ESAIM : Proceedings, Vol. 7, 1999, 24-35
Third International Workshop on Vortex Flows and Related Numerical Methods
http://www.emath.fr/proc/Vol.7/

A new formulation of the near-equilibrium theory of turbulence [1]

G.I. Barenblatt and Alexandre J. Chorin

Department of Mathematics
University of California
Berkeley, CA 94720–3840

Abstract.

We present a status report on a discrete approach to the the near-equilibrium statistical theory of three-dimensional turbulence, which generalizes earlier work by no longer requiring that the vorticity field be a union of discrete vortex filaments. The idea is to take a special limit of a dense lattice vortex system, in a way that brings out a connection between turbulence and critical phenomena. The approach produces statistics with basic features of turbulence, in particular intermittency and coherent structures. The numerical calculations have not yet been brought to convergence, and at present the results are only qualitative.

[1]Supported in part by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under contract DE–AC03–76–SF00098, and in part by the National Science Foundation under grants DMS94–14631 and DMS973-2710

Article published by EDP Sciences and available at http://www.edpsciences.org/proc or http://dx.doi.org/10.1051/proc:1999003

1 Introduction

In recent work on wall-bounded turbulence and local structure in fully developed turbulence [1,2,3,6] we have made extensive use of vanishing-viscosity asymptotics and of expansions in powers of the small parameter $\frac{1}{\ln Re}$ (Re is the Reynolds number) around the asymptotic, highly intermittent state $Re\infty$; we concluded in particular that the kolmogorov-Obukhov spectrum must be corrected when the viscosity is finite, and also that moments of the velocity field of order $p \geq 4$ diverge as the viscosity ν tends to zero. Our analysis used results of a near-equilibrium statistical theory of turbulence based on vortex representations [5,6]. In the present paper we present briefly a new derivation of the near-equilibrium statistical theory of turbulence, without assuming that the vorticity field can represented as a union of discrete vortex filaments, and discuss conclusions that can be drawn from it, in particular regarding intermittency, scaling, and the existence of coherent structures. The work includes preliminary numerical computations in three space dimensions, and while these computations are informative, they are still incomplete.

One of the keys to the understanding of turbulence is an explanation of what can be viewed as a statistical equilibrium in fluid mechanics. Statistical theories in both mechanics and physics are built along differing lines when they describe systems at equilibrium, near equilibrium, or far from equilibrium. The validity of the expansions we used in [1,2,3,6] is connected with specific properties of statistical equilibria in fluid mechanical vortex systems. A fluid mechanical system can be described macroscopically or microscopically; a macroscopical description requires in principle just a few global parameters, such as the energy per unit mass or the rate of energy dissipation per unit mass; A microscopical description purports to provide much of the detail in the system. As many microscopical descriptions are generally compatible with a single macroscopical picture, a microscopical description usually consists of an ensemble of possible realizations, all compatible with a fixed macroscopical picture, and each occurring with a certain probability. A statistical equilibrium is an ensemble of microscopical pictures, each with the probabilities it would have when the macroscopical, large scale, description has been kept constant for a long time. In other words, to find a statistical equilibrium, one fixes the large scale parameters, and, having waited a while for the system to settle down, one observes the resulting microscopic states and the probabilities of their occurrence. There are a number of recipes for constructing statistical equilibria. One is the so-called "microcanonical" ensemble, in which one considers all the states allowed by the macroscopic data and assumes that each occurs with equal probability; this recipe is suitable for an isolated system. An equivalent characterization can be given in terms of the canonical ensemble, in which the probability of a microscopic state is proportional to $\exp(-\beta E_s)$, where E_s is the energy of that state and β is a parameter. The principle of "equivalence of ensembles" asserts that these last two ensembles are equivalent in the sense that, when the parameter β is properly chosen and the system has many degrees of freedom, average properties calculated in either ensemble are equal.

