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GENERALIZED MULTIPLYING BOUNDARY CONDITIONS AND B-BOUNDED SEMIGROUPS *

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Abstract. A transport problem for the particle density function $n(x, v, t)$ in the bounded region $[-b, +b]$ is considered. The particle velocities are assumed to be unbounded. The boundary conditions are expressed by means of a general *multiplying* boundary operator Λ, relating the incoming and outgoing flux of particles. The generation of a B-bounded semigroup (see [1], [2], [3]) is proved, and the solution of the transport problem is explicitly given by means of the B-bounded semigroup.

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1. Introduction

The motion of a system of charged or uncharged particles in a bounded region and subject to absorption by the host medium is modeled by means of a transport equation equipped with some kind of boundary conditions (usually a convex combination of reflection and diffusion). When no external force acts on the system of particles, the transport equation is given by the sum of the Vlasov equation with null force term (see [8]) and the absorption term. Moreover, when the autoinduct "internal" forces are not considered, the coupling of the Vlasov equation with the Poisson or Maxwell equations is not taken into account.

In this paper, we are mainly interested in the semigroup generation properties of the transport operator L, sum of the free-streaming term and the absorption term, equipped with *general multiplying* boundary conditions (see also [7]). In particular, the multiplying boundary conditions are described by means of a linear bounded and positive operator Λ relating the incoming and outgoing fluxes of particles. The norm of the boundary operator Λ satisfies the condition $\|\Lambda f\|_{in} \geq \alpha \|f\|_{out}$, with $\alpha > 1$. This fact implies a *multiplication* of particles at the boundaries. The absorption term of the transport operator is then needed in order to avoid the *blow up* of the system of particles. Otherwise, the bounded region would be quickly saturated. Such a boundary operator Λ can model any kind of boundary conditions as diffusion or reflection at the boundaries ([10]).

The generation properties of the free-streaming operator defined in a slab and equipped with multiplying boundary conditions were first studied in [10]. Furthermore, the explicit form of the solution of the one-dimensional particle transport problem (including collisions) with multiplying boundary conditions was given in [6]. Both in [6] and [10] the particle velocities where assumed to be bounded. In this paper we shall assume

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that the particle velocities are unbounded (i.e. $v \in \mathbb{R}$). In other words, our goal is the study of a transport problem in a one-dimensional bounded region with unbounded coefficients.

In the quoted papers the absorption term has the form $\sigma n|v|$, where σ is the absorption cross section, $|v|$ is the modulus of the velocity and is fixed (e.g. $|v| = 1$), and n the density function. In this paper we shall assume that the absorption term is equal to $\sigma|v|n$ with $|v| \in \mathbb{R}$. In [7] was proved that such a transport operator equipped with multiplying boundary conditions generates a once-integrated semigroup, and also a C-semigroup, but not a C_0-semigroup. Moreover, the explicit form of the solution, under some specific assumptions on the initial data, was written by means of the integrated semigroup and of the C-semigroup. On the other hand, in the present paper we write the explicit form of the solution of the transport problem in terms of the initial data and of a B-bounded semigroup (or briefly B-semigroup). In fact, B-semigroups were first introduced in [3] and [6] in order to solve transport problems with multiplying boundary conditions (see also [1], [2] for more details on B-semigroups).

In this paper, we shall apply the method used in [4] and [6] in order to construct the B-semigroup. Moreover, we shall evaluate the number of particles in the considered region by means of the semigroup generated by the streaming operator and a suitable bounded operator B. The applied method, first introduced in [4], consists in splitting the density function n in n_j density functions, for $j = 0, 1, \ldots$, where j represents the number of interactions of the particles with the boundaries. The boundary conditions then read: $n_j^{in} = \Lambda n_{j-1}^{out}$. We shall call *mother* a particle just before the interaction with the boundaries and *daughter* a particle just after the interaction. This method yields directly to the definition of the bounded operator B, and thus to the B-semigroup.

The paper is organized as follows. In section 2, we define the operators and spaces needed in order to write the abstract form of the transport problem for the densities n and n_j. In section 3, we introduce the bounded operator B, we prove the existence of the B-bounded semigroup, and finally we obtain the solution of the transport problem.

