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On an Abstract Linear Elastic System with Indefinite Damping

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Abstract

In this paper we consider an abstract linear system with perturbation of the form

$$\frac{dy}{dt} = Ay + \varepsilon By$$

on a Hilbert space $\mathcal{H}$, where A is skew-adjoint, B is bounded, and ε is a positive parameter. Motivated by a result of Freitas and Zuazua on the one-dimensional wave equation with indefinite viscous damping [JDE, 1996], we obtain sufficient conditions for the exponential stability for the above system when B is not a dissipative operator. Our result is then applied to many elastic systems with indefinite viscous damping.

1 Introduction

We consider a linear evolution equation

$$\begin{cases} \dfrac{d}{dt}y(\cdot,t) = \mathcal{A}_\varepsilon y(\cdot,t) = (A+\varepsilon B)y(\cdot,t) \\ y(\cdot,0) = y_0 \end{cases} \tag{1.1}$$

on a Hilbert space $\mathcal{H}$ with the energy norm, where A is a densely defined, closed linear operator with domain $\mathcal{D}(\mathcal{A})$ which generates a C_0 semigroup $S(t)$ on $\mathcal{H}$. We assume that

(H1) A is skew adjoint, and has a compact resolvent.

(H2) B is a bounded linear operator with $\|B\| \leq b$.

By the perturbation theory of semigroups, the operator $\mathcal{A}_\varepsilon$ also generates a C_0 semigroup $S_\varepsilon(t)$ on $\mathcal{H}$. Our main interest is on the exponential stability of the above system, i.e., on whether there exist $\mu > 0$ and $M \geq 1$ such that

$$\|S_\varepsilon(t)\| \leq Me^{-\mu t}, \quad \forall\, t \geq 0. \tag{1.2}$$

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This problem has been investigated extensively for bounded and unbounded operator B under the assumption of *dissipativeness* of B,

$$\mathrm{Re}\langle By, y\rangle \le 0, \quad \forall\, y \in \mathcal{D}(A), \tag{1.3}$$

which implies that the energy of the system, $E(t)$, is a decreasing function of time. Clearly, this is not a necessary condition for $E(t)$ being bounded above by a function which tends to zero exponentially. A natural question to ask is the following: Without the dissipativeness of B, can we still obtain (1.2) under some extra conditions?

Such a question was first raised in [CFNS] for a one-dimensional wave equation

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) - d(x)w_t(t,x) & x \in (0,1), \quad t > 0, \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \end{cases} \tag{1.4}$$

where $d(x)$ is a smooth function and changes sign on $(0,1)$. It was conjectured that (1.2) holds if

$$I_n \equiv \int_0^1 d(x)\sin^2 n\pi x dx \ge c_0 > 0, \quad n = 1, 2, \cdots. \tag{1.5}$$

It turns out that (1.5) is not enough to ensure exponential stability. When $\|d(x)\|_{L^\infty}$ becomes large, there will be eigenvalues of the system (1.2) with positive real part (see [F]). Thus, in order to have exponential stability, the damping coefficient must not only satisfy (1.5), but also must have a small L^∞ norm. Later on, Freitas and Zuazua [FZ] considered the modified system of (1.4):

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) - \varepsilon d(x)w_t(t,x) & x \in (0,1), \quad t > 0, \\ w(0,t) = w(1,t) = 0, \\ w(0,t) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases} \tag{1.6}$$

They proved that if $d(x) \in BV(0,1)$ and the condition (1.5) holds, then there exist a positive constants ε_0, M, ω, depending only on the function $d(x)$, such that for all $0 < \varepsilon < \varepsilon_0$

$$E(t) = \int_0^1 (|w_x|^2 + |w_t|^2)dx \le Me^{-\varepsilon\omega t}E(0), \quad \forall t > 0, \tag{1.7}$$

for every finite energy solution of (1.6). Their result was further extended in [BR] to the equation

$$w_{tt} = w_{xx} - 2\varepsilon d(x)w_t - b(x)w \tag{1.8}$$

where $b(x) \in L^1(0,1)$ is a zero order potential.

