

# Optimal Regularity Results in Boundary Control of Elastic Systems with Fractional Order Damping

Scott W. Hansen\*

## Abstract

In this article a criterion known as the “operator Carleson measure criterion” is applied to obtain regularity results for a general class of second-order boundary control problems. As specific examples, optimal regularity results are obtained for a wave equation and a plate equation, both with fractional order damping and on a rectangular domain with one edge controlled.

## 1 Introduction and statement of the main results

We consider the second-order control system

$$w_{tt} + 2\gamma A^\alpha w_t + Aw = Bf, \quad (1.1)$$

where  $A$  is a positive operator which is an isomorphism  $Y_1 \rightarrow Y_0$ , for appropriately defined Hilbert spaces  $Y_1 \subset Y_0$ . The damping operator  $A^\alpha$ ,  $\alpha \geq 0$ , is well-defined by the spectral theorem and is an isomorphism  $Y_\alpha \rightarrow Y_0$ , where  $Y_\alpha$  is an appropriately defined intermediate space. We wish to consider a class of  $B$  operators motivated by boundary control problems. Thus  $B$  may be “unbounded” in the sense that its range lies in a larger space than  $Y_0$ . We assume the minimal condition that  $B$  is continuous from a Hilbert space  $U$  to  $Y_{-1} = Y_1'$ . The input  $f$  is assumed to belong to the class  $L^2(0, \infty; U)$ .

We are concerned with the problem of determining the optimal regularity of solutions of such control systems as a function of  $\alpha$  for damping exponents  $\alpha$  in the range  $0 < \alpha < 1/2$ . Since  $B$  maps continuously into  $Y_{-1}$ , mild solutions with zero initial data

$$w(0) = w_t(0) = 0 \quad (1.2)$$

*a-priori* possess the regularity

$$w \in C([0, T]; Y_{-1/2}) \cap C^1([0, T]; Y_{-1}), \quad \forall T > 0.$$

If in addition, for some  $\sigma \geq -1/2$

$$W \in C([0, T]; Y_\sigma) \cap C^1([0, T]; Y_{\sigma-1/2}), \quad \forall T > 0,$$

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\*Department of Mathematics, Iowa State University, Ames, Iowa 50011

then  $B$  is called *admissible* relative to the state space  $Y_\sigma \times Y_{\sigma-1/2}$ . The problem is to determine the optimal  $\sigma$  for which  $B$  is admissible.

Our approach is to first transform the above system to an equivalent first order, diagonal system on  $l^2$ , where  $A$  and  $B$  are represented by infinite matrices. We then apply the *operator Carleson measure criterion* (OCMC) of Hansen and Weiss [5] [6], which provides a sharp criterion for the admissibility for such systems.

We consider two main examples. Both concern boundary control of elastic systems with fractional order damping on rectangular domain  $\Omega$ .

**Example 1** First, consider the problem of a wave equation with fractional order damping subject to Dirichlet control on the boundary:

$$\begin{cases} w_{tt} + 2\gamma(-\Delta)^\alpha w_t - \Delta w = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ w = f & \text{on } \Gamma \times \mathbb{R}^+ \end{cases} \quad (1.3)$$

where  $\Gamma = \partial\Omega$ . We will see that the optimal regularity space for solutions to (1.2), (1.3), with  $0 \leq \alpha < 1/4$  is  $L^2(\Omega) \times H^{-1}(\Omega)$ ; i.e., over the class of controls  $f \in L^2((0, \infty) \times \Gamma)$  one has the optimal regularity

$$w \in C([0, \infty), L^2(\Omega)) \cap C^1([0, \infty), H^{-1}(\Omega)). \quad (1.4)$$

The regularity in (1.4) coincides with known optimal regularity results for the undamped case (with  $\gamma = 0$ ) obtained (in more generality) by Lasiecka, Lions and Triggiani, [9]. In this case the addition of fractional order damping does not change the optimal spaces.