2. Operators and spaces

We define the set $\Omega = ([-b, +b] \times \mathbb{R})$ and the incoming and outgoing sets:

$$\Omega^{in} = (\{-b\} \times (0, +\infty)) \times (\{+b\} \times (-\infty, 0)),$$

$$\Omega^{out} = (\{-b\} \times (-\infty, 0)) \times (\{+b\} \times (0, +\infty)).$$

Further, we consider the Banach space $X = L^1(\Omega)$ equipped with the usual norm:

$$\|f\| = \int_{-b}^{+b} dx \int_{-\infty}^{+\infty} |f(x, v)|\, dv,$$

and the incoming and outgoing Banach spaces $X^{in} = L^1(\Omega^{in}; |v|dv)$ and $X^{out} = L^1(\Omega^{out}; |v|dv)$ respectively equipped with the norms:

$$\|f\|_- = \int_0^{+\infty} |f(-b, v)| v\, dv + \int_{-\infty}^0 |f(+b, v)|\, |v|\, dv,$$

$$\|f\|_+ = \int_{-\infty}^0 |f(-b, v)|\, |v|\, dv + \int_0^{+\infty} |f(+b, v)| v\, dv.$$

Finally, we introduce the infinite product Banach spaces Y, Y^{in} and Y^{out} given by:

$$
\begin{aligned}
Y &= \left\{ f \in X \times X \times X \times \dots\,,\ f = (f_j),\ f_j \in X,\ \sum_{j=0}^{\infty} \|f_j\| < \infty \right\},\\
Y^{in} &= \left\{ f \in X^{in} \times X^{in} \times X^{in} \times \dots\,,\ f = (f_j),\ f_j \in X^{in},\ \sum_{j=0}^{\infty} \|f_j\|_- < \infty \right\},\\
Y^{out} &= \left\{ f \in X^{out} \times X^{out} \times X^{out} \times \dots\,,\ f = (f_j),\ f_j \in X^{out},\ \sum_{j=0}^{\infty} \|f_j\|_+ < \infty \right\},
\end{aligned}
$$

respectively equipped with norms:

$$
\|f\|_{_Y} = \sum_{j=0}^{\infty} \|f_j\|\ , \quad \left\|f^{in}\right\|_{in} = \sum_{j=0}^{\infty} \|f_j^{in}\|_-\ , \quad \left\|f^{out}\right\|_{out} = \sum_{j=0}^{\infty} \|f_j^{out}\|_+\ ,
$$

where $f \in Y$ is the infinite vector $f = (f_0, f_1, f_2, \dots) = (f_j)$ with $f_j \in X$ for all $j = 0, 1, 2, \dots$, and similarly for f^{in} and f^{out}.

Let $n = n(x, v, t) \in Y$ be the vector:

$$
n = \begin{pmatrix} n_0(x,v,t) \\ n_1(x,v,t) \\ n_2(x,v,t) \\ \vdots \end{pmatrix},
$$

where $n_j = n_j(x, v, t) \in X$ represents the density function of particles which at time $t > 0$ are in $x \in [-b, +b]$ with velocity $v \in \mathbb{R}$ and have undergone j interaction with the boundaries, for all $j = 0, 1, 2, \dots$.

The evolution problem describing the motion of particles in a bounded region $[-b, +b]$ and subject to absorption by the host medium and interactions with the boundaries can be written for every density function n_j as follows:

$$
\begin{cases} \dfrac{dn_j(t)}{dt} = L_j\, n_j(t)\ ,\ t \geq 0\,, \\ n_j(0) = n_{j,0} \end{cases} \tag{1}
$$

where $n_j(\cdot, \cdot, t)$ is a function, to be determined, from $[0, +\infty)$ with values in X, and the operators L_j are defined as follows, for $j = 0, 1, 2, \dots$:

$$
\begin{aligned}
L_j f_j &= -v \frac{\partial f_j}{\partial x} - |v| \sigma f_j\ ,\\
D(L_j) &= \left\{ f_j \in X,\ v\frac{\partial f_j}{\partial x} \in X,\ |v|\sigma f_j \in X,\ f_j^{in} \in X^{in},\ f_{j-1}^{out} \in X^{out},\ f_j^{in} = \Lambda f_{j-1}^{out} \right\}.
\end{aligned} \tag{2}
$$

In (2) f_j^{in} and f_j^{out} represents the incoming and outgoing fluxes of particles and are the traces of the function f_j on the spaces X^{in} and X^{out}:

$$
f_j^{in} = f_j|_{X^{in}}\ , \quad f_j^{out} = f_j|_{X^{out}}.
$$

Moreover, the operator Λ is assumed to be a linear, bounded and positive operator with $D(\Lambda) \subset X^{out}$ and $R(\Lambda) \subset X^{in}$. The boundary operator Λ may represents any kind of boundary conditions, as for example reflection or diffusion. The assumption on the norm the boundary operator Λ shall be specified later on.

