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Asymptotic behavior of coupled Euler-Bernoulli beams with dissipative joint

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Abstract

We consider an example of coupled Euler-Bernoulli beams with dissipative joint. We show that the exponential decay property holds for any position of the joint.

1 Introduction

In this paper we discuss the exponential stability of coupled Euler-Bernoulli beams with pointwise dissipation at the common end.

The equations considered in this paper are:

$$\partial_t^2 u + \Delta^2 u + \partial_t u(\xi,t)\,\delta_\xi - \partial_{xt}^2 u(\xi,t)\,\frac{d\delta_\xi}{dx} = 0 \quad \text{in } Q, \tag{1}$$

$$u = \Delta u = 0 \quad \text{on } \Sigma, \tag{2}$$

$$u(x,0) = u^0(x), \partial_t u(x,0) = u^1(x) \quad \text{in } \Omega, \tag{3}$$

where $Q = \Omega \times (0,\infty), \Omega = (0,1), \Sigma = \left\{0,1\right\} \times (0,+\infty)$, $\Delta = \partial_x^2$ and δ_ξ is the Dirac mass at the point $\xi \in \Omega$.
Equation (1) governs vibrations of coupled Euler-Bernoulli beams with a dissipative joint. The dissipation acts via the bending moments (i.e Δu) which is proportional to the angular velocity $\partial_{xt}^2 u(\xi,t)$ at the point ξ and by the shear force $\partial_x^3 u(\xi,t)$ which is proportional to the velocity $\partial_t u(\xi,t)$ at the joint ξ.

As far as we known this type of problem was not studied, in the case of feedbacks acting simultaneously in bending moments and in shear forces. In the case of only one feedback the problem was studied in [8].

Our main result is the exponential stability of the above system for any position ξ of the joint. To do so, we use a method similar to the one in [7],[8] and [6], by proving that the resolvent remains bounded on the imaginary axis. To show this result, we shall verify, following [8], that the transfer function (see [12]), which relates the input to the output, is bounded on the imaginary axis and on the other hand, that our system is stable and detectable.

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A different method for proving this result is used in [1] where the variable coefficients case is also tackled.

Now if we write a variational formulation of the problem (1)-(3), we notice that (1)-(3) can be reformulated in the following form:

$$\partial_t^2 u + \Delta^2 u = 0 \quad \text{in } Q_\xi, \tag{4}$$

$$u = \Delta u = 0 \quad \text{on } \Sigma, \tag{5}$$

$$[u]_\xi = [\partial_x u]_\xi = 0 \quad \text{in } (0, \infty), \tag{6}$$

$$[\partial_x^2 u]_\xi = \partial_{xt}^2 u(\xi, t) \quad \text{in } (0, \infty), \tag{7}$$

$$[\partial_x^3 u]_\xi = -\partial_t u(\xi, t) \quad \text{in } (0, \infty), \tag{8}$$

$$u(x, 0) = u^0(x), \partial_t u(x, 0) = u^1(x) \quad \text{in } \Omega, \tag{9}$$

where $Q_\xi = (\Omega \setminus \xi) \times (0, \infty)$ and $[f]_\xi = f(\xi^+) - f(\xi^-)$.

In order to prove existence and uniqueness of solutions of $(1) - (3)$, it is enough to apply standard semigroup techniques (see [2]). The following holds:

Proposition 1.1 *Let*

$$(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega).$$

Then the equations $(1) - (3)$ *admit a unique solution*

$$u \in C([0, \infty), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty), L^2(\Omega)).$$

Furthermore, if

$$(u^0, u^1) \in (H^4(\Omega \setminus \xi) \cap (H^2(\Omega) \cap H_0^1(\Omega))) \times (H^2(\Omega) \cap H_0^1(\Omega)),$$

then $(1) - (3)$ *admit a unique solution*

$$u \in C([0, \infty), H^4(\Omega \setminus \xi) \cap (H^2(\Omega) \cap H_0^1(\Omega))) \cap C^1([0, \infty), H^2(\Omega) \cap H_0^1(\Omega)),$$

where

$$H^4(\Omega \setminus \xi) = \Big\{ u \ \text{ such that } u_{|(0,\xi)} \in H^4(0, \xi),\ u_{|(\xi,1)} \in H^4(\xi, 1) \Big\}.$$

