

## NUMERICAL METHOD FOR VLASOV-LORENTZ MODELS.\*

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**Abstract.** The Lorentz operators are derived from either Boltzmann or Fokker-Planck collision operators when considering a mixture of species with disparate masses [10, 19]. This paper deals with a finite element numerical approximation of these operators. Numerical results on a simple model of photonics are presented.

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## 1. INTRODUCTION

In this paper, we are concerned with collision operators which are known as the Lorentz models. They are used, for example, to describe the effects of collisions of electrons with neutral, heavier particles. In first approximation, the electrons are assumed to diffuse with a stationary equilibrium distribution of target particles. The simplified Boltzmann-Lorentz (BL) model is then derived from the Boltzmann equation in the limit of small electron mass with respect to the mass of atoms. The BL model can also be found in the framework of semi-conductors (see [1]). More sophisticated models of similar form have also been recently studied in the context of wave-particle modelling [12], cometary plasma [11], quantum Boltzmann equation [7] or for polymers models [17]. In the same way, we can define a simplified Fokker-Planck-Lorentz (FPL) model which can be derived from the Fokker-Planck-Landau equation in the limit of small electron mass with respect to the mass of ions (see again [10]). Physical situations actually exist for which this operator appears as the leading order collision term in this asymptotic (see [19] for an example in the context of plasmas). We refer to [10] (respectively [6]) for a detailed presentation and for the justification of the disparate mass asymptotics (respectively the grazing collision limit) for Lorentz operators.

Lorentz collision operators describe the effects of elastic collisions on the distribution functions, only modifying the direction of the velocity of a particle and conserving its modulus or equivalently its (kinetic) energy. In other

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words, the Lorentz collision operators act independently on each sphere in the velocity space, that is for each value of the energy. Thus, they only change the repartition of the particles on each sphere. The equilibrium states are the isotropic functions, i.e. functions which are constant for each given velocity modulus. For the Lorentz models, the velocity modulus only acts as a multiplicative parameter and it can be taken into account by rescaling the time (see the factor  $\tau$  in this paper). Concerning the BL model, the associated partial differential equation is an integro-differential equation on the sphere of constant energy. Concerning the FPL model, the associated partial differential equation is the Laplace operator defined on the sphere of constant energy and usually called the Laplace-Beltrami operator. From now on, using a suitable normalization, we shall consider only one sphere with radius one.

A numerical method based on a cartesian discretization of the distribution function (in the velocity variable) and a finite difference scheme, following the same lines as previous works for the Fokker-Planck-Landau operator [3, 4, 5, 8, 18] have been investigated [2]. It turns out that such methods are not suitable for the Lorentz case since it is impossible to preserve the equilibrium states, i.e. the isotropic distribution functions. Due to the cartesian grid, the operator allows transfers of particle for neighbourhood cells which are not on the same sphere. This leads to an artificial thermalization of the distribution function: it converges towards a Maxwellian distribution function instead of converging to its projection onto isotropic functions, i.e. to its mean value with respect to the velocity direction. This artefact is reduced by refining the mesh size of the discretization but the computational cost becomes prohibitive. Therefore, such a cartesian approach is not suitable for the Lorentz operator. These difficulties leads us to consider another discretization based on finite elements on the sphere.

From a probabilistic point of view, the Fokker-Planck-Lorentz operator describes a brownian random walk of the particles on any sphere of constant energy. Note that the numerical simulations using probabilistic methods for the Boltzmann-Lorentz operator is often based on the approximation of this operator by such brownian model [14].

Our paper is organized as follows. In the first section, we describe the Lorentz operators. In section 2, we present the numerical method we have developed. This method is based on a splitting in time : one time step for the transport equation, one for the collision operator. We first focus on the collision part for which we develop a  $P1$  finite elements discretization of the sphere in the velocity variable and an implicit scheme in time. For the transport part, we use an uncentered explicit in time finite difference method in the space variable. In section 3, we present numerical results for a simplified model of photonics.

