VOL. 5, 1998, 69-74

SIMULTANEAOUS APPROXIMATION OF A FAMILY OF (STOCHASTIC) DIFFERENTIAL EQUATIONS

PHILIPPE CARMONA AND LAURE COUTIN

ABSTRACT. To approximate the fractional integral of order a in (0,1), we use an integral representation based on exponential functions introduced in a previous paper, and we present a scheme to approximate the whole family of associated linear differential equations:

dy(x,t)/dt=u-xy(x,t), for any x positive real.

We show how to extend these results to the stochastic case u="white noise", the fractional integration of which is a fractional brownian motion.

RÉSUMÉ. En vue d'approximer l'intégrale fractionnaire d'ordre a compris entre 0 et 1, nous utilisons une représentation intégrale à base d'exponentielles introduite dans un précédent papier, et présentons un schéma numérique permettant d'approximer simultanément la famille d'équations différentielles linéaires associées:

dy(x,t)/dt=u-xy(x,t), pour tout x reel positif.

Ensuite, nous montrons comment étendre ces résultats au cadre stochastique, u="bruit blanc", pour lequel l'intégrale fractionnaire est un mouvement brownien fractionnaire.

1. INTRODUCTION

Suppose that you want to compute the fractional integral

$$I^{\alpha}_{0,+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, u(s)\, ds \qquad (0<\alpha<1).$$

Since $(t-s)^{\alpha-1}$ explodes when s gets near t, usual time discretization schemes are

unsuitable. But if we combine the identity

$$\frac{1}{z^a} = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}e^{-zx}\, dx \quad (z>0, a>0)\,,$$

with Fubini's Theorem, we obtain, for $a = 1-\alpha$, another representation of the fractional integral:

$$I^{\alpha}_{0,+}u(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty dx\, x^{-\alpha}\int_0^t e^{-x(t-s)}\, u(s)\, ds\,.$$

More generally, one can consider the convolution integral

$$I^{h}_{0,+}u(t) = \int_0^t h(t-s)\, u(s)\, ds$$

where h is the Laplace transform of a sigma-finite measure μ

$$h(s) = \int e^{-sx}\, \mu(dx) \qquad (s>0),$$

P. Carmona, L. Coutin : Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4. E-mail :carmona@cict.fr coutin@cict.fr .

such that

$$\forall t > 0, \quad \int_0^t \int |u(s)| e^{-x(t-s)} \, \mu(dx) < +\infty \,,$$

(if μ is a measure on $\mathbb{R}_+$ and u is locally bounded, it is enough that $\int (1 \wedge x^{-1}) \, \mu(dx) < +\infty$). Then, we have the representation of $I^h_{0,+} u$ as the mixture

$$I^h_{0,+} u(t) = \int \mu(dx) y^x_t$$

where y^x is the solution of the ordinary (linear) differential equation

$$\dot{y^x} = u - x y^x \,, \qquad y^x(0) = 0 \,. \tag{1.1}$$

We are then naturally led to a two step approximation scheme.
The first step is the *spatial approximation*. The measure μ is approximated by a finite linear combination of Dirac masses

$$\mu \simeq \mu^\pi = \sum_{x \in \pi} c_x \delta_x \,,$$

where π is a finite subset. Thus,

$$I^h_{0,+} u(t) \simeq I^{h,\pi}_{0,+} u(t) = \int \mu^\pi(dx) \, y^x_t = \sum_\pi c_x y^x_t \,.$$

The second step is the *time discretization*. Each function y^x, $x \in \pi$, is approximated by a function ${}^{(n)}y^x(t)$ depending on a partition Δ_n of $(0, +\infty)$.
This approach is successful when in the spatial aproximation we manage to keep the dimension low, that is the cardinal of π small. It has been shown (see [2, 3]), that even if we replace u by a very rough signal, $u = \dot{B}$=" white noise", which is the case when we need to approximate fractional Brownian motion, then to obtain a precision of the order ϵ

$$\sup_{0 \le s \le t} \left| I^h_{0,+} u(s) - I^{h,\pi}_{0,+} u(s) \right| \simeq \epsilon \,,$$

one needed roughly $|\pi(\epsilon)| \simeq \frac{1}{\sqrt{\epsilon}} \log(\frac{1}{\sqrt{\epsilon}})$ points.