Remark 2.1. The boundary conditions appearing in the domain $D(L_j)$ are to be interpreted as follows: the incoming flux of particles f_j^{in} which have undergone j interactions with the boundaries is related to the outgoing flux of particles f_{j-1}^{out} which have undergone $j-1$ interactions with the boundaries by means of the operator Λ. The outgoing particles are called *mothers* and the incoming one are called *daughters*. For instance, f_0 is the density of particles which have never interacted with the boundaries, f_1 is the density of particles which have undergone one interaction with the boundaries, and so on. We shall impose that $f_{-1}^{out} = 0$. □

Introducing the operators L and $\hat{\Lambda}$:

$$\begin{aligned} Lf &= \begin{pmatrix} L_0 & 0 & 0 & \dots \\ 0 & L_1 & 0 & \dots \\ 0 & 0 & L_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} L_0 f_0 \\ L_1 f_1 \\ L_2 f_2 \\ \vdots \end{pmatrix} \\ D(L) &= \left\{ f \in Y,\ v\frac{\partial f}{\partial x} \in Y,\ |v|\sigma f \in Y,\ f^{in} \in Y^{in},\ f^{out} \in Y^{out},\ f^{in} = \hat{\Lambda} f^{out} \right\}, \end{aligned} \tag{3}$$

$$\hat{\Lambda}\, f = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \Lambda & 0 & 0 & \dots \\ 0 & \Lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda f_0 \\ \Lambda f_1 \\ \vdots \end{pmatrix}, \quad D(\hat{\Lambda}) = Y^{out}, \tag{4}$$

the abstract problem reads:

$$\begin{cases} \dfrac{dn(t)}{dt} = Ln(t)\,,\ t \geq 0 \\ n(0) = n_0 \end{cases} \tag{5}$$

where $n(\cdot,\cdot,t)$ is a function from $[0,+\infty)$ into Y, and n_0 is the initial condition:

$$n_0 = \begin{pmatrix} n_{0,0} \\ n_{1,0} \\ n_{2,0} \\ \vdots \end{pmatrix}.$$

We shall assume that there exists $\alpha > 1$ such that $\|\Lambda f\|_- \geq \alpha \|f\|_+$. Therefore, we have

Lemma 2.2. *The boundary operator $\hat{\Lambda}$ has norm grater than one: $\|\hat{\Lambda}\| > 1$.*

Proof: We first remark that the operator $\hat{\Lambda}$ acts from the product space Y^{out} into Y^{in}, while the operator Λ acts from the space X^{out} into X^{in}. Furthermore, for $f \in Y^{out}$, we have:

$$\|\hat{\Lambda} f\|_{in} = \sum_{j=0}^{\infty} \|\Lambda f_j\|_- \geq \sum_{j=0}^{\infty} \alpha \|f_j\|_+ = \alpha \|f\|_{out},$$

Concluding the proof. □

This assumption yields to a multiplication of particles at the boundary, and justifies, on the one hand, the absorption term in the transport operators L needed in order to avoid the blow up of the system of particles, and on the other hand, the name *multiplying* for such a kind of boundary conditions.

In [7], the existence and uniqueness of the solution of problem (5) were proved both in the case $\|\Lambda\| \le 1$ and $\|\Lambda\| > 1$. The explicit form of the solution was given by means of a once integrated semigroup generated by the operator L in the case of multiplying boundary conditions and by means of the strongly continuous semigroup generated by L in the case $\|\Lambda\| \le 1$ (under a suitable assumption on n_0). Concerning, multiplying boundary conditions, the solution was given also by means of a C-semigroup generated by the operator L. Our goal is to write the solution of problem (5) by means of a B-semigroup (see [1],[2] and [3]).

3. The operator B

We begin recalling the definition of B-bounded semigroup (see [1], [2] and [3] for more details).

