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Simulation of Heat–Vortex Interaction by the Diffusion Velocity Method

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Abstract

A new vortex scheme for simulating flows involving natural convection and interaction of temperature and vorticity is presented. The creation of vorticity from temperature is modeled either by creating a vortex pair from a temperature particle or by changing the strength of vortices according to the vorticity equation. The diffusion velocity method is used for simulating the diffusion of vorticity and temperature. The vortices of negative and positive strength are separately treated in diffusion process to avoid an unreasonably large diffusion velocity. Our results indicates that these techniques successfully simulate creation of vorticity from heat, diffusion and convection of temperature and vorticity, and interaction of them.

I. Introduction

The vortex methods have been applied to a variety of physical flows such as vortex sheets, shear layers, external and internal flows, or reactive flows[1,2]. In this paper, we present a new vortex scheme for simulating flows involving interaction of temperature and vorticity.

In this flow, vorticity is created from heat, and both the vorticity and the heat are transported with the convection velocity generated from the vorticity while each diffuses with a different diffusion coefficient. The extension of the vortex methods to the heat transfer problem requires models for

1. the representation of temperature/heat with particles,
2. the creation of vorticity from heat, and
3. the diffusion of heat and vorticity.

Ghoniem and Sherman[3] investigated one-dimensional and quasi-one-dimensional diffusion using temperature elements for representing temperature and random walks for diffusion. The creation of vortex from the temperature is taken into account. Also Ghoniem et al[4,5] studied shear layer and plume rise using the core spreading method for diffusion. The vortex strength is updated by the transport element method in which scalar gradients are used in the transport process. Smith and Stansby[6] treated one– and two–dimensional flows using temperature particles and the random walks. The creation of vorticity from heat is not considered.

In this paper, the temperature particles are used, and two models for vortex creation are presented. The first model is based on the direct interpretation of the vorticity equation and is considered to be an extension of the scheme adopted by Ghoniem and Sherman[3] to two-dimension. The second model is a new idea that one temperature particle creates one vortex pair (two vortices with the strength Γ and $-\Gamma$).

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For treating the diffusion, a deterministic Lagrangian technique based on a new concept is employed. Here let us imagine many particles, provided with a positive charge, floating in the air. They would exert repulsive force on each other according to the Coulomb's law and consequently they spread. It looks like diffusion but obviously it is not. However, with an appropriate acceleration, velocity, or else, the particles would spread in the way that the density of the particles satisfies the diffusion equation. This idea was realized by the concept of diffusion velocity[7,8].

This velocity is defined in order that the vorticity is conserved in the transfer of diffusion process as it is so in the convection process. Ogami and Akamatsu introduced this concept of the diffusion velocity and presented it as *the diffusion velocity method* [7,8] (the term *the diffusion velocity* is given in contrast with the convection velocity). With this method, the diffusion equation ($Re = 0$) , the boundary layer and two-dimensional flows around a circular cylinder ($Re = 40$ and 1200) are successfully treated. This method is also applied to a circular cylinder ($Re = 0.1 \sim 10^7$)[9], an aerofoil[10], the Burgers equation and the equations for a compressible fluid[11].

Prior to Ogami and Akamatsu, a similar Lagrangian method was presented by Fronteau and Combis[12] to solve Fokker–Planck equations. The concept of the diffusion velocity, however, was not given, and the velocity they used was a combination of the convection and the diffusion. It seems that they introduced their velocity as a mathematical artifice rather than a physical concept.

Regarding the diffusion velocity method, one serious problem was pointed out by Clarke and Tutty[14] that the diffusion is limited to regions where the vortices are overlapped. On the other hand, Kempka and Strickland[13] indicated that the core radius of the vortex has to vary with time because the diffusion velocity field is non-solenoidal. This can be an answer to the problem mentioned above. However, obtaining the core radius of each vortex which gives a smooth density distribution in multi–dimensions is not as easy as in one–dimensional case because the unevenness of the vortex distribution is more serious.

The *core* of the problem given by Clarke and Tutty is considered that the strength/circulation of each vortex is too large to represent the regions of small vorticity. Therefore, another solution to this problem would be to divide a vortex into multi vortices. For this purpose, we use the re–griding (re–meshing) technique. As is well known, by this technique, the particles in a segment (in one–dimension) or in a square region (in two–dimension) are divided, merged and re–positioned at the knots of the segment or at the corners of the square region resulting in a decrease of the particle number. It should be noted that this technique also works for portioning a vortex in regions where vortices are sparse and more vortices are required to make them overlapped and continue to diffuse.

In order to test our scheme, a boundary condition problem in one–dimension where the temperature of a wall is kept constant, and a two–dimensional problem with no boundary are considered.