## 2. THE LORENTZ MODELS.

Let us now present the BL and FPL models in a homogeneous case (i.e. independent on the space variable). As underlined in the introduction, the distribution function  $f$  depends on the time  $t$  and on the direction of the velocity  $\omega \in S^2$ . It satisfies the following Cauchy problem

$$\partial_t f = Q(f), \quad f(t=0) = f^0, \quad (1)$$

where  $f^0 > 0$  is the initial data, and  $Q$  represents either the BL operator, in the sequel denoted by  $Q^B$ , or the FPL operator, denoted by  $P$ .

Concerning the BL case, the elastic-collision operator  $Q^B$  reads :

$$Q^B(f)(\omega) = \int_{S^2} B(\omega - \omega') [f(\omega') - f(\omega)] d\omega', \quad (2)$$

where  $S^2$  denotes the unit sphere of  $\mathbb{R}^3$ . The cross section  $B = B(\omega - \omega')$  is a positive function which expression is directly connected to the type of interacting potential between the particles. More precisely,  $B$  only depends on  $\|\omega - \omega'\|$ , or in other words, on the scattering angle  $\theta$ , i.e. the angle of deviation undergone during a collision by a particle having  $\omega$  as initial velocity and  $\omega'$  as post-collisional velocity. We have, in fact,  $\|\omega - \omega'\|^2 = 2(1 - \cos \theta)$ .

In the examples we shall consider in section 3, the cross sections are assumed to be either constant or perfectly localized. The first limit case is the so called *isotropic* BL operator. It implies an exponential relaxation in time

of the function towards the equilibrium state (like for a BGK equation). Indeed, the solution of the Cauchy problem (1) with  $Q = Q^B$  and  $B = 1/\tau$  is given by

$$f(\omega, t) = f^0(\omega) \exp(-4\pi t/\tau) + F^0(1 - \exp(-4\pi t/\tau)), \quad t \geq 0, \omega \in S^2,$$

where  $F^0 = \frac{1}{4\pi} \int_{S^2} f^0(\omega') d\omega'$  is the constant equilibrium state.

In the second limit case, so called grazing collision limit (see [6]) for the BL model, the operator  $Q^B$  converges towards the FPL operator  $P$  (see the introduction and in [6] for further details on this asymptotic). We give here its form in spherical coordinates

$$P(f) = \Delta_\omega f = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial f}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \right]. \quad (3)$$

Let us consider the associated Cauchy problem (1) with an initial data  $f^0$  in  $H^1(S^2)$ . The variational formulation of this problem is given by

$$\partial_t \left( \int_{S^2} f \varphi d\omega \right) = - \int_{S^2} [\nabla_\omega f]^T \nabla_\omega \varphi d\omega, \quad (4)$$

for any test function  $\varphi \in H^1(S^2)$  where  $\nabla_\omega f$  denotes the projection of the gradient of  $f$  (with respect to the velocity variable) on the plane orthogonal to the sphere at point  $\omega$ . That can be interpreted as follows : there exists  $\bar{f}(v)$  such that  $f = \bar{f}|_{S^2}$  and  $\nabla_\omega f = S \nabla \bar{f}$  with  $S = Id - \omega \otimes \omega$  is the projection on the plane orthogonal to the direction  $\omega$ .

In both cases (i.e.  $Q^B$  or  $P$ ), the equilibrium state is given by  $F^0 = \frac{1}{4\pi} \int_{S^2} f^0(\omega') d\omega'$ . These Lorentz operators have the following properties: the conservation of mass and the exponential decay towards 0 of  $(f - F^0)$  (in  $L^2$  norm for example). We refer to [6] for a detailed spectral analysis of these operators.

**Remark:** For the quantum Boltzmann operator, the cross section  $B$  is given by the Born series, which can be approximated by  $\|\hat{V}\|^2$  where  $V$  is the potential of interaction . When  $V$  is localized,  $B$  tends towards a constant. On the contrary, when  $V$  becomes small with large support, its Fourier transform is localized and we obtain the grazing collision limit. We refer to [7] for details.