**Example 2** As our second example we consider a plate equation also with fractional order damping on a rectangular domain, but with moment control on the boundary:

$$\begin{cases} w_{tt} + 2\gamma(-\Delta)^{2\alpha} w_t + \Delta^2 w = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ w = 0, \quad \frac{\partial^2 w}{\partial n^2} = f & \text{on } \Gamma \times \mathbb{R}^+, \end{cases} \quad (1.5)$$

where  $\Gamma = \partial\Omega$  and  $n$  is the outward normal direction vector. In the case  $\alpha = 1$ , the damping present in (1.5) is known as Kelvin-Voigt damping, while in the case  $\alpha = 1/2$  the damping model has been referred to as “square-root damping” or “structural damping”; see [1]. In the case  $0 \leq \alpha \leq 1/2$  we obtain the optimal regularity result (for  $f \in L^2((0, \infty) \times \Gamma)$ ):

$$w \in C([0, \infty), X_{\hat{\alpha}+1}) \cap C^1([0, \infty), X_{\hat{\alpha}}), \quad (1.6)$$

where  $X_0 = L^2(\Omega)$ ,  $X_1 = H^2(\Omega) \cap H_0^1(\Omega)$  and  $\hat{\alpha} = \min\{-1/4, \alpha - 1/2\}$ .

The regularity in (1.6) coincides with known results (see e.g., Lasiecka and Triggiani [10]) for the undamped case if  $\alpha = 0$  where one has admissibility in the space  $H_0^1(\Omega) \times H^{-1}(\Omega)$ . However as  $\alpha$  is increased from zero to  $1/4$ , the corresponding space of admissibility of (1.6) also increases to  $(H^{3/2}(\Omega) \cap H_0^1(\Omega)) \times H^{-1/2}(\Omega)$ . For  $1/4 \leq \alpha < 1/2$  there is no increase in regularity. As  $\alpha \rightarrow 1/2$  the regularity agrees with the regularity obtained by Triggiani [12] for the case  $\alpha = 1/2$ .

Regarding the background of this problem, Chen and Russell [1] proved that systems of the form (1.1) for  $\alpha = 1/2$  are described by an analytic semigroup while the subsequent work of S. Chen and Triggiani [2-4] showed that the analyticity remains for  $1/2 \leq \alpha \leq 1$ , but not for  $0 \leq \alpha < 1/2$ . While this work did not specifically address boundary control, Triggiani [12] later obtained optimal regularity results for the case of boundary control as in Example 2 with  $\alpha = 1/2$ , as mentioned above.

The approach here, in contrast to direct estimates of the previously mentioned articles, utilizes the OCMC (described in Section 2) to obtain the regularity results. A disadvantage of this approach is that one needs to calculate the matrix form of the  $B$  operator in (1.1). However, if this can be accomplished, quite a range of  $A$  operators may be considered; for example, powers of the Laplacian, as is considered here.

## 2 Admissibility and the Operator Carleson Measure Criterion

We describe in this section some background material on admissibility and the operator Carleson measure criterion. The reader is referred to [5,6,13] for details.

### 2.1 Admissibility

Suppose that  $A$  is the generator of a strongly continuous semigroup  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on the Hilbert space  $V_0$ . Define the space  $V_{-1}$  as the completion of  $V_0$  with respect to the norm

$$\|x\|_{-1} = \|(\beta I - A)^{-1}x\|,$$

where  $\beta \in \rho(A)$  ( $V_{-1}$  does not depend upon  $\beta$ ). Then  $(\beta I - A)^{-1}$  extends to an isomorphism from  $V_{-1}$  to  $V_0$ .  $\mathbb{T}$  extends to a strongly continuous semigroup on  $V_{-1}$ , whose generator is an extension of  $A$  with domain  $V_0$ . The extended semigroup is isomorphic to the initial one. (We will not make notational distinctions for various extension operators.) Let  $Z_1$  be the Hilbert space obtained by endowing  $D(A^*)$  with the norm

$$\|x\|_1 = \|(\beta I - A^*)x\|.$$

When  $V_0$  is identified with  $V_0^*$  it follows that  $Z_1^* = V_{-1}$ .

Let  $U$  be a Hilbert space and suppose  $B \in \mathcal{L}(U, V_{-1})$ . Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \quad (2.1)$$

Given any  $x_0 \in V_0$  and any  $u \in L^2_{loc}([0, \infty), U)$  the unique solution (in  $V_{-1}$ ) of (2.1) is given by

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u$$

where  $\Phi_\tau u$  is defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) d\sigma.$$

In particular,  $x \in C([0, \infty), V_{-1})$ . It is not necessarily true that  $x \in C([0, \infty), V_0)$ .