These works lead us to the current study in this paper. Instead of working on a particular PDE system, we would like to obtain general result along these lines. Although the shooting method used in [FZ] and [BR] is no longer applicable to our problem, the analysis in these papers does provide us valuable information on how to impose additional conditions in order to guarantee (1.2).

2 Main Theorem

Under condition (H1), A has a complete system of orthonormal eigenfunctions

$$\{\phi_n^{(i_n)} \mid n = 1, 2, \cdots,\ 1 \le i_n \le \dim V_n = g_n\} \tag{2.1}$$

such that

$$\begin{cases} A\phi_n^{(i_n)} = \boldsymbol{i}\beta_n \phi_n^{(i_n)}, \quad n = 1, 2, \cdots,\ \beta_n \in \mathbb{R} \\ 0 \le |\beta_1| \le |\beta_2| \le \cdots \le |\beta_n| \le |\beta_{n+1}| \to \infty \end{cases} \tag{2.2}$$

with V_n being the eigenspace corresponding to the eigenvalue $\boldsymbol{i}\beta_n$, and $\beta_j \neq \beta_k$ if $j \neq k$.

We further assume that

(H3) The spectrum of A only consists of point spectrum and satisfies the gap condition

$$\inf\{|\beta_j - \beta_k| : j, k = 1, 2, \cdots;\ j \neq k\} \equiv \gamma > 0. \tag{2.3}$$

(H4) For any unit eigenfunction ψ of A, there exists a positive number c_0 independent of ψ such that

$$-\mathrm{Re}\langle B\psi, \psi\rangle \ge c_0. \tag{2.4}$$

We denote the type of the semigroup $S_\varepsilon(t)$ by

$$\omega_0(\mathcal{A}_\varepsilon) = \lim_{t\to\infty} \frac{\ln \|S_\varepsilon(t)\|}{t} \tag{2.5}$$

and the spectral bound of $\mathcal{A}_\varepsilon$ by

$$\sigma_0(A_\varepsilon) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathcal{A}_\varepsilon)\}. \tag{2.6}$$

To estimate $\sigma_0(A_\varepsilon)$, we shall use a result in [Pr] and [Hu], which states

$$\omega_0(\mathcal{A}_\varepsilon) = \inf\{s > \sigma_0(A_\varepsilon) \mid \sup_{Re\lambda = s} \|(\lambda I - \mathcal{A}_\varepsilon)^{-1}\| < +\infty\}. \tag{2.7}$$

Theorem 2.1 *Assume that the conditions (H1)-(H4) hold. For any given $c \in (0, c_0)$, let*

$$\varepsilon_0 = \frac{\gamma}{8b} \min\left\{1,\ \frac{c_0 - c}{b + \sqrt{b^2 + c_0(c_0 - c)}}\right\}. \tag{2.8}$$

Then, for each $\varepsilon \in (0, \varepsilon_0)$,

$$\omega_0(\mathcal{A}_\varepsilon) \le -\varepsilon c. \tag{2.9}$$

Remark: The above theorem not only states that $\omega_0(\mathcal{A}_\varepsilon)$ has a negative upper bound for small ε, but also gives explicit estimates of $\omega_0(\mathcal{A}_\varepsilon)$ and ε_0. Note that the rule of chosing c is to maximize the decay rate estimate εc. For a simple expression, we can fix c at $\frac{1}{2}c_0$ to get

$$\varepsilon_0 = \frac{\gamma}{8b} \min\left\{1,\ \frac{c_0}{2b + \sqrt{4b^2 + 2c_0^2)}}\right\}. \tag{2.10}$$

Proof of Theorem 2.1.