II. Governing Equations and Lagrangian Scheme

We consider the vorticity equation, the energy equation and the continuity equation in two dimension to briefly explain our method. The motion law of our Lagrangian scheme is found if these equations are put into a conservation form as

$$\frac{\partial \omega}{\partial t} + \mathrm{div}(\omega \boldsymbol{u}_c + \omega \boldsymbol{u}_\omega) = -g\beta \frac{\partial T}{\partial x} \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathrm{div}(T \boldsymbol{u}_c + T \boldsymbol{u}_T) = 0 \tag{2}$$

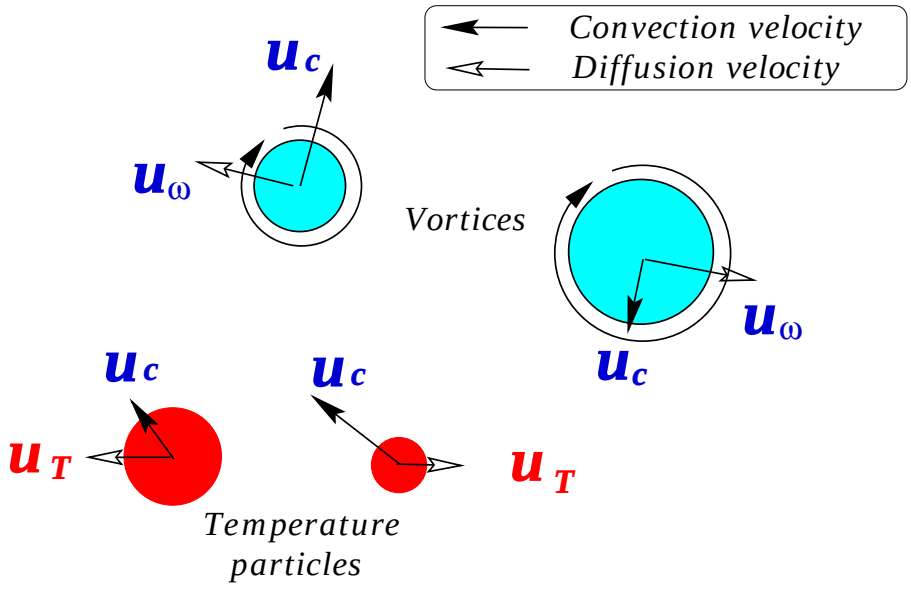

Figure 1: The convection velocity and the diffusion velocity

where

$$\boldsymbol{u}_c = (u, v) \tag{3}$$

$$\boldsymbol{u}_\omega = -\frac{\nu}{\omega}\nabla\omega \tag{4}$$

$$\boldsymbol{u}_T = -\frac{\alpha}{T}\nabla T \tag{5}$$

ω, T, ν, g, β and α are the vorticity, the temperature, the kinematic viscosity, the gravitational acceleration, the modulus of compressibility and the thermal diffusivity. u and v are the velocity components in x and y directions.

Equation (1) states that the vorticity, ω, moves both with the convection velocity, $\boldsymbol{u}_c$, induced by the vorticity (the Biot-Savart law), and with the diffusion velocity, $\boldsymbol{u}_\omega$, defined by Eq.(4)[7,8] (Fig.1). It also indicates that the strength of vorticity varies according to the right hand side of this equation. Similarly, Eq.(2) gives the law that the temperature distribution, T, moves both with the convection velocity, $\boldsymbol{u}_c$, and with the diffusion velocity, $\boldsymbol{u}_T$, defined by Eq.(5) (Fig.1).

One may be concerned about what happens if at the same position the gradient of vorticity or temperature is not zero but the value of it is zero. This would cause a prohibitively large diffusion velocity. To avoid this, the positive particles and the negative particles are separately treated when calculating the diffusion velocity. This may be allowed because the diffusion process is linear.

Incidentally, if the left hand side of Eq.(4) is not a velocity but an acceleration, we can treat a compressible fluid motion[11] .

As usual, the vorticity, $\omega(\boldsymbol{x})$, is expressed by a summation of Gaussian-cored vortices, located at

$\boldsymbol{x}_{\omega j}$, with the strength Γ_j and the core radius $\sigma_{\omega j}$ as

$$\omega(\boldsymbol{x}) = \sum_j \frac{\Gamma_j}{\pi {\sigma_{\omega j}}^2} \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{x}_{\omega j}|^2}{{\sigma_{\omega j}}^2}\right) \tag{6}$$

As mentioned before, we use temperature particles to calculate the temperature distribution as

$$T(\boldsymbol{x}) = \sum_j \frac{\Theta_j}{\pi {\sigma_{Tj}}^2} \exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{x}_{Tj}|^2}{{\sigma_{Tj}}^2}\right) \tag{7}$$

where Θ_j $[K{\cdot}L^2]$, σ_{Tj} and $\boldsymbol{x}_{Tj}$ are the strength, the core radius and the location of the temperature particle.