**Definition 2.1** *With the above notation, let  $U$  be a Hilbert space and  $B \in \mathcal{L}(U, V_{-1})$ . Then  $B$  is said to be an admissible control operator for  $\mathbb{T}$  relative to the space  $V_0$ , if for some  $\tau > 0$  and any  $u \in L^2([0, \infty), U)$  we have  $\Phi_\tau u \in V_0$ .*

If  $B$  is admissible then for any  $\tau \geq 0$ ,  $\Phi_\tau$  is a bounded linear operator from  $L^2([0, \infty), U)$  to  $V_0$  (this follows from the closed graph theorem). In other words, for each  $\tau \geq 0$  there is a  $k_\tau \geq 0$  such that

$$\|\Phi_\tau u\|_{V_0} \leq k_\tau \|u\|_{L^2} \quad \forall u \in L^2([0, \infty), U).$$

It follows that given any  $x_0 \in V_0$  and any  $u \in L^2_{loc}([0, \infty), U)$  the solution of (2.1) has the regularity

$$x \in C([0, \infty), V_0).$$

## 2.2 Operator Carleson Measure Criterion

The *Carleson measure criterion* of Ho and Russell, [8] and Weiss, [14] provides a necessary and sufficient condition for the admissibility of control operators for diagonal semigroups on  $l^2$  in the case of scalar inputs. In [5,6] a related condition, called the *operator Carleson measure criterion* (OCMC), was shown to provide a necessary and sufficient condition for admissibility in the case of  $l^2$ -valued inputs if certain spectral conditions on the semigroup generator are satisfied. We describe below some of the main results of [5,6].

We represent elements of  $l^2$  as infinite column matrices (of complex numbers) and elements of the dual  $l^{2*}$  as infinite row matrices.

The OCMC applies to systems of the form

$$\dot{x}(t) = \Lambda x(t) + \widehat{B}u(t), \tag{2.2}$$

with input  $u \in L^2([0, \infty), l^2)$  and where  $\Lambda$  and  $\widehat{B}$  are represented by infinite matrices.  $\Lambda$  is diagonal and its diagonal elements  $\lambda_k$  (its eigenvalues) are in a left half of the complex plane:

$$-\lambda_k \in \mathbb{C}_\varepsilon := \{s \in \mathbb{C} \mid \operatorname{Re} s > \varepsilon\} \quad \forall k \in \mathbb{N}. \tag{2.3}$$

For our purposes we may assume that  $\varepsilon$  is some positive number. (Otherwise Definition 2.2 below can be modified accordingly.) Therefore  $\Lambda$  generates a strongly continuous diagonal semigroup  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  on  $l^2$ :

$$(\mathbb{T}_t z)_k = e^{\lambda_k t} z_k, \quad \forall k \in \mathbb{N}, \tag{2.4}$$

where  $z_k$  denotes the  $k$ -th component of  $z \in l^2$ . The infinite matrix  $\widehat{B}$  may be unbounded in the sense that the image of  $l^2$  is not necessarily contained in  $l^2$ .

To recall the results of [5,6], we need a notation for certain rectangles in  $\mathbb{C}$ :

$$R(h, w) = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z \leq h, w - h \operatorname{Im} z < w + h\}, \tag{2.5}$$

for any  $h + iw \in \mathbb{C}_0 (i = \sqrt{-1})$ . We denote by  $b_k$  the  $k$ -th row of  $\widehat{B}$ . The  $b_k^* b_k$  is an infinite matrix of rank one.

**Definition 2.2** The infinite matrix  $\widehat{B}$ , with rows  $b_k$ , satisfies the operator Carleson measure criterion for the semigroup  $\mathbb{T}$  defined by (2.5) if  $b_k \in l^{2*}$  for any  $k \in \mathbb{N}$  and there is some  $M \geq 0$  such that for any  $h + iw \in \mathbb{C}_0$ ,

$$\left\| \sum_{-\lambda_k \in R(h,w)} b_k^* b_k \right\|_{\mathcal{L}(l^2)} \leq Mh. \quad (2.6)$$

The space of infinite matrices which satisfy the operator Carleson measure criterion for  $\mathbb{T}$  is denoted by  $OCM(\mathbb{T})$ . The following results were proved in [5], [6].

**Theorem 2.3** Assume  $\mathbb{T}$  is defined by (2.4) and the infinite matrix  $\widehat{B}$  represents an admissible control operator for  $\mathbb{T}$ . Then  $\widehat{B} \in OCM(\mathbb{T})$ .