2 Exponential stability

Before stating the exponential stability result of our system, we prove a strong stability result of solutions of $(1) - (3)$. More precisely, the following proposition holds:

Proposition 2.1 *For any* $\xi \in \Omega$, $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ *the solutions of* $(1) - (3)$ *satisfy:*

$$\lim_{t \to \infty} E(t, u) = 0,$$

where

$$E(t, u) = \int_\Omega \Big\{ (\Delta u)^2 + (\partial_t u)^2 \Big\}\, dx.$$

Proof. As the imbedding of the domain of the infinitesimal generator of the evolution equation in the energy space $[H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ is obviously compact , the strong stability estimate can be obtained by a simple application of Lasalle's invariance principale. This is why we skip here the proof and we refer, for instance, to [3].

Next, we state and prove the main result of the present paper.

Theorem 2.1 *For any* $\xi \in \Omega$, *there exist* $C, \gamma > 0$ *such that for any solution* u *of* (4) – (9) *in* $C([0,\infty), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty), L^2(\Omega))$, *we have:*

$$E(t,u) \leq C\, e^{-\gamma t} E(0,u), \; t > 0. \tag{10}$$

The proof of this theorem is based on crucial arguments given by Rebarber in [8] and also by [4],[10] and [11].

Proof. Let the space X be defined by:

$$X = \begin{array}{c} [H^2(\Omega) \cap H_0^1(\Omega)] \\ \times \\ L^2(\Omega) \end{array},$$

and equipped with the norm:

$$||(u,v)||_X = \Big(\int_\Omega \Big\{ (\Delta\, u)^2 + v^2 \Big\}\, dx \Big)^{1/2}.$$

Let the operator A_0 be defined on X by:

$$A_0 = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix},$$

with

$$D(A_0) = \Big\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{array}{c} H^4(\Omega \setminus \xi) \cap H^2(\Omega) \\ \times \\ H^2(\Omega) \end{array}, u = v = \Delta\, u = 0 \quad \text{on } \partial\Omega,$$
$$[\tfrac{d^3u}{dx^3}]_\xi = \tfrac{dv}{dx}(\xi) = 0 \Big\},$$

note that the spectrum of A_0 is on the imaginary axis.

We also need to introduce the operator A_1 defined on X by:

$$A_1 = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix}, \; x \neq \xi$$

with

$$D(A_1) = \Big\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \begin{array}{c} H^4(\Omega \setminus \xi) \cap H^2(\Omega) \\ \times \\ H^2(\Omega) \end{array}, u = v = \Delta u = 0,$$
$$\textit{on } \partial\Omega,\; [\tfrac{d^3u}{dx^3}]_\xi = -v(\xi),\; [\tfrac{d^2u}{dx^2}]_\xi = \tfrac{dv}{dx}(\xi) \Big\}.$$

Let $B_1, B_2 \in \mathcal{L}(\mathbb{R}, (D(A^*))')$ be two feedback operators given by:

$$B_1 = \begin{pmatrix} 0 \\ -\delta_\xi \end{pmatrix} \; and \; B_2 = \begin{pmatrix} \delta''(. - \xi^+) - \delta''(. - \xi^-) \\ 0 \end{pmatrix},$$

where A^* is the adjoint of A and $(D(A^*))'$ is the dual of $D(A^*)$ obtained by means of inner product in $L^2(\Omega)$.