### 3. FINITE ELEMENT DISCRETIZATION OF THE LORENTZ CAUCHY PROBLEM.

We first look at the FPL model. Consider a triangulation of the sphere, i.e. replace the sphere  $S^2$  by a polyhedral approximation  $S_h^2$ . This triangulation is first based on an icosahedron, then each of the 20 triangles are splitted in equilateral triangles  $T_k$  whose vertices are projected onto the sphere. We denote by  $N_T$  the number of triangles and by  $N_S$  the number of vertices (that are on the sphere). The surface is approximated by a union of plane triangles:  $S_h^2 = \bigcup_{k=1}^{N_T} T_k$ . Let us denote by  $\omega_i$  the vertices of the triangulation with  $i = 1, \dots, N_S$ . Note that a similar finite element method for the Laplace-Beltrami equation has been proposed in [16] for a model of mathematical meteorology. The author constructs a geometrically intrinsic base vector field for the Galerkin method. Our method (presented below) is derived in a simpler way by using barycentric coordinates that can be used for an arbitrary triangulation of the sphere and that is not restricted to icosahedron based triangulation.

We shall use a  $P1$  finite element method. Let us denote by  $\varphi^i$  the continuous function defined on  $S_h^2$ , affine on each  $T_k$ , and such that  $\varphi^i(\omega_j) = \delta_{i,j}$ . We note by  $H_h^1$ , the space spanned by the so called hat functions  $\varphi^i$

$$H_h^1(S_h^2) = \{ \varphi^h : S_h^2 \rightarrow \mathbb{R}, \varphi^h \in C^0(S_h^2), \varphi^h|_{T_k} \in P^1(T_k) \}. \quad (5)$$

The approximated distribution function  $f^h$  is entirely described by its value at the vertices as

$$f^h = \sum_{i=1}^{N_S} f_i \varphi^i, \text{ with } f_j = f^h(\omega_j).$$

The discretized (in  $\omega$ ) variational formulation of (4) can be written as

$$\partial_t \left( \int_{S_h^2} f^h \varphi^h d\omega \right) = - \sum_{k=1}^{N_T} \int_{T_k} [\nabla f^h(\omega)]^T S(\omega^k) \nabla \varphi^h(\omega) d\omega, \forall \varphi^h \in H_h^1 \quad (6)$$

where  $\omega^k$  is the unit normal to triangle  $T_k$  and  $S(\omega^k)$  the orthogonal projection on the plane of the triangle. Note that  $\omega \approx \omega^k$  for all  $\omega \in T_k$ . This modification ( $\omega \mapsto \omega^k$ ) is crucial in order to preserve the conservation properties at the discrete level. The variational discretized problem is then equivalent to : find  $(f_i(t))_{i=1 \dots N_S}$  such that

$$\begin{cases} \sum_j \partial_t f_j(t) A_{ij} = \sum_j f_j(t) B_{ij}, \forall i \in \{1, \dots, N_S\} \\ f(t=0) = f^0, \end{cases} \quad (7)$$

where

$$A_{ij} = \int_{S_h^2} \varphi^j(\omega) \varphi^i(\omega) d\omega$$

and

$$B_{ij} = \sum_k \int_{T_k} \nabla \varphi^j(\omega) S(\omega^k) \nabla \varphi^i(\omega) d\omega.$$

We note that  $A = (A_{ij})$  is the mass matrix whereas  $B = (B_{ij})$  is the matrix of the discrete Laplacian. The coefficients of these matrix are computed as follows. We define

$$A_{ij} = \sum_{k=1}^{N_T} A_{ij}(T_k), \quad A_{ij}(T_k) \stackrel{def}{=} \int_{T_k} \varphi^j \varphi^i d\omega$$

and the same notations for  $B$ . Moreover, we denote the 3 vertices of any triangle  $T_k$  as  $\omega_i, \omega_i^+, \omega_i^-$ . If  $i = j$ , we obtain

$$A_{ii}(T_k) = \frac{1}{6} A_k = \frac{1}{6} \|\omega_i^+ \wedge \omega_i^-\|,$$

where  $A_k$  is the area of the triangle  $T_k$ . On the other hand, when  $j$  is equal to  $i^+$  (or  $i^-$ ), we have  $A_{ij}(T_k) = \frac{1}{12} A_k$  and for any other value of  $j$ ,  $A_{ij}(T_k) = 0$ . We use a so called mass lumping approximation of  $A$ , i.e. we replace  $A$  by a diagonal matrix  $(a_i)$  (see [20] for this approximation).