Finally, the Lagrangian scheme for the heat–vortex interaction is given by the ordinary differential equations which determine the center point of ith vortex, $\boldsymbol{x}_{\omega i}$,

$$\frac{d\boldsymbol{x}_{\omega i}(t)}{dt} = \boldsymbol{u}_c(\boldsymbol{x}_{\omega i}, t) + \boldsymbol{u}_\omega(\boldsymbol{x}_{\omega i}, t) \tag{8}$$

and that of ith temperature particle, $\boldsymbol{x}_{Ti}$,

$$\frac{d\boldsymbol{x}_{Ti}(t)}{dt} = \boldsymbol{u}_c(\boldsymbol{x}_{Ti}, t) + \boldsymbol{u}_T(\boldsymbol{x}_{Ti}, t) \tag{9}$$

III. Creation of Vortex from Heat

The creation of vorticity from heat may be modelled in the following two ways.

Model 1 The strength of a vortex is increased at each time step Δt by the amount of $-g\beta\frac{\partial T}{\partial x}\Delta t \Delta S$, where ΔS the region the vortex occupies (Fig.2). This is a direct interpretation of Eq.(1), and is considered to be an extension of the scheme adopted by Ghoniem and Sherman[3] to two-dimension. This model may be suitable for the regions where both temperature particles and vortices exist. This is because in these regions new vortices need not to be created but the strength of the existent vortices have to be updated.

Model 2 One temperature particle creates one vortex pair at each time step. This idea is based on the fact that the slope of a temperature particle, $T_x = \partial T/\partial x$ (dashed line in Fig.3a), is precisely approximated by the density distribution of two Gaussian particles with opposite strength (the mark $\circ$ in Fig.3a): the first is a Gaussian particle with strength γ located on one side of the temperature particle with a certain horizontal distance apart and the second is a Gaussian particle with strength $-\gamma$ located on the opposite side of it. The relative error of this approximation is almost 0.3% when the strength of the particles is $1.058\Theta/\sigma_T$, the core radius is $0.930\sigma_T$ and the distance from the center of the temperature particle is $0.4385\sigma_T$ (these figures are chosen simply by a trial-and-error method so that this relative error will be lessened with better choice of parameters). Therefore, the strength of the vortex pair created from a temperature particle at each time step is $1.058g\beta\Delta t\Theta/\sigma_T$. Consequently, the buoyancy, which is one of the primary phenomena of heat transfer, is regarded as the acceleration (the unsteady convection velocity) caused on the temperature particles by the vortex pairs (Fig.3b). This vortex creation model may be suitable for the regions where only temperature particles exist and new vortices have to be created. This happens when the initial heat is applied to the fluid or temperature particles spread faster and wider than vortices ($\nu < \alpha$).

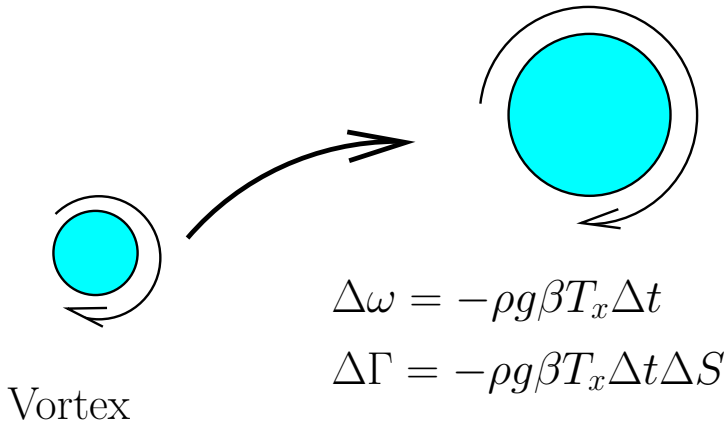

Figure 2: Vortex creation model 1

Also we need to adopt the re–griding technique to keep the vortex number from becoming too large.

IV. Test Problem and Numerical Method

To test the vortex creation model and the diffusion velocity method, we consider the following simple one-dimensional equations and compare the results with the analytical solutions.