**Theorem 2.4** Assume  $\mathbb{T}$  is defined by (2.4) and  $\widehat{B} \in OCM(\mathbb{T})$ . Then  $\widehat{B}$  represents an admissible control operator for  $\mathbb{T}$  if any one of the following conditions hold

- (1)  $\mathbb{T}$  is invertible (i.e., extends to a group),
- (2)  $\mathbb{T}$  is analytic,
- (3) There are numbers  $0, a \leq b$  and  $\alpha \geq 0$  such that

$$a|Im \lambda_k|^\alpha \leq -Re \lambda_k \leq b|Im \lambda_k|^\alpha \quad (2.8)$$

Then  $\widehat{B} \in OCM(\mathbb{T})$  if and only if  $\widehat{B}$  is an admissible control operator for  $\mathbb{T}$ .

The additional conditions (1), (2) and (3) in Theorem 2.4 are restrictions possible positions of the eigenvalues of the generator  $\Lambda$ . It is unknown whether Theorem 2.4 remains valid without *any* such restrictions; see [5, Conjecture 4.4].

### 3 Application to damped elastic systems

Let  $\Omega = \{x, y : 0 < x < 1, 0 < y < 1\}$  and let  $\Delta_0$  denote the Dirichlet Laplacian operator on the domain  $\Omega$ . Initially,  $\Delta_0$  is defined by its variational formulation on the Sobolev space  $H_0^1(\Omega)$ :

$$\langle \Delta_0 \psi, \phi \rangle_\Omega = - \int_\Omega \nabla \psi \cdot \nabla \phi d\Omega \quad \forall \phi \in H_0^1(\Omega). \quad (3.1)$$

The form  $\langle \cdot, \cdot \rangle$  in (3.1) denotes the duality pairing relative to the pivot space  $X_0 = L^2(\Omega)$ . It is well-known that  $-\Delta_0$  is positive, self-adjoint and provides an isomorphism (topologically and algebraically) from  $H_0^1$  to  $H^{-1}$ . As a consequence, a family of function spaces  $X_a, a \in \mathbb{R}$  can be constructed (by interpolation) so that  $-\Delta_0 : X_a \rightarrow X_{a-1}$  is an isomorphism. For example, since  $X_0 = L^2(\Omega)$  we have  $X_1 = H^2(\Omega) \cap H_0^1(\Omega)$ , while  $X_{-1/2} = H^{-1}(\Omega)$ . From the spectral theorem all powers of  $-\Delta_0$  are well-defined and thus for  $p > 0, (-\Delta_0)^p : X_{a+p} \rightarrow X_a$  is an isomorphism.

The control problems in Examples 1 and 2, (for simplicity, with control only on one edge  $\Gamma_0 = \{(x, y) : x = 0\}$  of the domain) can be written

$$w_{tt} + 2\gamma((-\Delta_0)^p)^\alpha w_t + (-\Delta_0)^p w = \delta'_0(y)f(x, t), \quad (x, y, t) \in \Omega \times \mathbb{R}^+. \quad (3.2)$$

In terms of the notation in (1.1), we are using  $A = (-\Delta_0)^p$ , where  $p = 1$  in Example 1 and  $p = 2$  in Example 2.  $B = \delta'_0(y)$ , where  $\delta'_0(y)$  denotes the functional (in  $X_{-1}$ ) that assigns to each function in  $X_1$  the normal derivative on  $\Gamma$ . Then one has  $B : U = L^2(0, 1) \rightarrow Y_{-1} = X_{-p}$  is continuous. The control function  $f$  in (3.2) is assumed to belong to the class  $L^2((0, 1) \times \mathbb{R}^+)$  so that  $\delta_0(y)f(x, t) \in L^2(0, T; X_{-1})$ . Then a solution of (3.2) is well-defined via the usual variation of constants formula. The representation (3.2) of the problems in Examples 1 and 2 is valid in the sense that the solution of (3.2) (defined by variation of constants) is the same as the transpositional (weak) solution of the boundary value problems in Example 1,2; see Lions and Magenes [11].

As mentioned earlier, it is known that when  $\alpha \geq 1/2$  the solutions of (3.2) are determined by an analytic semigroup and consequently hypothesis (2) in Theorem 2.4 is satisfied. In the case  $\alpha = 0$  the associated semigroup is invertible, so that hypothesis (1) of Theorem 2.3 applies. If  $0 < \alpha < 1/2$  the results of S. Chen and Triggiani [2,3] show that the associated semigroups are neither analytic nor invertible. We will see however, that hypothesis (3) of Theorem 2.4 is satisfied. In what follows we restrict our interest to the latter case:  $0 < \alpha < 1/2$ , although the same general approach is valid for if  $\alpha = 0$  or  $\alpha \geq 1/2$ .