Let us now turn to the matrix  $B$ . Since the functions used for the approximation are  $P1$  finite elements (i.e. affine functions), their gradients are constant on each triangle. This constant can be computed using a barycentric parametrization of the triangle  $T_k$ . We get:

$$\nabla \varphi^i |_{T_k} \stackrel{def}{=} \vec{C}_k^i = \frac{1}{V_k} \omega_i \vec{\omega}_i^+ \wedge \omega_i \vec{\omega}_i^-$$

where  $V_k = \det(\omega_i, \omega_i^+, \omega_i^-)$  is the volume of the tetrahedron based on the origin  $O$  (the center of the sphere) and  $T_k$ . Finally, the unitary normal to  $T_k$  is given by

$$\vec{\omega}_k = \frac{\omega_i \vec{\omega}_i^+ \wedge \omega_i \vec{\omega}_i^-}{A_k}.$$

Hence, we obtain, for any  $(i, j, k)$  such that  $\omega_i$  and  $\omega_j$  are vertices of  $T_k$

$$B_{ij}(T_k) = A_k \vec{C}_k^i{}^T S(\vec{\omega}_k) \vec{C}_k^j,$$

and 0 elsewhere.

We now present the discretization in time. In order to avoid time step restriction that will depend on the mesh size (and on the multiplicative factor when considering different modulus of the velocity, see the introduction) we use an implicit scheme in time

$$A(f^{n+1} - f^n)/\Delta t = Bf^{n+1}.$$

This linear system is solved using a conjugated gradient method preconditioned by an approximated/incomplete Choleski factorization (as explained in [20], chapter 4). This inversion is done once. We use a compressed sparse row storage for the matrices  $A$  and  $B$ . We have adapted a code described in [20].

This method verifies the maximum principle and the conservation of mass (and of the equilibrium state). Some preliminary numerical tests have been performed on one exact solution given by the spectral analysis [6].

Concerning the isotropic BL operator (with a constant cross section  $B = 1/\tau$ ), we use the explicit solution (that also satisfies conservation of mass and of equilibrium states)

$$f_i(t) = f_i^0 \exp(-A^h t/\tau) + F^h(1 - \exp(-t/\tau)), \tag{8}$$

where  $A^h$  is the trace of the mass matrix ( $A^h = \sum_i a_i \approx 4\pi$ ) and  $F^h$  is defined in the same way as for the continuous equation, i.e.  $F^h$  is the associated constant equilibrium state:

$$F^h = \sum_{i=1}^{N_s} f_i^0 a_i / A^h \approx F^0.$$

#### 4. NUMERICAL RESULTS.

The distribution function now depends also on the space variable, i.e.  $f = f(x, \omega, t)$  with  $\omega \in S^2$ ,  $x \in [0, 1]$  and  $t \geq 0$ . Moreover, it satisfies the scaled one-dimensional Vlasov-Lorentz equation :

$$\partial_t f + \omega^1 \partial_x f = \frac{1}{\tau} P(f), \quad f(t=0) = f^0, \tag{9}$$

where  $\omega^1 = \cos(\theta)$  is the first component of the velocity and  $\tau$  is the scaled relaxation time. This equation is supplemented by boundary conditions for entering particles, i.e. at  $x = 0$  with  $\omega^1 > 0$  and at  $x = 1$  with  $\omega^1 < 0$  we impose a condition (as described below).