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial x^2} - g\beta \frac{\partial T}{\partial x}, \qquad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad \frac{\partial v}{\partial y} = 0, \qquad u = 0 \tag{10}$$

The boundary condition problem where the temperature at the wall ($x = 0$) is kept constant is considered, that is

$$\begin{cases} T = 0°\mathrm{C} \text{ and } \omega = 0 \ \text{ at } \ x > 0 \ \text{ when } \ t = 0 \\ T = 20°\mathrm{C} \ \text{ at } \ x = 0 \ \text{ when } \ t \geq 0 \end{cases}$$

The temperature particles are created one by one near the wall at every time step, and the strength of each particle is given by the flux of heat,

$$\Theta = T u_T \Delta t = -\alpha \Delta t \frac{\partial T}{\partial x}\bigg|_{x=\delta} \tag{11}$$

where Eq.(5) has been employed and $\delta = \sqrt{\alpha \Delta t}$ is the position of the particles. The core radius of the particle, $\sigma_T = 2\sqrt{\alpha \Delta t}$, varies with time because the diffusion velocity field is non-solenoidal[13]. The derivative, $\partial T/\partial x$, is calculated from T created by the temperature particles, their image particles and the boundary temperature T_B. Namely, the temperature T is expressed by

$$T(x) = \sum_j \frac{\Theta_j}{\sqrt{\pi}\sigma_{Tj}} \exp\left(-\frac{(x - x_{Tj})^2}{{\sigma_{Tj}}^2}\right) - \sum_j \frac{\Theta_j}{\sqrt{\pi}\sigma_{Tj}} \exp\left(-\frac{(x + x_{Tj})^2}{{\sigma_{Tj}}^2}\right) + T_B \tag{12}$$

where T_B, shown below, is a practical substitution for the given boundary condition which has discontinuity.

$$T_B = T_0 \mathrm{erfc} \frac{x}{2\sqrt{\alpha \Delta t}} \tag{13}$$

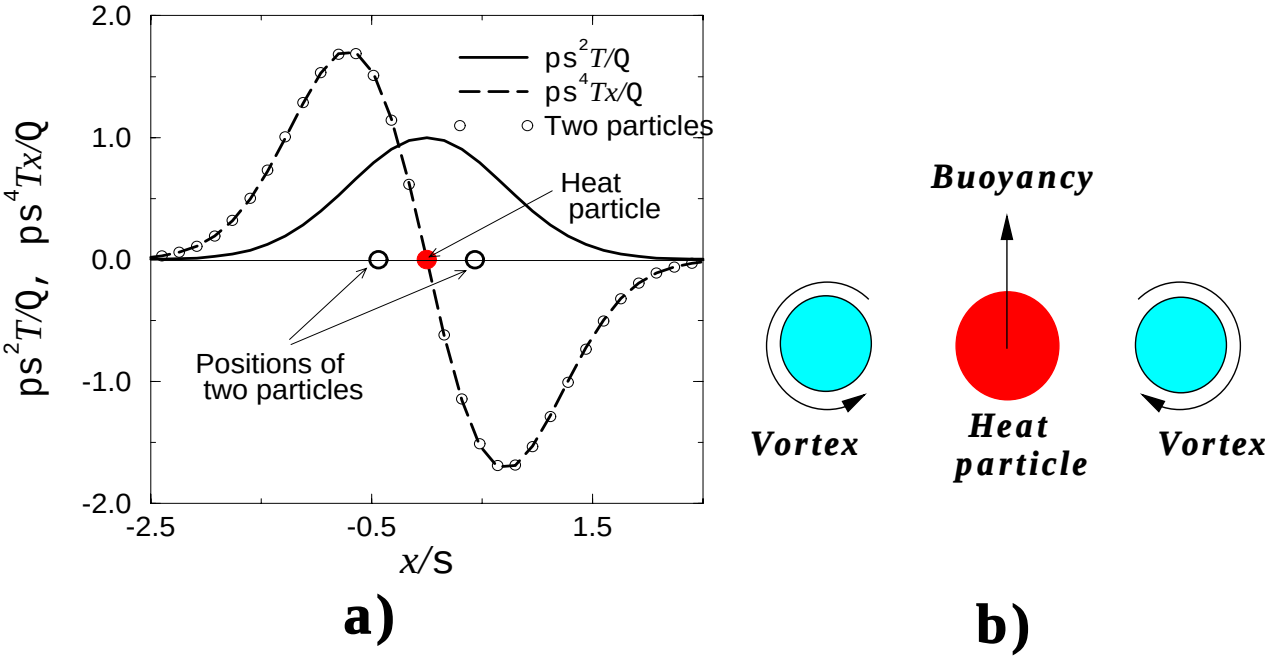

Figure 3: Vortex creation model 2: a) the slope of a temperature particle is approximated by two particles with opposite strength; and b) buoyancy is caused by a vortex pair.