### 3.1 Calculation of matrix representation

Letting  $\mathbf{w} = (w_1, w_2)^T = (w, w_t)^T$  we can rewrite (3.2) as

$$\frac{d\mathbf{w}}{dt} = \mathcal{A}\mathbf{w} + \mathcal{B}f \quad (3.3)$$

where

$$\mathcal{A}\mathbf{w} = \begin{pmatrix} w_2 \\ -(-\Delta_0)^p w_1 - 2\gamma((-\Delta_0)^p)^\alpha w_2 \end{pmatrix}, \quad \mathcal{B}f = \begin{pmatrix} 0 \\ \delta'_0 f \end{pmatrix}.$$

It is easy to show that  $\mathcal{A} : X_{p/2} \times X_0 \rightarrow X_0 \times X_{-p/2}$  is an isomorphism. Likewise, for any  $a \in \mathbb{R}$ ,  $\mathcal{A}$  can be extended continuously (or restricted) to an isomorphism:  $V_a \rightarrow V_{a-1}$ , where

$$V_a = X_{ap/2+p/2} \times X_{ap/2}.$$

The next task is to compute the matrix representations of  $\mathcal{A}$  and  $\mathcal{B}$  with respect appropriate bases. For  $j, k \in \mathbb{N}$  let  $e_{jk}(x, y) = 2 \sin j\pi y \sin k\pi x$ . Then

$$(-\Delta_0)^p e_{jk} = s_{jk}^p e_{jk}; \quad s_{jk} = \pi^2(j^2 + k^2).$$

The eigenvalues  $\lambda_{jk}^+$ ,  $\lambda_{jk}^-$  of  $\mathcal{A}$  are given in terms of  $s_{jk}$  by

$$\lambda_{jk}^\pm = -\gamma s_{jk}^{p\alpha} \pm i s_{jk}^{p/2} \sqrt{1 - \gamma^2 s_{jk}^{p(2\alpha-1)}} \simeq -\gamma s_{jk}^{p\alpha} \pm i s_{jk}^{p/2} \text{ as } |s_{jk}| \rightarrow \infty.$$

(We write  $f(s) \simeq g(s)$  as  $|s| \rightarrow \infty$  if  $f$  and  $g$  match to highest order as  $|s| \rightarrow \infty$ .)  
 Note that

$$\left(-\operatorname{Re} \lambda_{jk}^{\pm}\right) \simeq -\gamma |\operatorname{Im} \left(\lambda_{jk}^{\pm}\right)|^{2\alpha} \text{ as } |\lambda_{jk}^{\pm}| \rightarrow \infty \quad (3.4)$$

and consequently the condition (3) of Theorem 2.4 applies.

The eigenfunctions  $\Psi_{jk}^{\pm}$  corresponding to the eigenvalues  $\lambda_{jk}^{\pm}$  are

$$\Psi_{jk}^{\pm} = \frac{1}{\sigma_{jk}} \begin{pmatrix} 1 \\ \lambda_{jk}^{\pm} \end{pmatrix} e_{jk}$$

where  $\sigma_{jk}$  is picked so that the sequence  $(\Psi_{jk}^{\pm})$  is normalized in the space  $V_a$ . Since  $|\lambda_{jk}^{\pm}| \simeq s_{jk}$  as  $s_{jk} \rightarrow \infty$  we have

$$\|\Psi_{jk}^{\pm}\|_a = |\sigma_{jk}|^{-1} \left( \|A^{(a+1)/2} e_{jk}\|_{L_2} + \|(A)^{a/2} \lambda_{jk}^{\pm} e_{jk}\|_{L_2} \right) \simeq |\sigma_{jk}|^{-1} \left( 2s_{jk}^{(ap+p/2)} \right),$$

and hence

$$\sigma_{jk} = \left( 2s_{jk}^{p(a+1)/2} \right).$$

It can be shown (for example, by a proof similar to one in [7]) that with the above normalization, the eigenfunctions  $(\Psi_{jk}^{\pm})$  form a *Riesz basis* for  $V_a$ . This implies that the mapping:  $V_a \rightarrow l^2$  defined by  $\sum c_{jk}^{\pm} \Psi_{jk}^{\pm} \rightarrow (c_{jk}^{\pm})$  is an isomorphism.