It is clear that $\omega_0(\mathcal{A}_\varepsilon) \leq \varepsilon\|B\|$ by the theory of perturbations of C_0 semigroups. Let $-c \leq \sigma \leq \|B\|$. Suppose that

$$\sup_\tau \|((\varepsilon\sigma + \boldsymbol{i}\tau)I - \mathcal{A}_\varepsilon)^{-1}\| = \infty.$$

Then, there exist a sequence of real numbers τ_p and a sequence of unit vectors $y_p \in \mathcal{D}(\mathcal{A}_\varepsilon)$ such that

$$\lim_{p\to+\infty} \|((\varepsilon\sigma + \boldsymbol{i}\tau_p)I - \mathcal{A}_\varepsilon)y_p\|_{\mathcal{H}} = 0. \tag{2.11}$$

Taking the inner product of (2.11) with y_p in $\mathcal{H}$, we obtain

$$\sigma = \lim_{p\to+\infty} \mathrm{Re}\langle By_p, y_p\rangle. \tag{2.12}$$

Due to (2.11) again, there exists $N \in \mathbb{N}$ such that

$$\|\varepsilon(\sigma I + B)y_p + (\boldsymbol{i}\tau_p I - \mathcal{A})y_p\| \leq \varepsilon\|(\sigma I - B)\|, \quad \forall\, p > N,$$

which leads to

$$\|(\boldsymbol{i}\tau_p I - A)y_p\| \leq 2\varepsilon\|\sigma I - B\| \leq 4\varepsilon\|B\| = 4\varepsilon b, \quad \forall\, p > N. \tag{2.13}$$

Next, we expand y_p on the basis of the eigenfunction of A:

$$y_p = \sum_n \sum_{i_n=1}^{g_n} \langle y_p, \phi_n^{(i_n)}\rangle \phi_n^{(i_n)}. \tag{2.14}$$

Substituting (2.14) into the left side of (2.13) yields

$$\sum_n \sum_{i_n=1}^{g_n} |\tau_p - \beta_n|^2 |\langle y_p, \phi_n^{(i_n)}\rangle|^2 \leq 16\varepsilon^2 b^2. \tag{2.15}$$

As each y_p is a unit vector,

$$\sum_n \sum_{i_n=1}^{g_n} |\langle y_p, \phi_n^{(i_n)}\rangle|^2 = 1. \tag{2.16}$$

By the gap condition (H3), we claim that there exists an index $n(p)$ such that

$$\begin{cases} |\tau_p - \beta_n| \geq \frac{\gamma}{2}, & n \neq n(p) \\ |\tau_p - \beta_{n(p)}| < \frac{\gamma}{2}. & \end{cases} \tag{2.17}$$

Otherwise, (2.15) will be violated since $\varepsilon < \gamma/8b$. It follows from (2.15) and (2.17) that

$$\frac{\gamma^2}{4} \sum_{n\neq n(p)} \sum_{i_n=1}^{g_n} |\langle y_p, \phi_n^{(i_n)}\rangle|^2 + |\tau_p - \beta_{n(p)}|^2 \sum_{i_{n(p)}=1}^{g_{n(p)}} |\langle y_p, \phi_{n(p)}^{(i_{n(p)})}\rangle|^2 \leq 16\varepsilon^2 b^2. \tag{2.18}$$

Hence

$$\sum_{n\neq n(p)} \sum_{i_n=1}^{g_n} |\langle y_p, \phi_n^{(i_n)}\rangle|^2 \leq \frac{64b^2}{\gamma^2}\varepsilon^2. \tag{2.19}$$

This, combined with (2.16), leads to

$$\sum_{i_{n(p)}=1}^{g_{n(p)}} |\langle y_p, \phi_{n(p)}^{(i_{n(p)})}\rangle|^2 \geq 1 - \frac{64b^2}{\gamma^2}\varepsilon^2. \tag{2.20}$$

Define

$$\psi = \sum_{i_{n(p)}=1}^{g_{n(p)}} \langle y_p, \phi_{n(p)}^{(i_n(p))}\rangle \phi_{n(p)}^{(i_{n(p)})}. \tag{2.21}$$

Then,

$$\|y_p - \psi\|^2 = \| \sum_{n\neq n(p)} \sum_{i_n=1}^{g_n} \langle y_p, \phi_n^{(i_n)}\rangle \phi_n^{(i_n)}\|^2 \leq \frac{64b^2}{\gamma^2}\varepsilon^2. \tag{2.22}$$