The parameter β is generally called the "inverse temperature" of the system. In many physical systems $T = 1/\beta$ is indeed proportional to what one intuitively perceives to be the temperature, as it can be gauged by touching a system with one's finger. However, the parameter β can be viewed more abstractly, as the parameter that makes the two ensembles be equivalent; in incompressible turbulence, in which the hydrodynamical motion and the vibrations of the molecules that make up the fluid do not affect each other, β is unrelated to what one normally calls the temperature of the fluid; T does not have a readily intuitive interpretation; standard thermodynamics survives when one uses this less intuitive notion of temperature (see e.g. [5] Chap. 4). In a given system, β is a function of the energy kinetic energy E of the turbulence and of whatever else is needed to describe the system, just as in the kinetic theory of gases the temperature is defined in terms of a kinetic energy of the molecules. Our contention is that the intermediate scales of turbulence, those that are between the scales on which the turbulence is stirred and those at which turbulence is dissipated by viscosity, can be well-described by a theory with the following properties: (i) These intermediate scales are near a statistical equilibrium that is well-described as a canonical or microcanonical equilibrium ensemble, (ii) this equilibrium is characterized by values of macroscopical parameters such as energy per unit mass or squared vorticity per unit mass that define a critical state, and (iii) the properties of this equilibrium can be derived from the properties of certain discrete systems by taking a special limit that we shall describe.

2 Turbulence as a near-equilibrium process

Turbulence *as a whole* cannot be described as a microcanonical or canonical equilibrium, for if one stirs a box full of fluid and then isolates the resulting flow, the outcome after a long time is a state of rest; an isolated system does not allow for the presence of outside forces or an imposed shear that would maintain the flow. However, we apply canonical and microcanonical considerations only to the range of scales in turbulence intermediate between the dissipation scales and the scale of the stirring, and the relevant question is whether the motion on these intermediate scales has enough time to settle down to an equilibrium on a time scale in which one can assume that little energy is added or subtracted from this intermediate range, so that one can view it as being approximately isolated; in other words, we have to know whether the characteristic time of small-scale motion is short enough compared to the characteristic time over which the flow as a whole changes appreciably. According to the Kolmogorov scaling in homogeneous isotropic turbulence, the structure function $\langle (u(\mathbf{x}+\mathbf{r})-u(\mathbf{x}))^2\rangle$, where u is the velocity along $\mathbf{r}$, is proportional to $(\epsilon r)^{2/3}$, where $r = |\mathbf{r}|$ and ε is the rate of energy dissipation per unit mass. Thus the characteristic time (length/velocity) of an eddy of size r is proportional to $r/((\varepsilon r)^{1/3} = r^{2/3}\varepsilon^{-1/3}$. The time scale over which the flow changes appreciably (in the case of freely decaying turbulence, the characteristic time of over-all decay) is E/ε, where E is the energy per unit mass. The ratio of these time scales is proportional to $(\varepsilon r)^{2/3}/E$, which tends to zero as $r \to 0$, and thus small eddies (vortices) have time enough time to approach a statistical equilibrium. In some of the turbulence literature, statistical equilibrium is described by an equipartition ensemble, in which the energy is equally distributed on the average between Fourier components whose number tends to infinity. We have shown in earlier work (see [6]) that this equipartition ensemble has no relevance to fluid mechanics; its members have an infinite energy per unit mass, and the corresponding spectrum is proportional, in three space dimensions, to k^2, where k is a wave number. In contrast, when we construct equilibria below, we shall take great care *not* to distribute energy evenly between an increasing number of degrees of freedom. We shall instead look for a sequence of finite-dimensional equilibria and a limit designed so as to be meaningful when the number of variables increases.