If we order the basis  $(\Psi_{jk}^{\pm})$  as  $(\Psi_{11}^+, \Psi_{11}^-, \Psi_{21}^+, \Psi_{21}^-, \dots, \Psi_{12}^+, \Psi_{12}^-, \dots, \Psi_{13}^+, \dots)$  then the matrix representation  $A$  of  $\mathcal{A}$  with respect to this basis is

$$A = \operatorname{diag} (A_1, A_2, \dots), \quad \text{where } A_k = \operatorname{diag} (\lambda_{1k}^+, \lambda_{1k}^-, \lambda_{2k}^+, \lambda_{2k}^-, \dots).$$

To calculate the matrix representation  $B$  of  $\mathcal{B}$  we write

$$f(x, t) = \sum_{k \in \mathbb{N}} f_k(t) \sin k\pi x,$$

$$\delta'_0(y) = \sum_{j \in \mathbb{N}} \langle \delta'_0(y), \sqrt{2} \sin j\pi y \rangle \sqrt{2} \sin j\pi y = 2 \sum_{j \in \mathbb{N}} j\pi \sin j\pi y.$$

Therefore

$$\mathcal{B}f(x, y, t) = \begin{pmatrix} 0 \\ \sum_{j, k \in \mathbb{N}} j\pi f_k(t) e_{jk} \end{pmatrix} = \sum_{j, k \in \mathbb{N}} (b_{jk}^+ \Psi_{jk}^+ + b_{jk}^- \Psi_{jk}^-) f_k(t),$$

where  $b_{jk}^{\pm} \simeq \pm \sigma_{jk} j\pi / (2is_{jk}^{p/2})$  as  $|s_{jk}| \rightarrow \infty$ , and hence

$$|b_{jk}^{\pm}| \simeq j\sigma_{jk} \pi (2s_{jk}^{p/2})^{-1} = \pi j s_{jk}^{pa/2} \text{ as } |s_{jk}| \rightarrow \infty. \quad (3.5)$$

If we denote  $u = (f_1, f_2, \dots)^T$  then (3.3) is equivalent to a system of the form (1.1) with

$$B = \operatorname{diag} (B_1, B_2, \dots), \quad \text{where } B_k = (b_{1k}^+, b_{1k}^-, b_{2k}^+, b_{2k}^- \dots)^T.$$

### 3.2 Simplified form of the OCMC

Due to the known structure of the eigenvalues, the OCMC can be written in a much simpler form, using the following observations.

First, due to this block-diagonal structure of  $B$ , the infinite matrices (in the notation of (2.6))

$$\sum_{-\lambda_k \in R(h,w)} b_k^* b_k$$

are all diagonal, and hence to calculate their norms, one only needs to compute the supremum of the diagonal elements. Secondly, due to the conjugate symmetry in the eigenvalues, it will be enough to check the OCMC for  $w > 0$ . Furthermore, since there are only finitely many near the origin, it is enough to check the OCMC as  $w \rightarrow +\infty$ . Thus, for example,  $\text{Im } \lambda_{jk}^\pm \simeq s_{jk}^{p/2}$  as  $|\lambda_{jk}^\pm| \rightarrow \infty$ . Lastly, due to (3.4) we may assume the  $h$  in  $R(h, w)$  is algebraically related to  $w$  by

$$h = w^{2\alpha}. \tag{3.6}$$

This is due to the fact that taking  $h$  of smaller order causes the rectangles to eventually miss all the eigenvalues (and hence trivially satisfy the OCMC), while  $h$  of larger order is suboptimal due to the fact that for sufficiently large  $w_0$ , a rectangle with corresponding  $h_0$  of larger order than in (3.6) can be replaced by two or more (say  $K$ ) smaller disjoint rectangles  $R(h_1, w_1), R(h_2, w_2)$ , etc. with  $h_k = w_k^{2\alpha}$ . Then, denoting  $\mu(R(h, w))$  the left hand side of (2.6), one has  $\mu(R(h_0, w_0)) \leq \mu(R(h_1, w_1)) + \mu(R(h_2, w_2)) + \dots + \mu(R(h_K, w_K))$  so that if  $\mu(R(h_j, w_j)) \leq M_j h_j, j = 1, 2, \dots, K$ , then we also have

$$\mu(R(h_0, w_0)) \leq \sum_{j=1}^K M_j h_j \leq (\text{Max}_{j=1,2,\dots,K} M_j) h_0.$$

Thus the  $M$  (in (2.6)) associated with the larger order rectangle is bounded by the  $M$ 's associated with the smaller rectangles. Hence it is enough to check only rectangles  $R(h, w)$  with  $h = w^{2\alpha}$ .