By (H4), we have

$$-\mathrm{Re}\langle B\psi, \psi\rangle \geq c_0\|\psi\|^2 \geq \left(1 - \frac{64b^2}{\gamma^2}\varepsilon^2\right) c_0. \tag{2.23}$$

Therefore,

$$\lim_{p\to\infty} \mathrm{Re}\langle By_p, y_p\rangle - \mathrm{Re}\langle B\psi, \psi\rangle \geq \sigma + \left(1 - \frac{64b^2}{\gamma^2}\varepsilon^2\right) c_0. \tag{2.24}$$

On the other hand, using (2.22) we have

$$\begin{aligned} |\mathrm{Re}\langle By_p, y_p\rangle - \mathrm{Re}\langle B\psi, \psi\rangle| &= |\mathrm{Re}\langle By_p, y_p - \psi\rangle + \mathrm{Re}\langle B(y_p - \psi), \psi\rangle| \\ &\leq 2b\|y_p - \psi\| \leq \frac{16b^2}{\gamma}\varepsilon. \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25), and the choice of σ that guarantees $\sigma \geq -c$ we obtain that

$$-c \leq \sigma \leq -c_0 + \frac{64b^2 c_0}{\gamma^2}\varepsilon^2 + \frac{16b^2}{\gamma}\varepsilon. \tag{2.26}$$

This is a contradiction when $\varepsilon \in (0, \varepsilon_0)$.

So far we have proved

$$\sup_\tau \|((\varepsilon\sigma + \boldsymbol{i}\tau)I - \mathcal{A}_\varepsilon)^{-1}\| < \infty.$$

for each $\sigma \geq -c$. This implies (2.9) due to the result given in (2.7). □

3 Applications

In this section, we apply Theorem 2.1 to wave, beam, and 2-d Shrödinger equations with indefinite viscous damping.

Example 1: The 1-d wave equation

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) - \varepsilon d(x) w_t(t,x) & 0 < x < 1,\ t \geq 0 \\ w(0,t) = w_x(1,t) = 0, \\ w(0,t) = w_0(x), \quad w_t(x,0) = w_1(x) \end{cases} \tag{3.1}$$

The underlying Hilbert space is

$$\mathcal{H} = \left\{ \left[\begin{array}{c} w \\ v \end{array} \right] \in H^1(0,1) \times L^2(0,L) \;\middle|\; w(0) = 0 \right\},$$

with the inner product

$$\langle \left[\begin{array}{c} w_1 \\ v_1 \end{array} \right], \left[\begin{array}{c} w_2 \\ v_2 \end{array} \right] \rangle = \int_0^1 [w_1' \bar{w}_2' + v_1 \bar{v}_2] dx.$$

Define

$$A = \left[\begin{array}{cc} 0 & I \\ \partial_x^2 & 0 \end{array} \right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & -d(x) \end{array} \right]$$

A has a complete orthonormal set of eigenfunctions

$$\phi_n = \frac{1}{(n+\frac{1}{2})\pi} \left[\begin{array}{c} \sin(n+\frac{1}{2})\pi x \\ \pm \boldsymbol{i}(n+\frac{1}{2})\pi \sin(n+\frac{1}{2})\pi x \end{array} \right]$$

with eigenvalues

$$\boldsymbol{i}\beta_n = \pm \boldsymbol{i}(n+\frac{1}{2})\pi,$$

for $n = 0, 1, 2, \cdots$.

It is easy to see that (H1) and (H2) are satisfied. (H3) is also satisfied since

$$\gamma = \pi.$$

To verify (H4), we choose

$$d(x) = 1 + \alpha \cos k\pi x, \quad 1 < |\alpha| < 2. \tag{3.2}$$

Then,

$$c_0 = -Re\langle B\phi_n, \phi_n \rangle = \begin{cases} \frac{1}{2}, & k \text{ is even}, \\ \frac{1}{4}(2-\alpha), & k \text{ is odd}. \end{cases}$$

Therefore, $S_\varepsilon(t)$ is exponentially stable.