Since A_0 is anti-adjoint then for any $(u, v) \in D(A_0^*) = D(A_0)$ we have:

$$B_1^* \begin{pmatrix} u \\ v \end{pmatrix} = -v(\xi), \; B_2^* \begin{pmatrix} u \\ v \end{pmatrix} = \frac{d^2u}{dx^2}(\xi^+) - \frac{d^2u}{dx^2}(\xi^-).$$

Now we shall introduce the controlled and the observed systems. The controlled system is given by:

$$\partial_t^2 u + \Delta^2 u = 0 \quad \text{in } Q_\xi, \tag{11}$$

$$u = \Delta u = 0 \quad \text{on } \Sigma, \tag{12}$$

$$[u]_\xi = [\partial_x u]_\xi = 0 \quad \text{in } (0, \infty), \tag{13}$$

$$w_1(t) = [\partial_x^3 u]_\xi, \; w_2(t) = \partial_{xt}^2 u(\xi, t). \tag{14}$$

The observed system is the system defined by:

$$\partial_t^2 u + \Delta^2 u = 0 \quad \text{in } Q_\xi, \tag{15}$$

$$u = \Delta u = 0 \quad \text{on } \Sigma, \tag{16}$$

$$[u]_\xi = [\partial_x u]_\xi = 0 \quad \text{in } (0, \infty), \tag{17}$$

$$V_1(t) = C_1 U(t) = -\partial_t u(\xi, t), \; V_2(t) = C_2 U(t) = [\partial_x^2 u]_\xi, \tag{18}$$

where $C_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ is the observation operator, $B_0 = (B_1, B_2)$ is the control operator and

$$U = \begin{pmatrix} u \\ u' \end{pmatrix}.$$

We also introduce the controlled-observed system (C, A, B) given by:

$$U'(t) = A_0 U(t) + B_1 w_1(t) + B_2 w_2(t),$$
$$V_1(t) = C_1 U(t), V_2(t) = C_2 U(t).$$

This system is given by the following schem:

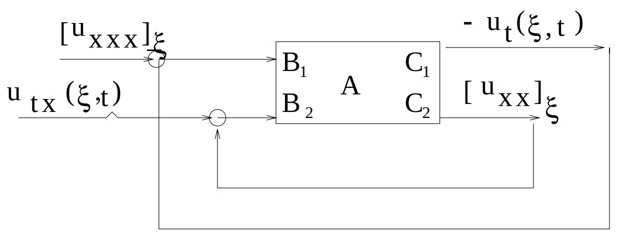

Figure 1 : *The controlled-observed system*

Theorem 2.2 *Assume that the system* (C, A, B) *is regular,* (A, B) *is stable and* (C, A) *is detectable. Then if* $G(\lambda)$*, the transfert function related to the system* (C, A, B)*, is bounded on* C_0^+*,* $R(\lambda, A)$ *is bounded on* C_0^+*. This implies that the semi-group generated by* A *is exponentially stable.*

Proof. We refer to [9] for the proof.

Theorem 2.1 is a consequence of the above theorem: since $R(\lambda, A_1)$ is bounded on $C_0^+ = \Big\{z \in \mathbb{C}\,,\, \mathrm{Re(z)} > 0\Big\}$, then following [6], inequality (10) holds.

Remark 2.1 *Theorem* 2.2. *is used to prove that* $R(\lambda, A_1)$ *is bounded on* $C_0^+ = \Big\{z \in C,\, \mathrm{Re(z)} > 0\Big\}$*. Since operator* A_1 *is dissipative then it is enough to show that* $R(\lambda, A_1)$ *is bounded on the imaginary axis.*

Before proving that the assumptions in Theorem 2.2 hold for our system we recall some basic definitions concerning regular systems, stable systems, detectable systems and transfer functions.

Definition 2.1 *We say that the system* (C_0, A_0, B_0) *is regular if:*
(1) (B_i) *are bounded operators from* $\mathbb{R}$ *into* X_{-1} *and satisfy the following condition: there exist* $t_1, \alpha > 0$ *such that for any* $w_i \in L^2((0, t_1), \mathbb{R})$*, we have:*

$$|| \int_0^{t_1} S(t-\tau) B_i w_i(\tau)\, d\tau ||_X \leq \alpha ||w_i||_{L^2((0,t_1),\mathbb{R})}\ for\ i = 1, 2.$$