It follows that an equivalent formulation of the OCMC is

$$\sup_{k \in \mathbb{N}, w > 1} \sum_{j: w \leq s_{jk}^{p/2} \leq w+h} |b_{jk}^\pm|^2 \leq Mh; \quad h = w^{2\alpha}.$$

For fixed  $k$  let  $J = \{j : w \leq s_{jk}^{p/2} \leq w + h\}$  and let  $j_0, j_1$  denote the minimal and maximal elements of  $J$ . (We only need to consider the cases where  $J$  is nonempty.) From (3.6) the growth of the size of the intervals  $[w, w + h]$  as a function of  $w$  is sublinear. It follows that for any number  $s$  in this interval, there exists  $C > 0$  so that  $w \leq s \leq Cw$ . Thus

$$\sum_{j: w \leq s_{jk}^{p/2} \leq w+h} |b_{jk}^\pm|^2 \simeq \pi^2 \sum_{j: w \leq s_{jk}^{p/2} \leq w+h} j^2 s_{jk}^{pa} \equiv w^{2a} \sum_{j=j_0}^{j_1} j^2,$$

where the symbol  $\equiv$  means that each side is bounded by some fixed multiple (as  $w \rightarrow \infty$ ) of the other. Thus, referring to (2.6) (divided through by the  $h$ ), an equivalent form of the OCMC is

$$\sup_{w>1, k \in \mathbb{N}} w^{2a-2\alpha} \sum_{j=j_0}^{j_1} j^2 < \infty. \quad (3.7)$$

For optimality of (3.7) we seek to find  $a$  so that the LHS of (3.7) is finite but nonzero.

### 3.3 The case $p = 1$

Let us now specialize to the case  $p = 1$ . We have from the definition of set  $J$

$$w^2 \leq \pi^2(j_0^2 + k^2) \leq \pi^2(j_1^2 + k^2) \leq (w + h)^2,$$

so that  $\pi^2(j_1^2 - j_0^2) \leq 2wh + h^2 = 2w^{1+2\alpha} + w^{4\alpha} \leq 3w^{1+2\alpha}$ . We thus have the estimate

$$j_1 \leq j_0(1 + R)^{1/2}, \quad \text{where } R = (3/\pi^2)w^{1+2\alpha}/j_0^2. \quad (3.8)$$

We find that

$$\begin{aligned} \sum_{j=j_0}^{j_1} j^2 &= \frac{j_1^3 - j_0^3}{3} + \frac{j_0^2 + j_1^2}{2} + j_1 - j_0 \leq \frac{4}{3}(j_1^3 - j_0^3) + w^2 \\ &\leq \frac{4}{3}j_0^3((1 + R)^{3/2} - 1) + w^2 \leq 2j_0^3(R + R^{3/2}) + w^2 \\ &\leq j_0w^{1+2\alpha} + w^{3(1+2\alpha)/2} + w^2 \leq w^{2+2\alpha} + w^{3(1+2\alpha)/2} + w^2. \end{aligned}$$

Thus, by (3.7), the OCMC will hold if the following three conditions hold:

$$(i) \ 2a - 2\alpha + 2 + 2\alpha \leq 0, \quad (ii) \ 2a + 2\alpha + 3/2 + 3\alpha \leq 0, \quad (iii) \ 2a - 2\alpha + 2 \leq 0.$$

Therefore admissibility holds if  $a \leq -1$ . For necessity, note that the condition  $a \leq -1$  is required by the inequality (i) above. Furthermore it is simple to check that all inequalities leading to (i) become sharp if one takes  $k$  bounded and  $\alpha < 1/2$  (so that  $R$  in (3.8) is small). Therefore  $a \leq -1$  is optimal for the problem (3.2) for  $0 < \alpha < 1/2$ . This implies (due to the previously discussed Riesz basis isomorphism) that the solutions of (3.2) satisfy the optimal regularity

$$w \in C([0, \infty), L^2(\Omega)) \cap C^1([0, \infty), H^{-1}(\Omega)), \quad (3.9)$$

which is in agreement with the optimal results for the undamped case (see [10]). Since the problem in Example 1 can be represented in the form (3.2) if  $0 \leq \alpha \leq 1/4$ , the regularity in (3.9) also applies to Example 1 for  $0 \leq \alpha \leq 1/4$ .