We can also choose $d(x)$ with local support, such as

$$d(x) = \begin{cases} \sin 4\pi x, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 2\sin 4\pi x, & \frac{1}{2} \le x \le \frac{3}{4}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

Then,

$$-Re\langle B\phi_n, \phi_n \rangle \ge c_0 = \frac{3}{20\pi}, \quad \forall\, n \in \mathbb{N}.$$

Example 2: The Euler-Bernoulli beam equation

$$\begin{cases} w_{tt}(t,x) = -w_{xxxx}(t,x) - \varepsilon d(x) w_t(t,x), 0 < x < 1, \ t \ge 0, \\ w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0, \\ w(0,t) = w_0(x), \ \ w_t(x,0) = w_1(x). \end{cases} \tag{3.4}$$

The underlying Hilbert space is

$$\mathcal{H} = (H_0^1(0,1) \cap H^2(0,1)) \times L^2(0,1)$$

with the inner product

$$\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \rangle = \int_0^1 [w_1'' \bar{w}_2'' + v_1 \bar{v}_2] dx.$$

Define

$$A = \begin{bmatrix} 0 & I \\ -\partial_x^4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -d(x) \end{bmatrix}.$$

A has a complete orthonormal set of eigenfunctions

$$\phi_n = \frac{1}{n^2\pi^2} \begin{bmatrix} \sin n\pi x \\ \pm \boldsymbol{i} n^2\pi^2 \sin n\pi x \end{bmatrix}$$

with eigenvalues

$$\boldsymbol{i}\beta_n = \pm \boldsymbol{i} n^2 \pi^2,$$

for $n = 1, 2, \cdots$. It is easy to see that (H1) and (H2) are satisfied. (H3) is also satisfied since

$$\beta_{n+1} - \beta_n = 2(n+1)\pi \to \infty.$$

To verify (H4), we choose the same $d(x)$ as in (3.2). Then,

$$c_0 = -Re\langle B\phi_n, \phi_n \rangle = \begin{cases} \frac{1}{2}, & k \text{ is odd}, \\ \frac{1}{4}(2-\alpha), & k \text{ is even}. \end{cases}$$

Therefore, $S_\varepsilon(t)$ is exponentially stable.

Example 3. Timoshenko beam equation

$$\begin{cases} \rho u_{tt} = K u_{xx} - K\phi_x - \varepsilon d_1(x) u_t, & 0 < x < 1, \ t \geq 0, \\ J\phi_{tt} = EI\phi_{xx} + K u_x - K\phi - \varepsilon d_2(x)\phi_t, & 0 < x < 1, \ t \geq 0, \\ u(0) = u(1) = 0, \quad \phi_x(0) = \phi_x(1) = 0, \end{cases} \tag{3.5}$$

where ρ, I, J, E, K are corresponding physical constants.

The underline Hilbert space is

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H^1(0,1) \times L^2(0,1)$$

with the norm

$$\|z\|^2 = \int_0^1 \left(K|u_x - \phi|^2 + EI|\phi_x|^2 + \rho|v|^2 + J|\psi|^2 \right) dx$$

where $z = (u, v, \phi, \psi)^T$. Define

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ \frac{K}{\rho}\partial_x^2 & 0 & -\frac{K}{\rho}\partial_x & 0 \\ 0 & 0 & I & 0 \\ \frac{K}{J}\partial_x & 0 & \frac{EI}{J}\partial_x^2 - \frac{K}{J} I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\rho}d_1(x) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{J}d_2(x) \end{bmatrix}.$$

It is easy to see that (H1) and (H2) are satisfied. To compute the eigenvalues of A, we solve

$$Az = \lambda z.$$

Eliminating the unknowns v, ϕ, ψ, we obtain

$$\begin{cases} EIKu_{xxxx} - (JK + \rho EI)\lambda^2 u_{xx} + \rho\lambda^2(J\lambda^2 + K)u = 0 \\ u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0. \end{cases}$$

A straight forward calculation leads to

$$\lambda^2 = \frac{-[n^2\pi^2(JK + \rho EI) + \rho K] \pm \sqrt{[n^2\pi^2(JK - \rho EI) + \rho K]^2, +4n^2\pi^2 EIK\rho^2}}{2\rho J}$$

for $n = \pm 1, \pm 2, \cdots$.