##### 4.1. Boundary conditions

The boundary conditions can be at each border  $x = 0$  or  $x = 1$  either

- a fixed injection profile

$$(i) \quad f(x, \omega, t) = g(\omega),$$

which is called free boundary condition when  $g = 0$  (no particles enters the domain for this side)

- an accommodation model that describes the re-emission of the outgoing particles according to a given profile

$$(ii) \quad f(x, \omega, t) = K(t)g(\omega)$$

where the operator  $K$  is chosen such that the flux of outgoing particles (e.g.  $\int_{\omega^1 < 0} f(0, \omega, t) \omega^1 d\omega$ ) equals the flux of re-entering particles.

- a specular boundary condition

$$(iii) \quad f(x, \omega, t) = f(x, \omega^*, t)$$

where  $\omega^* = (-\omega^1, \omega^2, \omega^3)$ . This can be easily used since the triangulation we consider is symmetric with respect to the mapping  $\omega \mapsto \omega^*$ .

- periodic boundary condition (the space variable being on the torus), the distribution function at boundary are equals i.e.

$$(iv) \quad f(0, \omega, t) = f(1, \omega, t), \quad \forall \omega$$

## 4.2. A simple photonics model

Let us consider a very simple model of photonics. We shall consider a focussed beam of photons that enters from an empty bounded region where it interacts with particles and scatters. The boundary condition, corresponding to the injection of a mono-kinetic beam in the direction of the  $x$ -axis, are given by

$$f(0, \omega^1 > 0, t) = \delta_{e^1},$$

where  $\delta_{e^1}$  denotes a delta measure on the velocity in the direction of the  $x$ -axis, and

$$f(1, \omega^1 < 0, t) = 0,$$

i.e. the particles that exit never return (type (i) with "free boundary" at  $x = 1$ ). The results are obtained for the initial data equal to 0 (or almost zero, a uniform background with small density). Up to our knowledge, there are no analytical solution of this problem even in the stationary case.

The approximated distribution function is represented by its value on a regular mesh in space  $x_j = j/N$  with  $j = 0, \dots, N$  and on the vertices  $\omega_i$  of the triangulation, described in the preceding section. The value of the function at time  $t^n$ , position  $x_j$  and velocity  $\omega_i$  is denoted  $f_{i,j,n}$ . We use a splitting in time of the transport and of the collision part. For the collision part, we use the method described in section 3. For the transport part, we use an uncentered explicit scheme, the evolution is independent for each  $i$ . For any  $i$  such that  $\omega_i^1 > 0$

$$\frac{f_{i,j,n+1} - f_{i,j,n}}{\Delta t} + \omega_i^1 \frac{f_{i,j,n} - f_{i,j-1,n}}{\Delta x} = 0,$$

and the converse finite difference scheme is used for any  $i$  such that  $\omega_i^1 < 0$ .

## 4.3. Reflexion, transmission and focalization.

Let us now define the coefficients of reflexion  $R$ , transmission  $T$ , the focalization  $F$  and the density  $n$  as

$$\begin{aligned} R(t) &\stackrel{def}{=} - \int_{\omega^1 < 0} f(x=0, \omega, t) \omega^1 d\omega, \\ T(t) &\stackrel{def}{=} \int_{\omega^1 > 0} f(x=1, \omega, t) \omega^1 d\omega, \\ F(t) &\stackrel{def}{=} \int_{\omega^1 > \omega^F} f(x=1, \omega = e_1, t) \omega^1 d\omega, \\ n(x, t) &\stackrel{def}{=} \int_{\omega \in S^2} f(x, \omega, t) d\omega. \end{aligned}$$

The flux of incoming particles is normalized as follows  $\int_{\omega^1 > 0} f(0, \omega) \omega^1 d\omega = 1$ . The first coefficient,  $R$ , represents the flux of reflected particles; the second one,  $T$ , the flux of outgoing particle at  $x = 1$ , that is the flux of transmitted particles. The third coefficient,  $F$ , is the flux of particles that exit the domain with a velocity

almost parallel to the direction of the beam ( $\omega^F$  being close to 1, when  $\omega^F \rightarrow 1$ , only particles with velocity equal  $e_1$  are taken into account in the integral). This last parameter  $F$  measures the focalization of the beam ( $F = 1$  for all  $\omega^F$  means that all the particles exit the domain with velocity  $e_1$ ). When  $t \rightarrow \infty$ , assuming there exists a stationary state  $f^\infty$  and that  $f(t) \rightarrow f^\infty$ , the reflexion and transmission coefficients tend to constant values such that  $R + T = 1$  that only depends of the relaxation time  $\tau$ . The last definition,  $n$ , gives the usual density number that depends on space and on time.