(B_i) *are then called admissible control operators;*
(2) (C_i) *are bounded operators from* $D(A_0)$ *into* $\mathbb{R}$ *satisfying the following: there exist* $t_1, \alpha > 0$ *such that for any* $x \in D(A_0)$*, we have:*

$$||C_i S(t) x||_{L^2([0,t_1),\mathbb{R})} \leq \alpha ||x||_X,\ i = 1, 2.$$

(C_i) *are called admissible observation operators;*
(3) the range of $R(\lambda, A_0) B_0$ *belongs to* $D(C_L)$ *for some* $\lambda \in \rho(A_0)$*;*
(4) $C_{0L} R(\lambda, A_0) B_0$ *is bounded in the half plane*

$$C_\alpha^+ = \Big\{z \in C,\, \mathrm{Re(z)} > \alpha\Big\},$$

for some $\alpha \in \mathbb{R}$*,*
where $S(t) = e^{tA_0}$ *and* C_{0L} *is the Lebesgue extension of* C_0

$$C_{0L} x = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau C_0 S(t) x\, dt$$

for $x \in X$ *where the limit exists in* X*,*
X_{-1} *the closure of* X *in the norm* $||.||_{X_{-1}} = ||(\beta I - A_0)^{-1}.||_X$ *for* β *in the resolvent (the definition of the space* X_{-1} *is independent of* β*, see Weiss [13]).*

Definition 2.2 $G = (G_{ij})_{1 \leq i,j \leq 2}$ *is called the transfer function of the system* $(C_i, A_0, B_j)_{1 \leq i,j \leq 2}$ *if it is given by the following equation:*

$$\widehat{V}_i(\lambda) = G_{ij}^0(\lambda) \widehat{w}_j(\lambda),\ i, j = 1, 2$$

where $\widehat{.}$ *denotes the Laplace transform.*

Definition 2.3 *The system (A_0, B_0) is stable if there exists $F = (F_1, F_2)$ such that*
(1) the system (F, A_0, B_0) is regular;
(2) $(I - G_F)$ is invertible in H^∞_∞;
where G_H is the transfert function related to the system (F, A_0, B_0), H^∞_α is the space of anlytical functions bounded on C^+_α (this is a Hilbert space equipped with the norm L^∞) and

$$H^\infty_\infty = \cup_{\alpha \in \mathbb{R}} H^\infty_\alpha;$$

(3) $A_0 + B_0 F$ is the generator of an exponentially stable semigroup of X.

Definition 2.4 *The system (C_0, A_0) is detectable if there exists $H = (H_1, H_2)$ such that*
(1) (C_0, A_0, H) is regular;
(2) $(I - G_H)$ is invertible in H^∞_∞;
where G_H is the transfert function related to the system (C_0, A_0, H),
(3)$A_0 + HC_{0L}$ is the generator of an exponentially stable semigroup of X exponentiellement stable.

To verify that our system satisfies the assumptions of Theorem 2.2 we need the following technical result:

Lemma 2.1 *(1) Let G^1 be the transfert function of the following system:*

$$\partial_t^2 u + \Delta^2 u = 0 \quad \text{in } Q_\xi, \tag{19}$$

$$u = \Delta u = 0 \quad \text{on } \Sigma, \tag{20}$$

$$[u]_\xi = [\partial_x u]_\xi = 0 \quad \text{in } (0, \infty), \tag{21}$$

$$([\partial_x^3 u]_\xi + \partial_t u(\xi, t), [\partial_x^2 u]_\xi - \partial_{xt}^2 u(\xi, t)) = (q_1(t), q_2(t)) = Q(t), \tag{22}$$

$$C_0 U(t) = (-\partial_t u(\xi, t), [\partial_x^2 u]_\xi). \tag{23}$$

Then

$$\lim_{\lambda \to \infty} G^1(\lambda) = 0, \ \lambda \in \mathbb{R}$$

(2) $A_1 = A_0 - B_1 B_{1L}^* - B_2 B_{2L}^*$.