3 Turbulence as a critical phenomenon

Before discussing our construction in detail, a key observation must be made regarding the relation between turbulence and critical phenomena. We shall construct our ensemble and its probability density as the limit of a sequence of ensembles of discrete vortex filaments living on a lattice with mesh length h; in each discrete ensemble each filament has a circulation Γ, The energy per unit mass is E, and the total length of the filaments in the volume occupied by a unit mass is L. It is known [5] that, for a given choice of Γ and E and for each mesh spacing h, the filaments break up into a collection of isolated vortex loops when the dimensionless parameter $\Lambda = L\sqrt{E}/\Gamma$ is small, or consist of a dense collection of smooth vortex lines when Λ is large. The transition between these cases, generated by varying either E, L, or Γ but keeping h fixed, has all the hallmarks of a phase transition in statistical mechanics: At the points in the Γ, E plane that correspond to the transition the fluctuations in various mean quantities are large, the derivative of E with respect to the corresponding β is large (i.e., a large change in E changes the corresponding β only a little), and the correlation length (the length over which the vortex configurations are correlated) is also large, as one expects from the general theory of critical phenomena. This phase transition is related to the other, well-studied phase transitions in statistical mechanics [5]. We claim that, to the extent that a lattice vortex system can be generated by a fluid mechanical agency, the parameters E, Γ, L must place it near this phase transition. We shall list now some obvious reasons, leaving some deeper ones for the sequel.

(i) A key property of turbulence at large Re is that it is subject to scaling laws [1,2,3]. Scaling laws can hold for a many-particle or many- vortex system only near phase transition points; indeed, the existence of scaling laws is a general way of characterizing phase transition points. Phase transition points where a scaling law holds are known as "critical points".

(ii) Another salient feature of turbulence is that it dissipates energy. A system near equilibrium dissipates energy efficiently if the autocorrelation time of the fields that describe it is long [8]. The autocorrelation time of a vorticity or velocity field is largest near "critical" phase transition points, and this is where energy dissipation can be substantial. A version of this argument of particular relevance to vortex dynamics is as follows: A major mechanism of energy transfer from large to small scales in three space dimensions is vortex intersection, which reduces the scale of the vortical structures in the fluid. On the large Λ side of the phase transition the vortex lines are smooth and there are few vortex intersections, while on the small Λ side of the phase transition the vortices are small and isolated, and it is only on the phase transition line that the vortices are both long enough and crumpled enough to create a substantial number of intersections [5].

(iii) The hydrodynamical statistical equilibria have been discussed so far without reference to how they may be produced by a fluid flow. In three space dimensions vortex lines stretch and fold; numerical calculations (see [5]) show that vortex stretching pushes vortex systems to the neighborhood of the phase transition. Furthermore, a relatively simple calculation ([5], page 142) shows that, when the fluid system is on the phase transition line, the energy spectrum fluid system at the phase transition is the Kolmogorov spectrum.

4 A discrete equilibrium model and its continuum limit

We now describe in the discrete systems whose limit will be taken. Consider vortex loops like the one in Figure 1, with circulation Γ, the same for all the loops; these loops are elements of impulse [4]; they are located on a $N \times N \times N$ mesh of mesh size h, $Nh = 1$. An incompressible velocity field is created by solving the equations

$$\Delta_h \boldsymbol{\psi} = -\boldsymbol{\xi}, \qquad \mathbf{u} = \nabla_h \times \boldsymbol{\psi} \ , \tag{1}$$

where $\boldsymbol{\xi}$ is the vorticity field, Δ_h is the standard 6-point Laplacian in three space dimensions, $\boldsymbol{\psi}$ is the vector potential, ∇_h is the differencing operator implemented with forward differences; equation (1) is the discrete analog of the usual definition of the velocity in terms of the vorticity; the domain is assumed to be periodic with period 1. The kinetic energy of the system is defined as

$$E_h = \frac{1}{2}\Sigma(u_1^2 + u_2^2 + u_3^2)h^3 \ , \tag{2}$$

where u_1, u_2, u_3 are the components of $\mathbf{u}_h$ and the summation is over all the nodes in the lattice. The energy E_h being fixed, ther appropriate equilibrium ensemble is the microcanonical ensmeble. For a given h, the various configurations of the equilibrium ensemble can be sampled by "microcanonical sampling" [7]. The parameter β in the equivalent canonical ensemble can be determined in the course of calculating averages.