2. The time discretization schemes

In this communication we shall focus our interest on the time discretization step. It is natural to use an Euler's scheme:

$$\Delta y^x(t) = y^x(t + \Delta t) - y^x(t) = (u(t) - x y^x(t)) \Delta t \,.$$

Keeping in mind the example of $u = \dot{B}$= "white noise", we introduce the primitive of u,

$$U(t) = \int_0^t u(s) \, ds \,,$$

and we prefer to use the scheme

$$\Delta y^x(t) = \Delta U(t) - x y^x(t) \, \Delta t \,.$$

To be more precise we consider a regular partition of the interval $[0, T]$: $(t_k = Tk/n, 0 \le k \le n)$ and

$${}^{(n)}y^x(t_{k+1}) = {}^{(n)}y^x(t_k) + (U(t_{k+1}) - U(t_k)) - x \, {}^{(n)}y^x(t_k)(t_{k+1} - t_k) \,.$$

A classical Gronwall's type upper bound is (see, e.g. Bulirsch and Stoer [1], section 7.2)

$$\left| {}^{(n)}y^x(t) - y^x(t) \right| \le \frac{e^{xt}}{xt} (1 + x) \sup \left\{ |U(r) - U(s)|, |r - s| \le 1/n, 0 \le r, s \le t \right\}.$$

This may not be a very good upper bound. Indeed, in the case of fractional Brownian motion, U is Brownian motion, and therefore locally Hölder continuous of index $\gamma \in (0, \frac{1}{2})$. The upper bound we obtain on the interval $[0, 1]$ is, up to multiplicative constants, $e^x n^{-\gamma}$. We have to take into account that to obtain a precision of order ϵ, one has to consider a partition $\pi(\epsilon)$ with end point $\sup(\pi(\epsilon)) \simeq \epsilon^{-\frac{2}{2\alpha+3}}$; therefore, n should be, very roughly, $n \simeq \exp(\epsilon^{-\frac{2}{2\alpha+3}})$ which is clearly inacceptable.
Hence, we have to build a scheme adapted to the approximation of the whole family of linear differential equations (1.1). More precisely, we let $h = T/n$ and define

$$y^x(t) = e^{-xt}\, z^x(t), \quad z^x(t) = \int_0^t f^x(s) u(s)\, ds\,, \quad {}^{(n)}y^x(t) = e^{-xt}\ {}^{(n)}z^x(t),$$

$${}^{(n)}z^x(t) = \begin{cases} \int_0^t f_n^x(s)\, u(s)\, ds & \text{if } xh \leq \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

$$f_n^x(s) = e^{xt_k} \quad \text{if } t_k \leq x < t_{k+1}, \quad f^x(s) = e^{xs}.$$

Our main result for deterministic u is the

THEOREM 2.1. *Assume that for a $p > 1$, u is in every $L^p(0,t)$, $t > 0$. Then*

$$\sup_{x>0} \sup_{t \leq T} \left| {}^{(n)}y^x(t) - y^x(t) \right| \leq C(q,T) n^{-1/q}\,,$$

where q is the conjugate exponent of p, $1 = 1/p + 1/q$, and

$$C(q,T) = (2T/q)^{1/q} \sup((q+1)^{-1/q}, \frac{1}{2} e^{\frac{1}{2}}) \|u\|_{L^p(0,T)}\,.$$

If we combine the spatial approximation for a geometric subdivision $\pi = \{r^i\}_i$ of $[1/N, N]$ of geometric ratio $r > 1$, with a time discretizationof step $1/n$, we get, for fractional integration

COROLLARY 2.2. *Assume that for a $p > 1$, u is in every $L^p(0,t)$, $t > 0$. Then,*

$$\left| I^h_{0,+} u(t) - I^{h,\pi,n}_{0,+} u(t) \right| \leq c \|u\|_{L^p(0,t)} (N^{-(1-\alpha)} + N^{-(\alpha+1+1/q)} + (r-1)^2 + n^{-1/q} N^{1-\alpha})\,.$$

The proof of Theorem 2.1 relies heavily on the upper bounds given in the

LEMMA 2.3.

$$\sup(\left| {}^{(n)}z^x(t) \right|, |z^x(t)|) \leq \|u\|_{L^p(0,t)} \left(\frac{e^{qxt} - 1}{qx} \right)^{1/q} \tag{2.1}$$

$$\left| z^x(t) - {}^{(n)}z^x(t) \right| \leq \|u\|_{L^p(0,t)} (xT/n) e^{xT/n} (q(q+1))^{-1/q} x^{-1/q}\, e^{xt}\,. \tag{2.2}$$