The vortices are created by Model 2 stated in the previous section. The vorticity is calculated by the vortices and their image vortices as

$$\omega(x) = \sum_j \frac{\Gamma_j}{\sqrt{\pi}\sigma_{\omega j}} \exp\left(-\frac{(x - x_{\omega j})^2}{{\sigma_{\omega j}}^2}\right) + \sum_j \frac{\Gamma_j}{\sqrt{\pi}\sigma_{\omega j}} \exp\left(-\frac{(x + x_{\omega j})^2}{{\sigma_{\omega j}}^2}\right) \tag{14}$$

Note that the sign of the image particle for the temperature particle is negative while that for the vortex is positive. Since the particles of the two species move only with the diffusion velocity in this one-dimensional case, the positions of the particles are determined by

$$\frac{dx_{\omega i}(t)}{dt} = u_\omega(x_{\omega i}, t), \qquad \frac{dx_{Ti}(t)}{dt} = u_T(x_{Ti}, t) \tag{15}$$

To keep the vortex number from becoming too large, the vortices in the segment of width $2\sqrt{\alpha\Delta t}$ are merged into one vortex at every 20 steps. Also to prevent the vorticity, in Eq.4, from becoming too small (namely, to prevent the diffusion velocity from becoming too large), the positive vortices and the negative vortices are separately treated when calculating the diffusion velocity. This may be allowed because the diffusion process is linear.

V. Result and Discussion

In the following simulations, the parameters, ν, β and α, are those of air at $T = 20°$C, and dependency on the temperature is not considered for simplicity. The time step Δt is 0.1s.

Figure 4a shows that the temperature particles are created and placed one by one near the wall at every time step, and that they are spreading along x-axis due to the diffusion velocity. Figure 4b

shows excellent agreement between the temperature distributions obtained by these temperature particles and the analytical solutions (the relative error is 0.88%). This indicates that the introduction of the temperature particles are reasonable and that the diffusion velocity technique can treat the diffusion process quite successfully.

Figure 4c and 4d compare the vorticity and the velocity, both calculated by the present method, with the analytical solutions. Agreement is excellent again except the vorticity close to the wall (the relative error of the vorticity is 2.18% and that of the velocity is 0.51%). We may say that our temperature–vortex model is a reasonable representation of the vortex creation process.

Figure 5 shows the vortices of positive strength (a) and of negative strength (b). It can be seen that at times 2 and 4 (at 20 steps and 40 steps) the vortices are merged and redistributed resulting in a decrease of the vortex number (e.g. 210 positive vortices decreases to 47 at time 2, and 636 positive ones to 159 at time 4). Conservation of only the vortex strength is considered in this process.

To demonstrate how the temperature and the vorticity interact with one another, the following initial value problem in two-dimension is considered.

$$\begin{cases} T = 20^\circ\mathrm{C} \ \text{ and } \ \omega = 0 \ \text{ everywhere} \\ \text{except } \ T = 20^\circ\mathrm{C} + \Delta T \ \ (-1\mathrm{cm} \le x \le 1\mathrm{cm}, \ -1\mathrm{cm} \le y \le 1\mathrm{cm}) \end{cases}$$