If one decreases h while keeping the energy E_h fixed the "temperature" $T = 1/\beta$ of the system decreases; heuristically, T is the energy per degree of freedom, and as the number of degrees of freedom increases, there is less energy for any one of them – this is exactly the situation in the "equipartition" ensemble discussed above. The limiting ensemble does not exist. On the other hand, suppose one decreases h while keeping fixed not only the energy but also some measure of the total vorticity – for example, the sum $Z_2 = h^3 \sum |\boldsymbol{\xi}|^2$, where $\boldsymbol{\xi}$ is the vorticity field formed by the sides of the vortex loops described above and in Figure 1, and the summation is over all the nodes in the lattice. Z_2 cannot be viewed as an accurate approximation of the enstrophy (we would have to know how the vorticity is distributed across each leg of the loops), but it is a sensitive gauge of the total vorticity in the system. If one decreases h while keeping both E_h and Z_2 constant, as one can easily do with microcanonical sampling, the temperature of the system tends to infinity and beyond; indeed, temperatures beyond infinity, the so-called "negative temperatures", are typical of two dimensional turbulence [5] where additional integrals of motion constrain the vorticity. The heuristic explanation of this phenomenon is that in the presence of two constraints there are ever fewer states that satisfy both constraints; the effective number of degrees of freedom tends to one and the temperature increases. In the limit $h \to 0$ one finds a probability measure concentrated on a single organized structure, which is reasonable in two dimensions but not in three. Thus a fixed bound on Z_2 produces too much organization while an absence of bounds on the vorticity produces too little organization; the ensemble we seek should be between those two extremes.

What we shall do is pick some finite value of β and adjust Z_2 so that the system assume this inverse temperature β for each h. We already showed elwewhere [6] that this can be done in two space dimensions. The automatic way of finding the appropriate value of Z_2 described in [6] turned out to be impractical in three space dimensions, and we proceed by simple tabulation: Pick various bounds for Z_2, calculate the corresponding values of β and home in on the value of β that one wants to have by successive improvement guided by a human. The ensemble produced in this way is intermittent- the vorticity occupies only a fraction of the available sites, and has hidden coherent structures- the temperature being finite, there is effectively a finite number of degrees of freedom, fixed as $h \to 0$, but their nature and location is not explicitly known. The finiteness of β ensures that one can define a probability density in the limit state.

The choice of the fixed β is arbitrary, but as $h \to 0$ the physical results become independent of that fixed β because the resulting systems converge to the critical line: A refinement of the mesh is tentamount to a change in the scale of the vortex configurations; the critical points are the loci of the states that are invariant under a change in scale, and attract all other states as $h \to 0$, as long as their temperature is finite and non-zero. Controlling the rate of growth of the enstrophy as $h \to 0$ is consistent with the physics of three-dimensional turbulence: Turbulence is intermittent, and vorticity concentrates on small scales. As h is decreased ever more vorticity is revealed, at a rate that determines the vorticity spectrum and hence the energy spectrum.

5 Some numerical results

We now display some results obtained in calculations carried out as was just explained. The lattices we could afford so far are small, and the results are not converged; indeed, if one views the mesh size h as an analog of a viscous length, the scaling derived in [1,2,3] shows that the effect of the mesh size decreases like $\frac{1}{\ln|h|}$; this is the rate of convergence of analogous computations in two space dimensions as well [6], and even in two dimensions the calculations cannot be brought to convergence without relying on explicit renormalization group parameter flow which we do not have in three dimensions. We expect here only qualitative results. Analogous, more complete computations in simpler models have been displayed elsewhere.