When U is not deterministic anymore, but a Brownian motion, this Lemma has a stochastic analog. Let

$$\begin{aligned} Y^x(t) &= e^{-xt}\, Z^x(t), & Z^x(t) &= \int_0^t f^x(s) dB_s\,, \\ {}^{(n)}Y^x(t) &= e^{-xt}\ {}^{(n)}Z^x(t), & {}^{(n)}Z^x(t) &= \int_0^t f_n^x(s)\, dB_s\,. \end{aligned} \tag{2.3}$$

LEMMA 2.4.

$$\sup(\left\| \sup_{s \leq t} \left| {}^{(n)}Z^x(s) \right| \right\|_2, \left\| \sup_{s \leq t} |Z^x(s)| \right\|_2) \leq 2 \left(\frac{e^{2xt} - 1}{2x} \right)^{\frac{1}{2}}. \tag{2.4}$$

$$\left\| \sup_{s \leq t} \left| {}^{(n)}Z^x(s) - Z^x(s) \right| \right\|_2 \leq \frac{2}{\sqrt{6}} (xT/n) e^{xT/n} x^{-\frac{1}{2}}\, e^{xt}\,. \tag{2.5}$$

We cannot obtain a uniform result, as in the deterministic case, because in a stochastic framework, supremums don't behave well. More precisely you can have two random variables X_1 and X_2 such that $\|X_i\|_2 \le c$, but $\|\sup(X_1, X_2)\|_2 > c$ (take Bernoulli random variables $X_2 = 1 - X_1$). Therefore we have to loose something in the process of obtaining good upper bounds on the Y's based on upper bounds on the Z's.

Given a process $(U(t)\,; t \ge 0\,)$ and $x > 0$, we consider $V(t) = e^{-t}U(t)$,and the two quantities

$$U^*(t) = \sup_{s \le t} |U(s)|\,, \qquad V^*(t) = \sup_{s \le t} |V(s)|\,.$$

LEMMA 2.5. *Assume that for a $p \in (1, +\infty)$, and a constant $C > 0$, we have $\|U^*(t)\|_p \le Ce^{xt}$. Then,*

$$\|V^*(t)\|_p \le C(1 + xt)e\,.$$

From Lemmae 2.4 and 2.5 we immediately deduce the

THEOREM 2.6. *For the stochastic scheme (2.3), we have the upper bounds:*

$$\sup\Big(\Big\|\sup_{s\le t}\Big|{}^{(n)}Y^x(s)\Big|\Big\|_2, \Big\|\sup_{s\le t}|Y^x(s)|\Big\|_2\Big) \le \sqrt{\frac{2}{x}}(1 + xt)\,. \tag{2.6}$$

$$\Big\|\sup_{s\le t}\Big|{}^{(n)}Y^x(s) - Y^x(s)\Big|\Big\|_2 \le \frac{2e}{\sqrt{6}}(xT/n)e^{xt/n}x^{-\frac{1}{2}}(1 + xt)\,. \tag{2.7}$$

3. PROOFS

Proof of Lemma 2.5. Without loss in generality, we may assume $x = 1$. For every integer $n \ge 1$, we have

$$\begin{aligned}
\|V^*(t)\|_p &= \Big\|\sup_{0\le i<n}\Big(\sup_{it/n\le s\le(i+1)t/n} e^{-s}|U(s)|\Big)\Big\|_p \\
&\le \Big\|\sum_{0\le i<n}\sup_{it/n\le s\le(i+1)t/n} e^{-s}|U(s)|\Big\|_p \\
&\le \sum_{0\le i<n}\Big\|\sup_{it/n\le s\le(i+1)t/n} e^{-s}|U(s)|\Big\|_p \\
&\le \sum_{0\le i<n} e^{-it/n}\|U^*((i+1)t/n)\|_p \\
&\le \sum_{0\le i<n} e^{-it/n}Ce^{(i+1)t/n} = Cne^{t/n} \\
&\le e(1 + t)C
\end{aligned}$$

by taking $n = 1 + \lfloor t \rfloor$. □

Proof of Lemma 2.4. Since $(Z^x(t)\,; t \ge 0\,)$ is a square integrable martingale, Doob's L^2 inequality implies that

$$\Big\|\sup_{s\le t}|Z^x(s)|\Big\|_2 \le 2\|Z^x(t)\|_2 = 2\|f^x\|_{L^2(0,t)} = 2\Big(\frac{e^{2xt} - 1}{2x}\Big)^{\frac{1}{2}}.$$