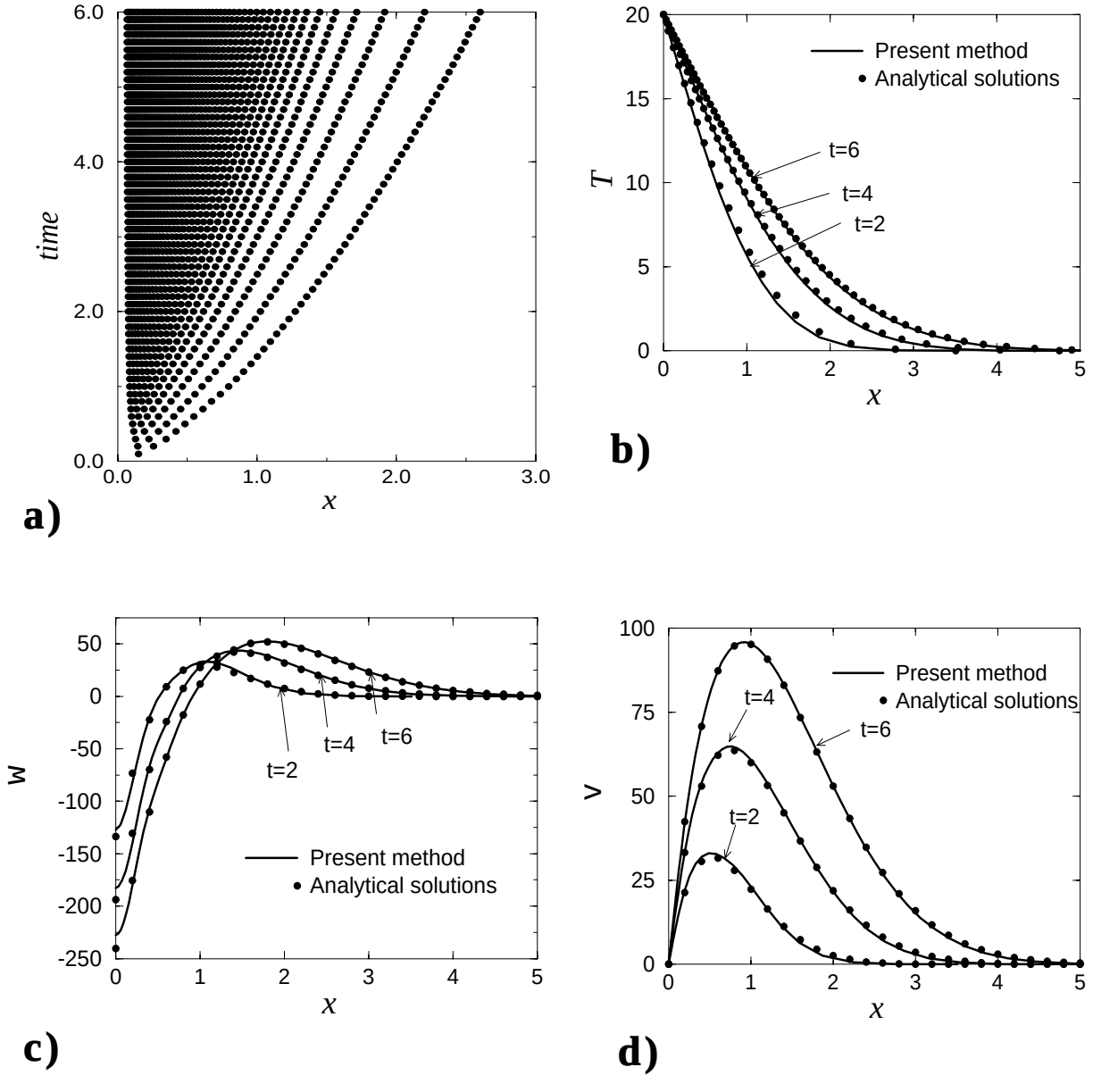

Figure 4: Results of one-dimension: a) temperature particles created near the wall spreading rightward due to diffusion; b) temperature; c) vorticity; and d) velocity.

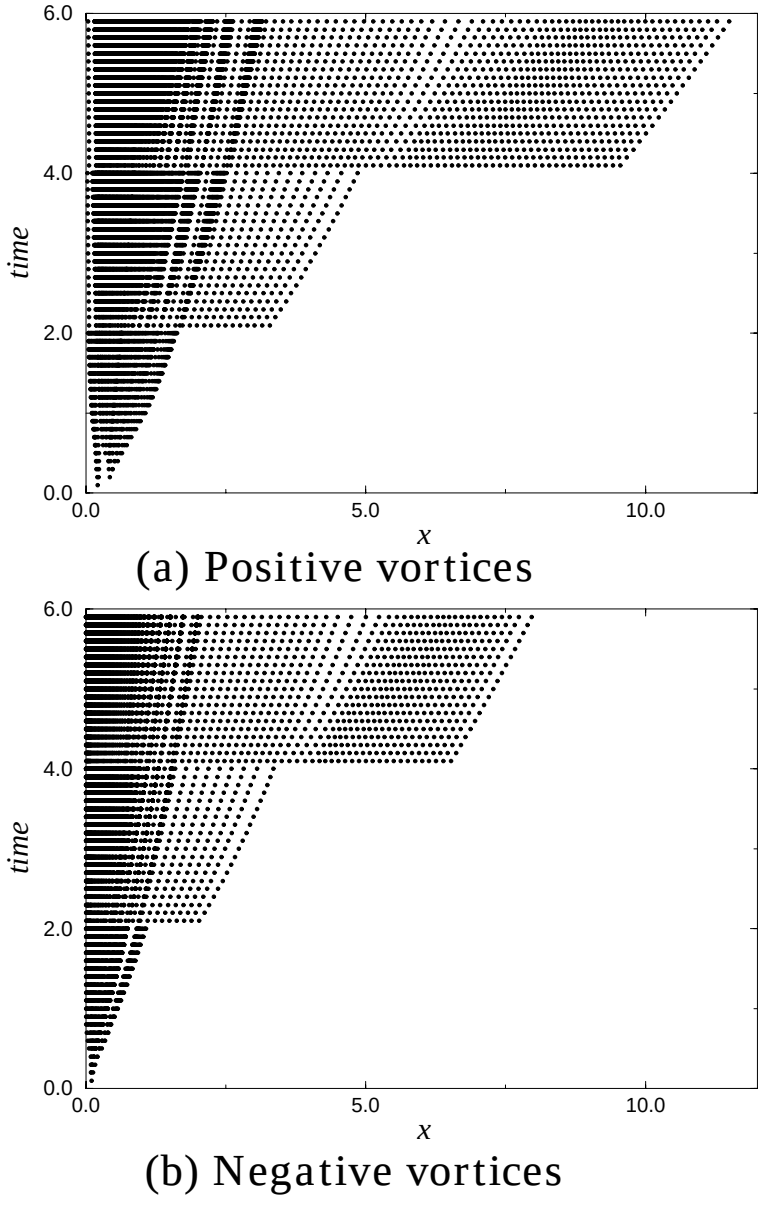

Figure 5: Vortices of positive strength (a) and of negative strength (b)