In Figure 2 we display the variation of the computed $\beta = T^{-1}$ as a function of the lattice size N for a fixed energy $E_h = 100$, with no restrictions on the "enstrophy" Z_2. The graph shows that β is proportional to N^3, as one can expect from equipartition; without constraints on the vorticity and at a fixed, finte energy, the temperature of the system tends to zero. In Figure 3 we exhibit the values of β obtained by bounding Z_2 with $N = 16$: We first found the value $Z_{2,0}$ of Z_2 that corresponded to equilibrium with no bound on Z_2, and then repeated the calculation requiring that $Z_2 \leq Z_{20} - \delta Z_2$ for various values of δZ_2. This graph shows that bounding Z_2 does indeed keep the "temperature" $T = \beta^{-1}$ away from zero.

In figure 3 we display the shape of the second order structure function $R(r) = C\langle(u(\mathbf{x}+\mathbf{r})-u(\mathbf{x}))^2\rangle$ as a function of the separation $r = |\mathbf{r}|$, with $N = 16$, with u the velocity component in the direction of $\mathbf{r}$, at two values of β: $\beta = 40.2$ and $\beta = 3$ obtained with differing values of δZ_2. The constant C equals $(4/3)/ < (u(\mathbf{x}))^2\rangle$ so as to make $R(r) = 1$ for large r. While no convergence can be expected on this crude grid, the graph does show that controlling β produces more plausible structure functions, in particular structure functions that do not tend to a δ function, as one would get in an equipartition ensemble. The number of Monte-Carlo steps needed for convergence is large, and a finer grid could not be used in the present state of our algorithm.

One point we can hope to clarify with the model we have is whether the higher moments of the velocity field remain bounded- a major issue in the scaling of the local structure of turbulence. We have deduced in [1,2,3], from scaling and heuristic arguments, that the moments of order $p \leq 3$ of the velocity field (and in the case of wall-bounded turbulence, the mean of the velocity gradient) can be obtained by an expansion about a vanishing viscosity limit (in powers of $\frac{1}{\log Re}$, where Re is the Reynolds number. We conjectured that the reason this apparently could not be done for higher moments was that the higher order moments, of order $p \geq 4$, diverged as the visosity tended to zero. Heuristic argument why this should be so were given in [1,2,3,6]. If this conecjture is true, the Kolmogorov scaling in the local structure of turbulence holds as long as the vanishing viscosity limit exists. The calculations described here, with $N = 16$ and $E_h = 100$, give for $< u_1^2 >$, where u_1 is a component of the velocity $\mathbf{u}$, the value $\langle u_1^2 \rangle = 66$, (a value dictated by the imposed E_h); however, $\langle u_1^4 \rangle = 1.3 \cdot 10^5$, $\langle u_1^6 \rangle = 4.2 \cdot 10^7$, with a standard deviation of 10%. The odd moments are zero up to a few standard deviations; however, one should note that for the fifth moment the standard deviation is large, of order 10^5, reflecting the very large value of the tenth moment. These values change little with lattice size N, as one expects from the analysis in [1,2,3] which shows that a significant change in the moments requires a significant change in $\log N$. (Note that an analogous argument applies to experimental data; a variation of the higher moments as the viscosity changes may be very hard to detect). There preliminary results are consistent with our earlier conjectures.

6 Acknowledgements

We would like to thank Prof. P. Colella, Dr. A. Kast and Dr. R. Kupferman for helpful discussions and comments.

References

1. G.I. Barenblatt and A.J. Chorin, Scaling laws and vanishing viscosity limits for wall-bounded shear flows and for local structure in developed turbulence, *Comm. Pure Appl. Math.* **50**, 381–391 (1997).

2. G.I. Barenblatt and A.J. Chorin, New perspectives in turbulence: Scaling laws, asymptotics and intermittency, *SIAM Review* **40**, 265–291 (1998)

3. G.I. Barenblatt, A.J. Chorin, and V.M. Prostokishin, Scaling laws in fully developed turbulent pipe flow, *Appl. Mech. Rev.*, **50**, 413–429, (1997).