Before proving this lemma we decompose B_0^* on a basis of eigenvectors of A_0.
Let $(\Phi_w)_w$ be a basis of eigenvectors of A_0 in X,
where

$$\Phi_{\pm w}(x) = C_w^{-2} \begin{pmatrix} \phi_w(x)/\pm i w^2 \\ \phi_w(x) \end{pmatrix},$$

with

$$C_w^{-2} = \int_0^1 \left\{ (\frac{\Delta \phi_w}{w^2})^2 + \phi_w^2 \right\} dx.$$

If we decompose the operator B_0^* we arrive at:

$$B_0^* = (\sum_w b_w^1 \Phi_w^*, \sum_w b_w^2 \Phi_w^*),$$

where

$$b_w = \begin{pmatrix} b_w^1 \\ b_w^2 \end{pmatrix}, \; b_{\pm w} = B_0^* \Phi_{\pm w}(\xi) =: \begin{pmatrix} B_1^* \Phi_{\pm w} \\ B_2^* \Phi_{\pm w} \end{pmatrix}$$

and

$$\Phi_w^* = C_w^{-2}(\phi_w(x)/\pm iw^2, \phi_w(x)).$$

Therefore, we can represent the operator B_0 by the following vector of $X \times X$:

$$[b_w^1, b_w^2]_w.$$

To compute the coefficients b_w we need the following lemma.

Lemma 2.2 *The eigenvalues and eigenvectors of the operator A_0 are given by:*
(1) $\lambda = 0$ is an eigenvalue of A_0 with mutiplicity 1 and its associated eigenvector is $(\phi_0, 0)$
where

$$\phi_0(x) = x^3 + (3\xi^2 - 6\xi + 2)x, \; x \in [0, \xi),$$
$$\phi_0(x) = (x-1)^3 + (3\xi^2 - 1)(x-1), \; x \in (\xi, 1];$$

(2) If ξ satisfies $cos n\pi\xi = 0$ for some $n \in \mathbb{Z}$ then $\lambda = \pm i(n\pi)^2$ are the eigenvalues of A_0, which are with multiplicity 1 and their associated eigenvectors are given by:

$$\Phi_{\pm n}(x) = \begin{pmatrix} \phi_n(x)/\pm i(n\pi)^2 \\ \phi_n(x) \end{pmatrix},$$

where

$$\phi_n(x) = sin(n\pi x), \; x \in [0, 1];$$

(3) Let w be a positive real such that

$$g(w) = th(w\xi) - tg(w\xi) - th[w(\xi - 1)] + tg[w(\xi - 1)] = 0. \tag{24}$$

Then $\lambda = \pm iw^2$ are the eigenvalues of A_0, which are with multiplicity 1 and their associated eigenvectors are given by:

$$\Phi_{\pm w}(x) = \begin{pmatrix} \dfrac{\pm i\phi_w(x)}{w^2\Big\{\xi\Big(\frac{1}{cos^2(w\xi)} - \frac{1}{ch^2(w\xi)}\Big) + (\xi-1)\Big(\frac{1}{ch^2[w(1-\xi)]} - \frac{1}{cos^2[w(1-\xi)]}\Big)\Big\}^{1/2}} \\ \dfrac{\phi_w(x)}{\Big\{\xi\Big(\frac{1}{cos^2(w\xi)} - \frac{1}{ch^2(w\xi)}\Big) + (\xi-1)\Big(\frac{1}{ch^2[w(1-\xi)]} - \frac{1}{cos^2[w(1-\xi)]}\Big)\Big\}^{1/2}} \end{pmatrix},$$

where

$$\phi_w(x) = \tfrac{sh(wx)}{ch(w\xi)} - \tfrac{sin(wx)}{cos(w\xi)}, \; x \in [0, \xi),$$
$$\phi_w(x) = \frac{sh[w(x-1)]}{ch[w(\xi-1)]} - \frac{sin[w(x-1)]}{cos[w(\xi-1)]}, \; x \in (\xi, 1].$$

Proof. We refer to [8] for the proof of this lemma.