Equations (8) and (9) including both the convection velocity and the diffusion velocity are solved so that the interaction of the temperature and the vorticity is simulated. The parameters of water are used and dependency on the temperature is considered this time. The time step is 0.0025s. The high temperature region ($-1\text{cm} \leq x \leq 1\text{cm}, \ -1\text{cm} \leq y \leq 1\text{cm}$) is expressed by 30×30 temperature particles. The initial vortices are created by Model 2 (see III) and after this the strength of these vortices are updated by Model 1.

The re–griding is done at every 100 steps. The ratio of the initial particle distance Δx to the particle radius σ is set 0.5 as the overlapping condition.

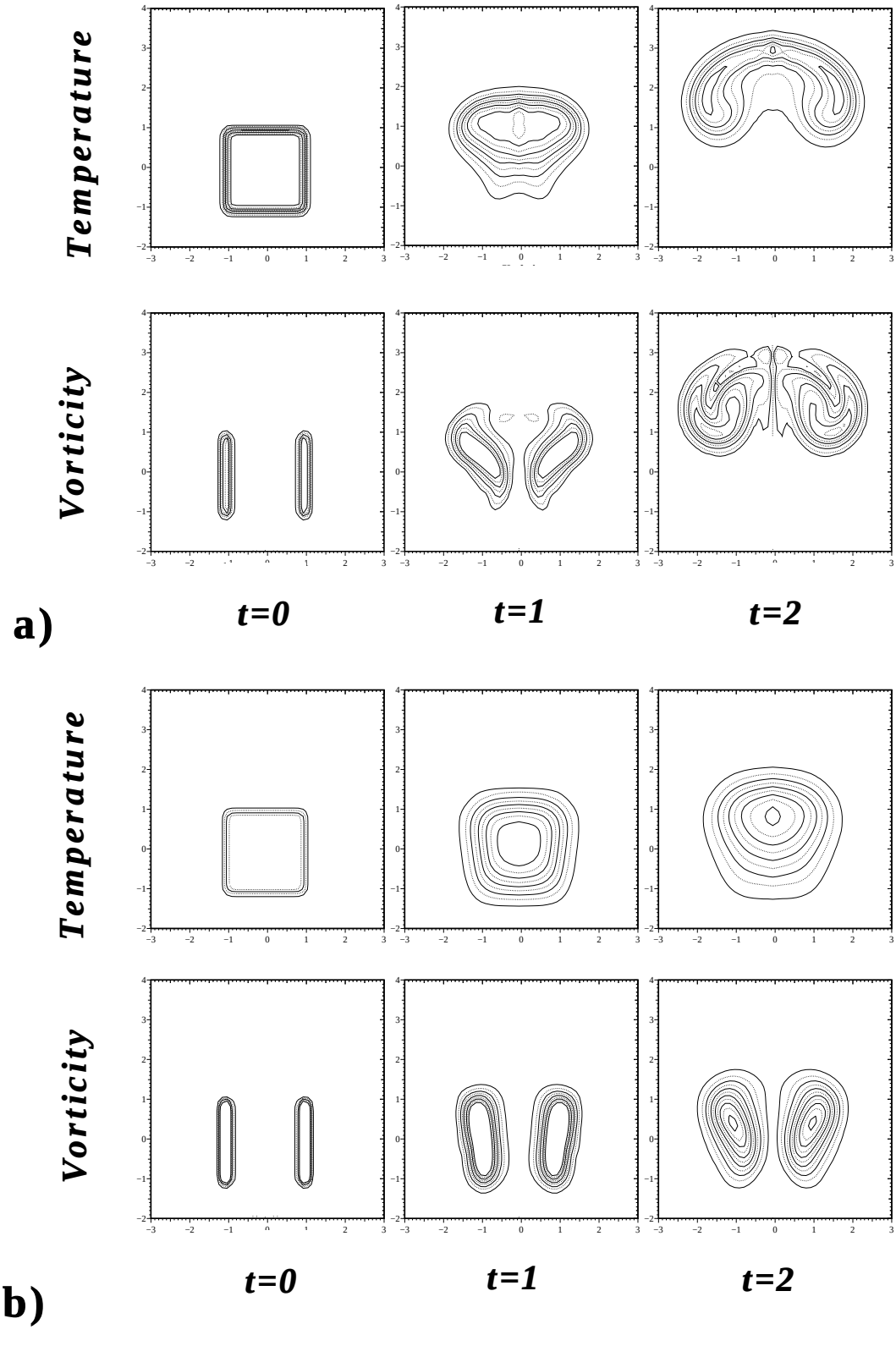

Figure 6: Evolution of temperature distribution and vorticity distribution: a) $\Delta T = 20°\mathrm{C}$; and b) $\Delta T = 5°\mathrm{C}$.