4. T. Buttke and A. Chorin, Turbulence calculations in magnetization variables, *Appl. Num. Math.* **12**, 47–54 (1993)

5. A. J. Chorin, *Vorticity and Turbulence*, Springer, 1994.

6. A.J. Chorin, New perspectives in turbulence, *Quart. Appl. Math.,* 1998, in press.

7. H. Gould and J. Tobochnik, *An Introduction to Computer Simulation Methods, Applications to Physical Systems*, part 2, Addison-Wesley, Reading, 1988.

8. L. Landau and E. Lifshitz, *Statistical Physics*, Vol. 1, Pergamon Press, Oxford, 1980, pp. 368–389.

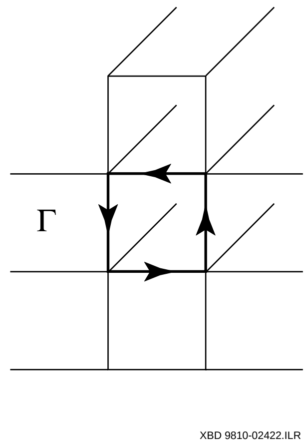

Figure 1: A basic vortex loop.

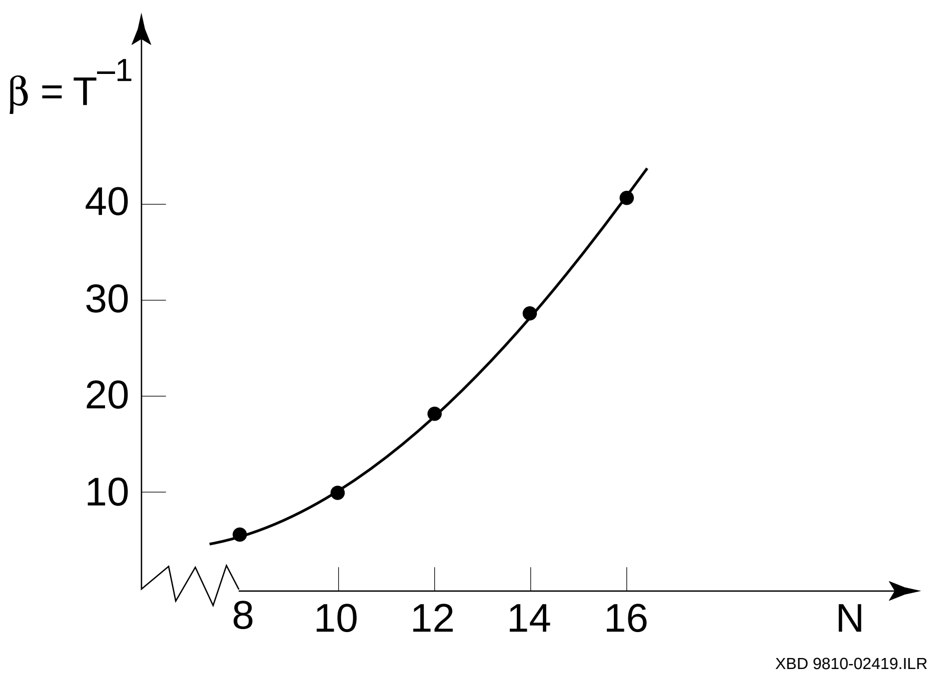

Figure 2: The variation of the inverse temperature β with N in the absence of vorticity constraints.