Remark 2.2 *Let*

$$I = \Big\{ w \geq 0 \ \ such\ that\ g(w) = 0 \Big\},$$

then since $g(w)$ is analytic, I is a countable family. Thus, we denote by

$$\lambda_{\pm k} = \pm i w_k^2, k \in \mathbb{N}\backslash\{0\} \ and\ w_0 = 0.$$

Now we are ready to compute the coefficients $(b_w)_w$.
If $\lambda = 0$ then:

$$b_0 = C_0^2 \begin{pmatrix} 0 \\ -6 \end{pmatrix}.$$

If $w \in I\backslash\{0\}$ then:

$$b_w = \begin{pmatrix} \dfrac{-\frac{sh(w\xi)}{ch(w\xi)} + \frac{sin(w\xi)}{cos(w\xi)}}{\Big\{\xi\Big(\frac{1}{cos^2(w\xi)} - \frac{1}{ch^2(w\xi)}\Big) + (\xi-1)\Big(\frac{1}{ch^2[w(\xi-1)]} - \frac{1}{cos^2[w(\xi-1)]}\Big)\Big\}^{1/2}} \\ \dfrac{2\frac{sh[w(\xi-1)]}{ch[w(\xi-1)]} - 2\frac{sh(w\xi)}{ch(w\xi)}}{\Big\{\xi\Big(\frac{1}{cos^2(w\xi)} - \frac{1}{ch^2(w\xi)}\Big) + (\xi-1)\Big(\frac{1}{ch^2[w(\xi-1)]} - \frac{1}{cos^2[w(\xi-1)]}\Big)\Big\}^{1/2}} \end{pmatrix}.$$

If $cos(n\pi\xi) = 0$ for some $n \in \mathbb{Z}$ then:

$$b_{\pm n} = \begin{pmatrix} \mp sin(n\pi\xi) \\ 0 \end{pmatrix}.$$

Note that $b_{\pm w} = \pm b_w$.

Proposition 2.2 *The operator B_0 is an admissible control operator and B_0^* is an admissible observation operator.*

Proof. It is clear from the expression of b_w that the sequence $(b_w)_w$ is bounded. To conclude that B_0 is an admissible control operator, we use the Carleson measure criterion (see [5]). This criterion implies in our case (the eigenvalues of A_0 are on the imaginary axis) that for $h > 0$, the number N_h of the eigenvalues of A_0, in the region

$$\Big\{z \in \mathbb{C}, 0 \leq \mathrm{Re(z)} \leq \mathrm{h}, \mathrm{a-h} \leq \mathrm{Im(z)} \leq \mathrm{a+h}\Big\},$$

satisfies

$$N_h \leq Mh, \tag{25}$$

for some $M < \infty$ independent of a. Following Lemma 3.6. in [8], for $\xi \in \Omega$, there exist a, b such that:

$$k\pi + b \leq w_k \leq k\pi + a, \ \forall\, k \in \mathbb{N}\backslash\{0\}.$$

It follows that the number of w_k which are in intervals of length M is more than $3 + \frac{M}{\pi}$, and then the number of eigenvalues of A_0 in

$$S_{a,M} = \Big\{z \in \mathbb{C}, \mathrm{Re(z)} = 0, \mathrm{a}^2 \leq \mathrm{Im(z)} \leq (\mathrm{a+M})^2\Big\},$$

is less than $3+\frac{M}{\pi}$. Thus $N_h \leq Mh$. This implies that B_0 is an admissible control operator for $S(t)$ and by duality, we obtain that B_0^* is an admissible observation operator for $S(t)$.

Proof of Lemma 2.1.

We know that B_0 can be represented as follows:

$$B_0 = (\sum_k b^1_{w_k}\Phi_{w_k}, \sum_k b^2_{w_k}\Phi_{w_k}),$$

and then

$$R(\lambda, A_0)B_0 = (\sum_k \frac{b^1_{w_k}}{\lambda-\lambda_k}\Phi_{w_k}, \sum_k \frac{b^2_{w_k}}{\lambda-\lambda_k}\Phi_{w_k}).$$

Introduce the following sum:

$$(\frac{b_0^1}{\lambda}\begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix}, \frac{b_0^2}{\lambda}\begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix}) + (\sum_{k\neq 0}\frac{b^1_{w_k}}{\lambda-\lambda_k}\begin{pmatrix} b^1_{w_k} \\ b^2_{w_k} \end{pmatrix}, \sum_{k\neq 0}\frac{b^2_{w_k}}{\lambda-\lambda_k}\begin{pmatrix} b^1_{w_k} \\ b^2_{w_k} \end{pmatrix}).$$

If λ goes to $+\infty$ in the above sum and $\lambda \in \mathbb{R}$, it turns out that the limit is 0. This gives that the system (B^*_{0L}, A_0, B_0) is regular.