Figure 6a ($\Delta T = 20°$C) and 6b ($\Delta T = 5°$C) show the evolution of the temperature contours and the vorticity contours. At time 0, the two oval vorticity regions, in other words the two large-scale, counter-rotating vortices are created from the initial square temperature. We can see that with the larger initial temperature difference the stronger vortices are created, and thus both the vortices and the temperature particles are moved upper and the shapes of these distributions are changed more significantly.

VI. Conclusion

A new vortex scheme for simulating flows involving interaction of temperature and vorticity is presented. The creation of vorticity from temperature is modelled either by creating a vortex pair from a temperature particle or by changing the strength of vortices. The diffusion velocity method is used for simulating the diffusion of vorticity and temperature. Our results indicates that these techniques bring reasonable solutions. Also it is found that the re–griding technique is useful both for keeping the vortex number from becoming too large, and for portioning a vortex in regions where vortices are sparse. This division compensates for the defect of the diffusion velocity method that the diffusion is limited to regions where the vortices are overlapped. The vortices of negative and positive strength are separately treated in diffusion process to avoid an unreasonably large diffusion velocity.

References

1 Sarpkaya, T.: Computational Methods with Vortices — the 1988 Freeman Scholar Lecture. J. Fluids Engineering. **111** (1989), 5–52.

2 Sethian, J.I.: A Brief Overview of Vortex Methods. Vortex Methods and Vortex Motion, SIAM. Philadelphia. (1991), 1–32

3 Ghoniem, A. F. and Sherman F. S.: Grid-free Simulation of Diffusion Using Random Walk Methods. J. Comp. Phys. **61** (1985), 1–37.

4 Ghoniem, A. F., Heidarinejad, G. and Krishan A.: Numerical Simulation of a Thermally Stratified Shear Layer Using the Vortex Element Method. J. Comp. Phys. **79** (1988), 135–166.

5 Zhang X. and Ghoniem, A. F.: A Computational Model for the Rise and Dispersion of Wind-Blown, Buoyancy-Driven Plumes – I. Neutrally Stratified Atmosphere. Atmospheric Environ. **27A** (1993), 2295–2311.

6 Smith P. A. and Stansby P. K.: An Efficient Surface Algorithm for Random-Particle Simulation of Vorticity and Heat Transport. J. Comp. Phys. **81** (1989), 349–371.

7 Ogami, Y and Akamatsu, T.: Numerical Study of Viscous Flow around a Circular Cylinder by Vortex Method with Diffusion–Velocity (in Japanese). 1st Symposium of Computational Fluid Dynamics. Tokyo. (1987), 279–282.

8 Ogami, Y. and Akamatsu, T.: Viscous Flow Simulation using the Discrete Vortex Method — the Diffusion Velocity Method. Computers & Fluids. **19**. (1991), 433–441.

9 Ogami Y.. and Ayano Y.: Flows around a Circular Cylinder Simulated by the Viscous Vortex Model – the Diffusion Velocity Method. Computational Fluid Dynamics JOURNAL. **4**. (1995), 383–399.

10 Shintani M. and Akamatsu T.: Investigation of Two–Dimensional Discrete Vortex Method with Viscous Diffusion Model. Computational Fluid Dynamics JOURNAL. **3**. (1994), 237–254.

11 Ogami, Y. and Cheer A.: Simulations of Unsteady Compressible Fluid Motion by an Interactive Cored Particle Method. SIAM J. Appl. Math. **55**. (1995), 1204–1226.

12 Fronteau, J. and Combis P.: A Lie-Admissible Method of Integration of Fokker–Planck Equations with Non–Linear Coefficients (Exact and Numerical Solutions). Hadronic J. **7**. (1984), 911–930.

13 Kempka, S.N. and Strickland, J.H.: A Method to Simulate Viscous Diffusion of Vorticity by Convective Transport of Vortices at a Non-Solenoidal Velocity. SAND93-1763. UC-700 (1993).

14 Clarke N. R. and Tutty O. R.: Construction and Validation of a Discrete Vortex Method for the Two-Dimensional Incompressible Navier–Stokes Equations. Computers & Fluids. **23**. (1994), 751–783.