### 3.4 The cases $p > 1$

We find more interesting behavior when  $p > 1$ . Nevertheless, most of the steps are the same and so we follow the same steps and notation used for the case  $p = 1$ , only indicating the differences.

Set  $J$  is defined by  $J = \{j : w \leq s_{jk}^{p/2} \leq w + h\}$  so that

$$w^{2/p} \leq \pi^2(j_0^2 + k^2) \leq \pi^2(j_1^2 + k^2) \leq (w + h)^{2/p}.$$

Thus we have  $\pi^2(j_1^2 - j_0^2) \leq (w + h)^{2/p} - w^{2/p} \simeq (w^{2/p}h/w)f'(1)$  as  $w \rightarrow \infty$  where  $f(t) = t^{2/p}$ . Therefore we obtain  $(j_1^2 - j_0^2) \leq Cw^{2/p+2\alpha-1}$  where  $C$  is independent of  $w$ . It follows we again obtain an estimate like (3.8):

$$j_1 \leq j_0(1 + R)^{1/2}, \quad \text{where } R = Cw^{2/p+2\alpha-1}/j_0^2.$$

From the same chain of inequalities following (3.8) we obtain

$$\begin{aligned} \sum_{j=j_0}^{j_1} j^2 &\leq 2j_0^3(R + R^{3/2}) + w^{2/p} \leq C\{j_0w^{2/p+2\alpha-1} + w^{3/p+3\alpha-3/2} + w^{2/p}\} \\ &\leq C\{w^{3/p+2\alpha-1} + w^{3/p+3\alpha-3/2} + w^{2/p}\}. \end{aligned}$$

Hence from (3.7) the OCMC holds if the following three conditions hold:

- (i)  $2a - 2\alpha + 3/p + 2\alpha - 1 = 2a + 3/p - 1 \leq 0$
- (ii)  $2a - 2\alpha + 3/p + 3\alpha - 3/2 = 2a + \alpha + 3/p - 3/2 \leq 0$
- (iii)  $2a - 2\alpha + 2/p \leq 0.$

One sees that for  $\alpha$  near zero inequality (iii) implies the other two, while for  $\alpha \geq (p - 1)/2p$  the first implies the other two. Retracing the inequalities involved, one again sees that all the inequalities involved in obtaining (i) and (iii) are reversible (up to a multiplicative constant) for  $k$  bounded. Hence  $\hat{a}$ , the optimal value of  $a$  is given by

$$\hat{a} = \begin{cases} \alpha - 1/p & \text{if } 0 < \alpha \leq \frac{p-1}{2p} \\ 1/2 - 3/2p & \text{if } 1/2 > \alpha \geq \frac{p-1}{2p}. \end{cases}$$

In Example 2  $p = 2$  and hence the system is admissible if

$$a \leq \hat{a} = \min\{-1/4, \alpha - 1/2\}.$$

It follows that solutions of (3.2) (and hence also the boundary value problem in Example 2) have the regularity

$$w \in C([0, \infty), X_{\hat{a}+1}) \cap C^1([0, \infty), X_{\hat{a}}).$$

It is interesting to note that the regularity improvement due to the damping ceases when  $\alpha = 1/4$ . This is perhaps surprising since analyticity is obtained only when  $\alpha = 1/2$ . The same general behavior is seen for all values of  $p > 1$ .

## 4 Conclusion

In this article we have indicated an approach based upon the Operator Carleson Measure Criterion to obtain sharp regularity results for systems of the form (1.1). Such systems include some relevant damped elastic systems such as those mentioned

in Examples 1 and 2. However this method is *directly* applicable only to special domains, such as rectangles or parallelepipeds, since one must be able to calculate Fourier coefficients corresponding to the control input operator.

The analysis here indicates that for  $p = 1$  there is no increase in regularity as the damping exponent  $\alpha$  is increased. On the other hand, for all  $p > 1$  there is a linear increase in regularity up to some critical  $\alpha_c$ , after which there is no further increase in regularity. It is particularly interesting that analyticity occurs (for all  $p$ ) when  $\alpha = 1/2$ , nevertheless  $\alpha_c$  can assume any value between 0 and  $1/2$ .

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