According to [8] we have $A_1 = A_0 - B_1 B^*_{1L} - B_2 B^*_{2L}$.

Proposition 2.3 *The system (B^*_{0L}, A_1, B_0) satisfies the assumptions of Theorem* 2.2.

Proof. The transfert function of the system (B^*_{0L}, A_1, B_0) is bounded on the imaginary axis. In fact, if we consider the following identity:

$$B^*_{0L}(\lambda I - A_1)^{-1}B_0(I + B^*_{0L}(\lambda I - A_0)^{-1}B_0) = B^*_{0L}(\lambda I - A_0)^{-1}B_0,$$

then since the transfer function of the system (B^*_{0L}, A_0, B_0) which is nothing else than $B^*_{0L}(\lambda I - A_0)^{-1}B_0$ is bounded on the imaginary axis and from $Re(B^*_{0L}(\lambda I - A_0)^{-1}B_0) = 0$, we deduce that the quantity $B^*_{0L}(\lambda I - A_1)^{-1}B_0$ is bounded on the imaginary axis and its limit when λ goes to infinity with $\lambda \in \mathbb{R}$ is equal to zero (this implies that (B^*_{0L}, A_1, B_0) is regular). It remains to show that the system (A_1, B_0) is stabilizable and the system (B^*_{0L}, A_1) is detectable. The following technical result holds:

Lemma 2.3 *For any $\xi \in \Omega$, there exist $m_1, m_2 > 0$ such that:*

$$m_1 < |b_{w_k}| < m_2, \; for\, any\, k \in \mathbb{N}$$

where

$$|b_{w_k}|^2 = |\begin{pmatrix} b^1_{w_k} \\ b^2_{w_k} \end{pmatrix}|^2 = |b^1_{w_k}|^2 + |b^2_{w_k}|^2.$$

Proof. For $\xi \in \Omega$ having the coprime factorization $\frac{p}{q}$, where q is odd, Rebarber proved in [8] that the sequence $(b^2_{w_k})_k$ has a lower bound which implies that the sequence $(b_{w_k})_k$ has a lower bound. Furthermore, for $\xi \in \Omega$ and ξ having the coprime factorization $\frac{p}{q}$, where q is even, we have:

$$|b_k| = 1.$$

This gives that $(b_k)_k$ has a lower bound.

Let $\eta \in (0,1)$, we denote by

$$A_\eta = \Big\{ k| \, |cos(w_k\xi)| \geq \eta \Big\}$$

and

$$B_\eta = \Big\{ k| \, |sin(w_k\xi)| \geq \sqrt{1-\eta^2} \Big\}.$$

So, we shall study the case where $\xi \in \Omega \setminus Q$.
Since we have:

$$\begin{aligned} |b_{w_k}|^2 &= |b^1_{w_k}|^2 + |b^2_{w_k}|^2 \\ &= \frac{\Big\{ \Big(\frac{sh(w_k\xi)}{ch(w_k\xi)} - \frac{sin(w_k\xi)}{cos(w_k\xi)} \Big)^2 + 4 \Big(\frac{sh[w_k(1-\xi)]}{ch[w_k(1-\xi)]} + \frac{sh(w_k\xi)}{ch(w_k\xi)} \Big)^2 \Big\}}{\xi \Big(\frac{1}{cos^2(w_k\xi)} - \frac{1}{ch^2(w_k\xi)} \Big) + (\xi-1) \Big(\frac{1}{ch^2[w_k(\xi-1)]} - \frac{1}{cos^2[w_k(\xi-1)]} \Big)}, \end{aligned}$$