When $\frac{K}{\rho} = \frac{EI}{J}$ and $\sqrt{K/EI}$ is not a multiple of π, we have two sequences of eigenvalues, $\lambda_{n,1}$ and $\lambda_{n,2}$. They can be written as

$$\lambda_{n,1} = \boldsymbol{i}\left(\sqrt{\frac{K}{\rho}}(n\pi - \frac{\sqrt{K/EI}}{2}) + O(\frac{1}{n})\right),$$

$$\lambda_{n,2} = \boldsymbol{i}\left(\sqrt{\frac{K}{\rho}}(n\pi + \frac{\sqrt{K/EI}}{2}) + O(\frac{1}{n})\right).$$

It is not difficult to see that the eigenvalues in each sequence are distinct. If there are multiple eigenvalues, they must be $\lambda_{n,1} = \lambda_{m,2}$ for some $n \neq m$. Since there exists a constant $\gamma_1 > 0$ such that $|\lambda_{n,1} - \lambda_{m,2}| \geq \gamma_1$ for all n, m large enough, the gap condition (H3) follows.

The corresponding unit eigenfunctions have the form

$$z_{n,1} = \begin{bmatrix} O(\frac{1}{n})\sin n\pi x \\ \boldsymbol{i}O(1)\sin n\pi x \\ O(\frac{1}{n})\cos n\pi x \\ \boldsymbol{i}O(1)\cos n\pi x \end{bmatrix}, \qquad z_{n,2} = \begin{bmatrix} O(\frac{1}{n})\sin n\pi x \\ \boldsymbol{i}O(1)\sin n\pi x \\ O(\frac{1}{n})\cos n\pi x \\ \boldsymbol{i}O(1)\cos n\pi x \end{bmatrix}.$$

We now verify condition (H4). In the case that all eigenvalues are simple, we can take $d_1(x), d_2(x)$ to be of the form of (3.2) or (3.3). Then,

$$-\langle Bz_{n,i}, z_{n,i}\rangle = |O(1)|\int_0^1 d_1(x)\sin^2 n\pi x dx + |O(1)|\int_0^1 d_2(x)\cos^2 n\pi x dx + O(\frac{1}{n}).$$

It follows that

$$-\langle Bz_{n,i}, z_{n,i}\rangle \geq c_0 > 0, \tag{3.6}$$

for $i = 1, 2$ and for all n. For double eigenvalues, we need verify condition (H4) for

$$z = c_1 z_{n,1} + c_2 z_{m,2}, \qquad \text{for all } |c_1|^2 + |c_2|^2 = 1.$$

Take $d_1(x), d_2(x)$ of the form

$$1 + \alpha\cos k\pi x, \qquad 1 < |\alpha| < 2, \tag{3.7}$$

where k is an odd integer larger than $n+m$ for all double eigenvalues $\lambda_{n,1} = \lambda_{m,2}$. Therefore,

$$\int_0^1 d_i(x) \cos n\pi x \cos m\pi x dx = \int_0^1 d_i(x) \sin n\pi x \sin m\pi x dx = 0, \quad i = 1, 2.$$

Then using (3.6), we have

$$\begin{aligned} -\langle Bz, z\rangle &= -c_1^2 \langle Bz_{n,1}, z_{n,1}\rangle - c_2^2 \langle Bz_{n,2}, z_{n,2}\rangle \\ &\geq (|c_1|^2 + |c_2|^2) c_0 = c_0 > 0. \end{aligned} \tag{3.8}$$

Therefore, $S_\varepsilon(t)$ is exponentially stalble.