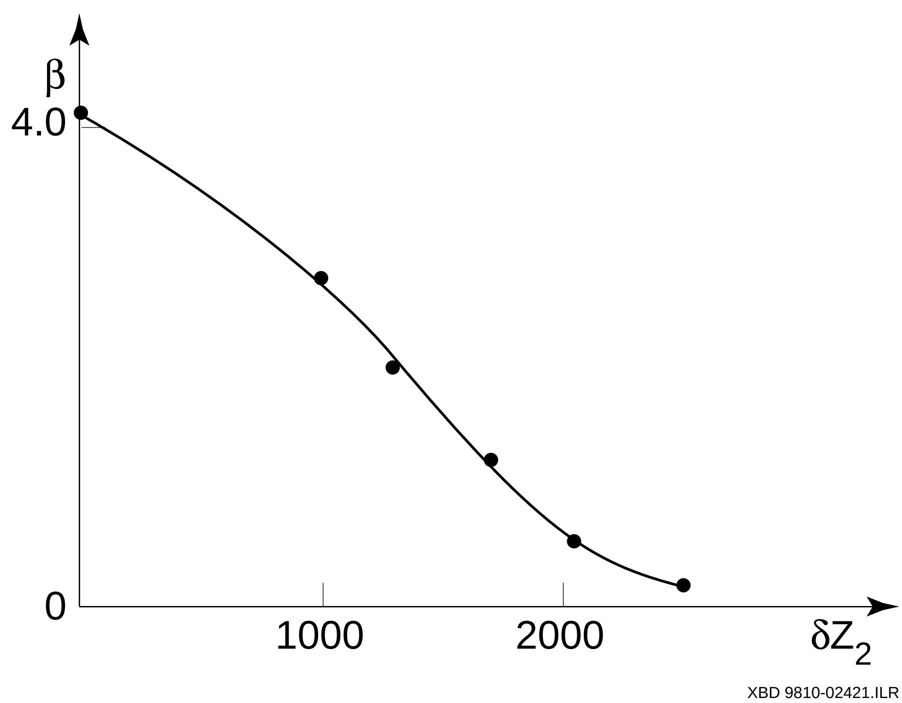

Figure 3: The increase of the temperature as the bound on the enstrophy is decreased (The notation is explained in the text).

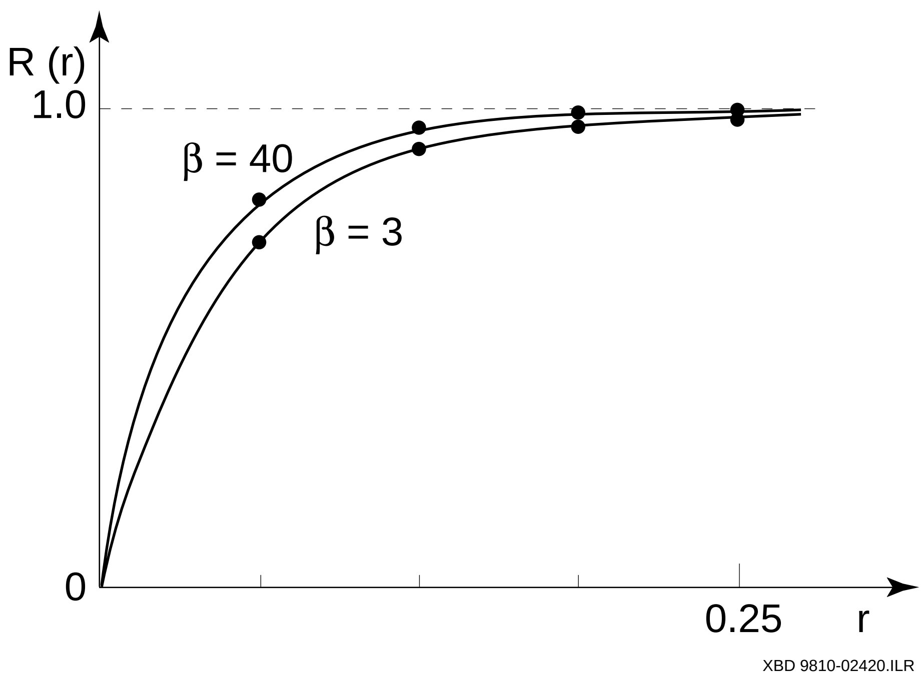

Figure 4: Scaled structure function $R(r) = C\langle(u(\mathbf{x}+\mathbf{r}) - u(\mathbf{x}))^2\rangle$, $C = 4/(3\langle u^2(\mathbf{x})\rangle$, $r = |\mathbf{r}|$, for $N = 16$, with $\beta = 3$ and with $\beta = 40.3$ (Small β corresponds to a large temperature T and requires a small bound on the enstrophy Z_2).