**Remark :** Note that one can also consider for the focalization coefficient only the outgoing particles that have exactly the right velocity i.e. that do not suffer any collision. This leads to consider only the measure part of the distribution on  $\omega = e_1$ . For the BL case, one can estimate this coefficient since, due to the integral structure of the operator, it is expected that the distribution function remains of the form of the entering Dirac mass that will decrease exponentially and the remaining part of the distribution that is smooth. In our case, the focalisation (with  $\omega^F = 1$ ) for the BL case should be  $\exp(-4\pi/\tau)$ . For the FPL case, the Dirac mass should be instantaneously smoothed (by analogy with the heat equation) and thus the exact focalisation (i.e. with  $\omega^F = 1$ ) vanishes. For discrete model, one observes that this quantity goes to 0 when refining the triangulation of the sphere in the FPL case whereas it converges toward some fixed value for the BL case. This explains the reason why we decided to take  $\omega^F \approx 1$  instead of  $\omega^F = 1$ .

#### 4.4. Choice of the relaxation time

We shall consider collision times (i.e. the multiplicative factor in front of the collision operator that can be related to the modulus of the velocity in the context of plasma physics as explained in the introduction or in [6]) of the form

$$\frac{1}{\tau}Q(f), \quad \text{with } \tau = 2^k$$

and we consider various value of  $k$ . Note that the large (positive) value of  $k$  gives large collision time (that have to be compared with the time needed to cross the domain that is assumed of length 1) or equivalently, a collisionless or free transport regime. On the other hand, for large (negative) value of  $k$ , the collisions dominate.

Moreover, we consider the two models (FPL  $P$  or BL  $Q^B$ ) with a normalization factor such that the number of collisions is the same. More precisely, we consider a multiplicative coefficient, that is related to the second moment of the cross section, as explained in [6], for the FPL operator corresponding to the grazing collision of the BL operator with a uniform cross section equal to 1. In practice, we replace the time relaxation  $\tau$  in the explicit solution (8) for the BL operator by  $\frac{3\tau}{4\pi}$ .

Note that the modification of the time relaxation can be interpreted in another way. Using a scaling, it can be easily seen that the solution  $f^L$  for a domain of length  $L$  with  $\tau = 1$  and the solution  $f_\tau$ , described above (for the domain with fixed  $L = 1$  and a relaxation time  $\tau$ ) are linked by the relation

$$f^L(x, \omega, t) = f_\tau(x/L, \omega, t/\tau), \quad x \in [0, L], t \geq 0, \omega \in S^2,$$

provided that  $L\tau = 1$ . The evolution of the solution versus  $\tau$  for a given domain (of length 1) is thus related to its evolution versus the domain size (with a relaxation time equal to 1).

#### 4.5. Parameters of the simulations

We performed a lot of tests with different numbers of triangles on the sphere in the velocity space and/or of space discretization points in  $x$ . We present results with a rather fine triangulation and coarse grid in  $x$  since we are interested more specifically with the collision operator. The number of grid points for the space variable is equal to 20 in all the presented simulations, the number of triangles in the triangulation of the sphere is equal to 1280. The time steps are chosen from the convective transport time using a CFL of 1 (the maximal velocity being 1, and the space step equal to  $1/20$ , we obtain  $\Delta t = 1/20$ ).