We would like to point out here that condition (H4) still holds when $d_1(x), d_2(x)$ are different, such as having disjoint local support on the interval $(0,1)$, or one of them even being zero. This means that the damping to the displacement (rotation angle) is also effective to the rotation angle (displacement) in a Timoshenko beam when the two wave speeds are the same, and the quantity $\sqrt{K/EI}$ is not a multiple of π.

Example 4. Two-dimensional damped Schrödinger equation:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = \boldsymbol{i}\Delta y(x,t) - \varepsilon d(x) y(x,t), \;\; x \in \Omega = (0,a) \times (0,b), \; t > 0, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{3.9}$$

Let

$$\mathcal{H} = L^2(\Omega)$$

with the standard L^2 inner product. Define

$$\begin{aligned} A &= \boldsymbol{i}\Delta, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ B &= -d(x). \end{aligned}$$

The operator A is skew adjoint, and has eigenvalues

$$\lambda_{l,m} = \boldsymbol{i}\beta_{l,m} = \boldsymbol{i}\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2, \quad l, m \in \mathbb{N},$$

and corresponding unit eigenfunctions

$$\phi_{l,m}(x) = \frac{2}{\sqrt{ab}} \sin\frac{l\pi x_1}{a} \sin\frac{m\pi x_2}{b}, \quad l, m \in \mathbb{N}.$$

When $a = b$, the gap condition (H3) is satisfied since

$$|\lambda_{l,m} - \lambda_{p,q}| \geq \frac{\pi^2}{a^2}$$

provided that they are distinct.

Note that we have multiple eigenvalues. To check condition (H4), we need verify eigenfunctions in the form

$$\phi_n(x) = \sum_{k=1}^{g_n} c_k \frac{2}{a} \sin\frac{l_k \pi x_1}{a} \sin\frac{m_k \pi x_2}{a}$$

subject to

$$\sum_k |c_k|^2 = 1,$$

$$l_p^2 + m_p^2 = l_q^2 + m_q^2, \quad p, q = 1, \cdots, g_n.$$

Choosing $d(x)$ same as in (3.2):

$$d(x) = 1 + \alpha \cos \frac{k\pi x}{a}, \qquad \text{k is odd.} \tag{3.10}$$

tnen we have

$$\begin{aligned} c_0 &= -Re\langle B\phi_n, \phi_n\rangle \\ &= \frac{4}{a^2} \int_0^a \int_0^a d(x) \left[\sum_{k=1}^{g_n} c_k \sin \frac{l_k \pi x_1}{a} \sin \frac{m_k \pi x_2}{a} \right]^2 dx_1 dx_2 \\ &= \frac{2}{a} \sum_{k=1}^{g_n} |c_k|^2 \int_0^a d(x) \sin^2 \frac{l_k \pi x_1}{a} dx_1 \\ &= 1. \end{aligned}$$

Therefore, $S_\varepsilon(t)$ is exponentially stable.

We can also choose a $d(x) \equiv d(x_1)$ only supported on a vertical strip, such as in (3.3) to verify condition (H4). If a/b is a rational number, the above analysis is still valid. If a/b is an irrational number, the gap condition (H3) is no longer true.

4 Conclusions

We have obtained a general result on the exponential stability of a linear conservative system with indefinite damping. It is shown that when a linear conservative system is perturbed along certain directions with small magnitude, exponential stability is still achievable even if the perturbation operator is not dissipative. This extends the result by Zuazua and Freitas [FZ] on the one dimensional wave equation. Our theorem not only covers many systems in elasticity, but also gives explicit estimates on the decay rate of the energy and the admissible values of the parameter ε. However, the gap condition (H3) is very restrictive. This limits the application primarily to one spatial dimension problems. Actually, even some one dimensional problems can have eigenvalues without a gap. For instance, in example 3 of the last section, when $EI/J \neq K/\rho$, one can check that the gap condition fails. Another example is a problem of 1-d wave equation with interior point masses studied in [HZ]. Thus, it is desirable to relax the gap condition. We will address this issue in our forthcoming paper.

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