Similarly,

$$\left\|\sup_{s\le t}\left|{}^{(n)}Z^x(s)\right|\right\|_2 \le 2\left\|{}^{(n)}Z^x(t)\right\|_2 = 2\|f_n^x\|_{L^2(0,t)}$$
$$= 2\left(\sum \int_{t_k\wedge t}^{t_{k+1}\wedge t} e^{2xt_k}\,ds\right)^{\frac12} \quad \le 2\|f^x\|_{L^2(0,t)} = 2\left(\frac{e^{2xt}-1}{2x}\right)^{\frac12}.$$

and

$$\left\|\sup_{s\le t}\left|{}^{(n)}Z^x(s)-Z^x(s)\right|\right\|_2^2 \le 4\left\|\left|{}^{(n)}Z^x(t)-Z^x(t)\right|\right\|_2^2$$
$$= 4\|f^x-f_n^x\|_{L^2(0,t)}^2 = 4\sum\int_{t_k\wedge t}^{t_{k+1}\wedge t}(e^{xs}-e^{xt_k})^2\,ds$$

Taking into account that for $0\le a\le b$, $0\le e^b-e^a\le(b-a)e^b$, we have

$$\left\|\sup_{s\le t}\left|{}^{(n)}Z^x(s)-Z^x(s)\right|\right\|_2^2 \le 4\sum\frac13(t_k\wedge t-t_{k+1}\wedge t)^3e^{2x(t_{k+1}\wedge t)}x^2$$
$$\le (4/3)(xh)^2\int_0^{t+h}e^{2xs}\,ds$$
$$\le (4/6))(xh)^2e^{2xh}x^{-1}e^{2xt}\,.$$

□

Proof of Lemma 2.3. By Hölder's inequality,

$$|z^x(t)| \le \|u\|_{L^p(0,t)}\|f^x\|_{L^q(0,t)}$$
$$= \|u\|_{L^p(0,t)}\left(\frac{e^{qxt}-1}{qx}\right)^{1/q}.$$

Similarly,

$$\left|{}^{(n)}z^x(t)\right| \le \|u\|_{L^p(0,t)}\|f_n^x\|_{L^q(0,t)}$$

and

$$\int_0^t f_n^x(s)^q\,ds = \sum\int_{t_k\wedge t}^{t_{k+1}\wedge t} e^{xt_k}\,ds$$
$$\le \sum\int_{t_k\wedge t}^{t_{k+1}\wedge t} e^{xs}\,ds \qquad\qquad = \frac{e^{qxt}-1}{qx}\,.$$

Observe that, since if $0\le a\le b$, $0\le e^b-e^a\le(b-a)e^b$, if $xh\le\frac12$, then

$$\|f_x-f_n^x\|_{L^q(0,t)}^q = \sum\int_{t_k\wedge t}^{t_{k+1}\wedge t}(e^{xs}-e^{xt_k})^q\,ds$$
$$\le \sum\int_{t_k\wedge t}^{t_{k+1}\wedge t}(s-t_k)^qx^qe^{qxt_{k+1}}\,ds$$
$$\le \frac{1}{q+1}\sum(t_{k+1}\wedge t-t_k\wedge t)^{q+1}x^qe^{qxt_{k+1}}$$
$$\le \frac{1}{q+1}x^qh^q\frac{\exp(qx(t+h))-1}{qx}$$

Therefore

$$\begin{aligned}\left|{}^{(n)}z^x(t) - z^x(t)\right| &\leq \|u\|_{L^p(0,t)} \|f_n^x f^x\|_{L^q(0,t)} \\ &\leq \|u\|_{L^p(0,t)} (q(q+1))^{-1/q} (xh) e^{xh} x^{-1/q} e^{xt} \\ &\leq \|u\|_{L^p(0,t)} (q(q+1))^{-1/q} \frac{1}{2} e^{\frac{1}{2}} (2T)^{1/q} n^{-1/q}\, e^{xt}\,.\end{aligned}$$

□

Proof of Theorem 2.1. From Lemma 2.3 we obtain, if $xh \leq \frac{1}{2}$,

$$\left|{}^{(n)}z^x(t) - z^x(t)\right| \leq \|u\|_{L^p(0,t)} (q(q+1))^{-1/q} \frac{1}{2} e^{\frac{1}{2}} (2T)^{1/q} n^{-1/q}\,.$$

And, if $xh > \frac{1}{2}$,

$$\left|{}^{(n)}z^x(t) - z^x(t)\right| = |z^x(t)| \leq \|u\|_{L^p(0,t)} (qx)^{-1/q} \leq \|u\|_{L^p(0,t)} (2T/q)^{1/q} n^{-1/q}$$

□

References

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