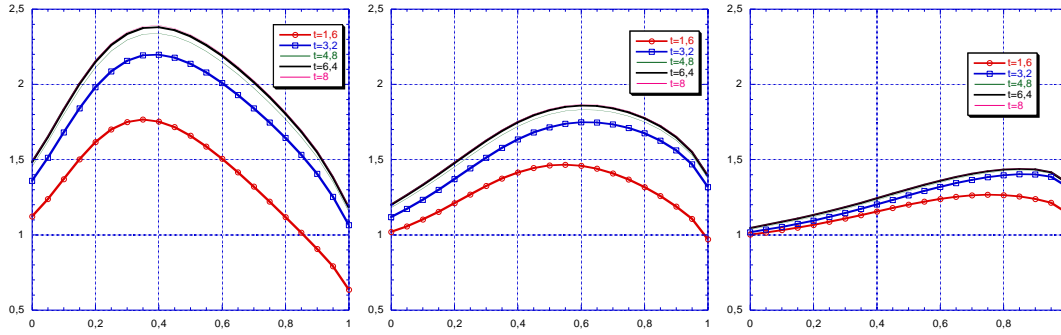


Fig. 1 : Density versus  $x$ , various time,  $k = 1$  ,  $k = 2$  ,  $k = 3$

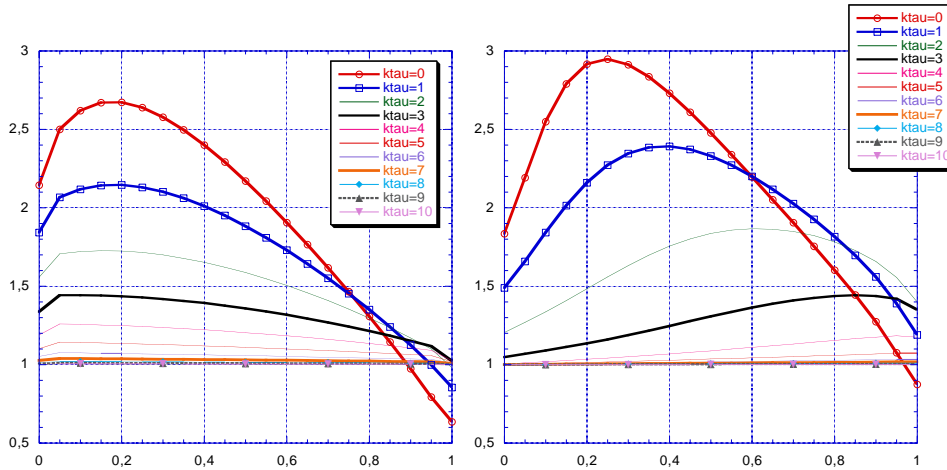


Fig 2 : Density versus  $x$ , various  $k$ ,  $t = 160$  for FPL (left), BL (right)

#### 4.6. Description of the results

On the three first figures, we plot the evolution in time of the density (which is a function of the space variable) obtained with the diffusion model for different values of  $k$ . We only plot the intermediate case ( $k = 1, 2, 3$ ). Indeed for  $k \rightarrow \infty$  (collisionless limit), the density converges to a uniform density that is the solution of the free transport problem :

$$f(x, v, t) = \delta_{e_1}(v)g(x, t)$$

with  $g(x, t) = 1$  if  $x < t < 1$  or if  $t \geq 1$ , 0 elsewhere. On the contrary, for the limit  $k \rightarrow -\infty$  (collision dominated case), a boundary layer appears near the injection boundary : particles are almost immediately reflected since the collision operator is very efficient and it ejects the particles out of the domain on a small characteristic length (that can be interpreted as a length of penetration of the beam in the domain). This phenomenon is not illustrated in the present paper because it requires to refine the mesh in the space variable  $x$  to get an accurate representation of the solution in this boundary layer. Therefore, the results presented for  $k < 0$  are not really reliable. For the difficulties arising in this dominated collision case or diffusion limit for a 2D neutronics problem, we refer to [15].

On figure 2, we plot the density obtained after large simulation time ( $t = 160$ ) i.e. the "stationary" state for different values of the time relaxation parameter  $k$  for either the BK or FPL collision operator.