and

$$|b_{w_k}|^2 = \frac{\Big\{ \Big(\frac{sh[w_k(1-\xi)]}{ch[w_k(1-\xi)]} - \frac{sin[w_k(1-\xi)]}{cos[w_k(1-\xi)]} \Big)^2 + 4 \Big(\frac{sh[w_k(1-\xi)]}{ch[w_k(1-\xi)]} + \frac{sh(w_k\xi)}{ch(w_k\xi)} \Big)^2 \Big\}}{\xi \Big(\frac{1}{cos^2(w_k\xi)} - \frac{1}{ch^2(w_k\xi)} \Big) + (\xi-1) \Big(\frac{1}{ch^2[w_k(\xi-1)]} - \frac{1}{cos^2[w_k(\xi-1)]} \Big)}, \tag{26}$$

it follows that:

$$\begin{aligned} |b_{w_k}|^2 \geq \; & inf(cos^2(w_k\xi), cos^2[w_k(1-\xi)]) \\ & \Big\{ th^2(w_k\xi) + tg^2(w_k\xi) + 4\,th^2[w_k(1-\xi)] + 4\,th^2(w_k\xi) + \\ & 4\,th(w_k\xi) \Big(2\,th[w_k(1-\xi)] - \tfrac{1}{2} tg(w_k\xi) \Big) \Big\}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} |b_{w_k}|^2 \geq \; & inf(cos^2(w_k\xi), cos^2[w_k(1-\xi)]) \\ & \Big\{ th^2(w_k\xi) + tg^2(w_k\xi) + 4\,th^2[w_k(1-\xi)] + 4\,th^2(w_k\xi) + \\ & 4\,th[w_k(1-\xi)] \Big(2\,th(w_k\xi) - \tfrac{1}{2} tg[w_k(1-\xi)] \Big) \Big\}. \end{aligned} \tag{28}$$

We consider then two cases:
First case:

$$\text{if } tg(w_k\xi)\, tg[w_k(1-\xi)] < 0,$$

then according to (27) and (28) we have

$$\begin{aligned} |b_{w_k}|^2 \geq \; & inf(cos^2(w_k\xi), cos^2[w_k(1-\xi)]) \Big(\tfrac{1}{2} tg^2(w_k\xi) + \\ & + \tfrac{1}{2} tg^2[w_k(1-\xi)] + 4th^2(w_k\xi) + 4th^2[w_k(1-\xi)] - th(w_k\xi)tg(w_k\xi) - \end{aligned}$$

$$- th[w_k(1-\xi)]tg[w_k(1-\xi)]\Big).$$

Which implies according to (24)

$$|b_{w_k}|^2 \geq \begin{cases} sin^2(w_k\xi), \text{ if } cos^2(w_k\xi) < cos^2[w_k(1-\xi)], \\ sin^2[w_k(1-\xi)], \ else. \end{cases}$$

We conclude in this case that $(b_{w_k})_{k\in B_\eta}$ has a lower bound.
Second case:

$$\text{if } tg(w_k\xi)\, tg[w_k(1-\xi)] \geq 0,$$

then we obtain according to (24) and (26)

$$\begin{aligned} |b_{w_k}|^2 &\geq inf(cos^2(w_k\xi), cos^2[w_k(1-\xi)])\Big(tg^2(w_k\xi) + tg^2[w_k(1-\xi)]\Big) \\ &\geq \begin{cases} sin^2[w_k(1-\xi)], \text{ if } cos^2(w_k\xi) > cos^2[w_k(1-\xi)], \\ sin^2(w_k\xi), \ else. \end{cases} \end{aligned}$$

So, according to (24) we conclude in the two cases that the sequence $(b_{w_k})_{k\in B_\eta}$ has a lower bound. Lemma 3.5 in [8] implies that $(b_{w_k})_{k\in A_\eta}$ has a lower bound. Furthermore, we know that the sequence $(b_{w_k})_k$ is bounded. This gives the claim and ends the proof of the lemma.

End of the proof of Proposition 2.3.
As a consequence of the fact that the sequence $(b_{w_k})_k$ has an upper and lower bounds is that the system (A_1, B_0) is stabilizable and the system (B^*_{0L}, A_1) is detectable (for more details we refer to Lemmas 2.10.,3.10. in [8]). This completes the proof of Proposition 2.3.

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