On figure 3, we plot the coefficient of reflexion  $R$  and transmission  $T$  for both the FPL and the BL operator. These coefficients are nothing but the outgoing flux at the boundary  $x = 0$  and  $x = 1$  respectively. In the



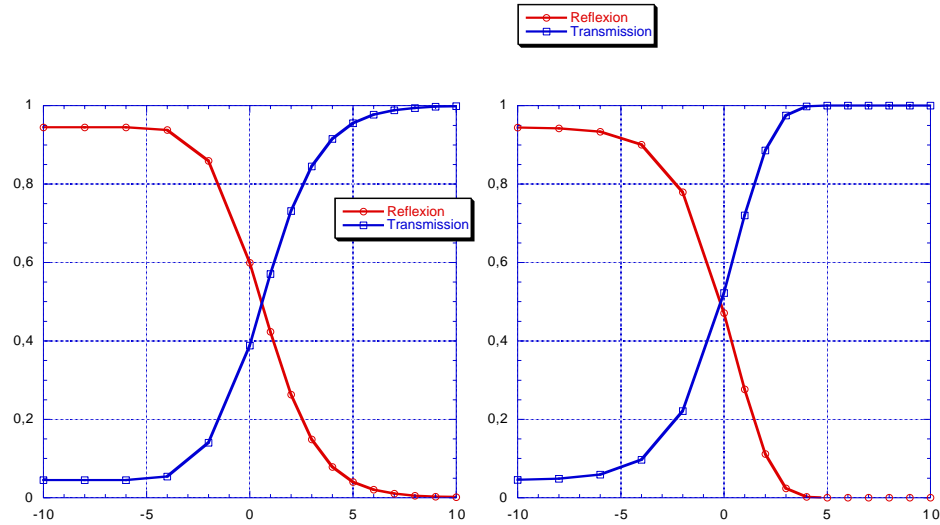


Fig 3: Reflexion/Transmission coefficients for FPL (left), BL(right) for  $k = -10, \dots, 10$

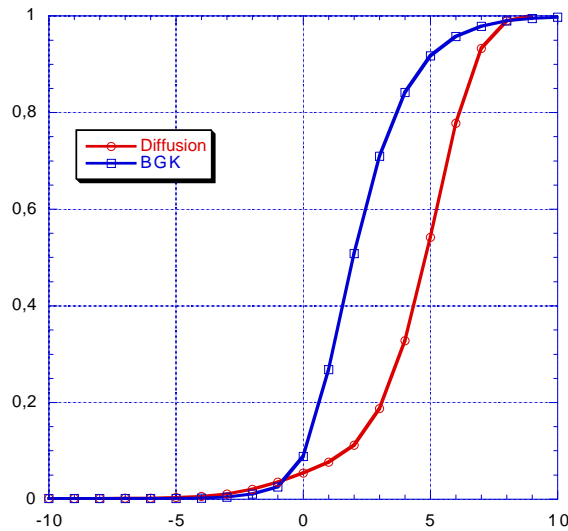


Fig 4: Focalization versus  $k$  at  $t = 160$ , comparison BL-FPL

stationary case, these fluxes are such that the mass in the domain is constant and we obtain that  $R + T = 1$ . For large negative values of  $k$ , the trend to equilibrium is very slow and we stop the computation up to  $t = 480$ , although the stationary state is not reached ( $R + T$  being less than 1, implies that the particles still fill the domain).

On the last figure, we draw the focalization parameter  $F$  with  $\omega^F = 0,95$  in order that there is a significant number (about 40) of points in the integral. We compare the two models of collisions. The FPL operator has a stronger effect on the focalization than the BL operator in particular for intermediate values of  $k$  between 0 and 5.

In conclusion, the finite element method has been implemented for the FPL operator on the sphere. The first results on this simple model are cheering. More works need to be done for a general BL operator that is with an arbitrary cross section. Other future possible works are concerned with the collision dominated limit

(or the diffusion limit), the comparison of this finite element method with other approaches (finite volumes, particle methods, probabilistic approaches) and other applications that are listed in the introduction. For the first aspect, we could wonder what are the constraints on the repartition of the points on the sphere in order to have the correct diffusion limit : an answer has been given for the BL case in [15] in two dimensional case, this can be considered either for the 3D case or for the FPL equation.

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