Definition 3.1. Let A and B be two linear operators on the Banach space X, such that:
i) B is bounded;
ii) $D(A) \subset X,\ R(A) \subset X,\ \rho(A) \supset (0, +\infty)$ where $\rho(A)$ is the resolvent set of A, and the resolvent $R(zI - A)$ is such that for every $z > 0$, $R(zI - A) = X$.
Then the family $\{Z(t), t \ge 0\}$ is a B-bounded semigroup if it satisfies:
a) $\{Z(t), t \ge 0\}$ is bounded and $\|Z(t)f\| \le \|Bf\|$ for every $t \ge 0,\ f \in X$,
b) $\phi(t) = Z(t)f \in C([0, +\infty), X)$ for every $f \in X$,
c) $Z(t)f = Bf + \int_0^t Z(s)Af\,ds$ for every $f \in D(A)$,
where the integral in c) is a strong integral. □

Remark 3.2. We note that assumption ii) is satisfied if the operator A is a closed operator. Moreover, if A is the generator of a strongly continuous semigroup $exp(tA)$ and A and B commute, then $Z(t) = Bexp(tA)$ is a B-semigroup. □

Let us now define the linear operator B as follows:

$$B\,f = \begin{pmatrix} \beta^0 & 0 & 0 & \dots \\ 0 & \beta^1 & 0 & \dots \\ 0 & 0 & \beta^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ \beta f_1 \\ \beta^2 f_2 \\ \vdots \end{pmatrix}, \quad D(B) = Y, \tag{6}$$

where $0 < \beta < 1/\|\Lambda\|$. This assumption implies the following results.

Lemma 3.3. *The operator B defined by (6) is bounded and $\|B\| < 1$.*

Proof: We first remark that our assumption on the norm of Λ implies that $\beta < 1$. Moreover, we have that:

$$\|B\,f\|_{Y} = \sum_{j=0}^{\infty} \|\beta^j f_j\| < \sum_{j=0}^{\infty} \|f_j\| = \|f\|_{Y}.$$

□

Lemma 3.4. *The operators $\hat{\Lambda}$ and B defined by (4) and (6) respectively, verify the following relation:*

$$B\,\hat{\Lambda} = \beta \hat{\Lambda}\,B.$$

Proof: We have :

$$\hat{\Lambda}\, B = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \Lambda & 0 & 0 & \dots \\ 0 & \beta\Lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Since:

$$B\,\hat{\Lambda} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \beta\,\Lambda & 0 & 0 & \dots \\ 0 & \beta^2\,\Lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \beta \begin{pmatrix} 0 & 0 & 0 & \dots \\ \Lambda & 0 & 0 & \dots \\ 0 & \beta\,\Lambda & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

the Lemma follows immediately. □

Remark 3.5. It is not difficult to prove that the operator B is invertible and that its inverse B^{-1} is given by:

$$B^{-1}\,f = \begin{pmatrix} 1/\beta^0 & 0 & 0 & \dots \\ 0 & 1/\beta^1 & 0 & \dots \\ 0 & 0 & 1/\beta^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ \beta^{-1} f_1 \\ \beta^{-2} f_2 \\ \vdots \end{pmatrix}, \quad D(B^{-1}) = R(B) = Y.$$

Moreover, we note that B^{-1} is not bounded. □

Our goal is to construct a new operator $\hat{L}$ such that it generates a strongly continuous semigroup of contractions and such that the composition $exp(\hat{L})B$ is a B-bounded semigroup.

By multiplying each equation of (1) by β^j, for $j = 0, 1, \dots$, we have:

$$\begin{cases} \dfrac{d\beta^j n_j(t)}{dt} = L_j\, \beta^j n_j(t)\,,\ t \geq 0\,, \\ \beta^j n_j(0) = \beta^j n_{j,0}. \end{cases} \tag{7}$$

and the boundary conditions give:

$$\beta^j n_j^{in} = \beta^j \Lambda n_{j-1}^{out} = \beta\Lambda\beta^{j-1} n_{j-1}^{out}. \tag{8}$$

Furthermore, by introducing the functions $N_j(t) = \beta^j n_j(t)$, $N_j^{in} = \beta^j n_j^{in}$ and $N_j^{out} = \beta^j n_j^{out}$, and defining the operator $\hat{L}_j$ as follows:

$$\hat{L}_j f_j = L_j f_j\,,$$
$$D(\hat{L}_j) = \left\{ f_j \in X,\ v\frac{\partial f_j}{\partial x} \in X,\ |v|\sigma f_j \in X,\ f_j^{in} \in X^{in},\ f_{j-1}^{out} \in X^{out},\ f_j^{in} = \beta\Lambda f_{j-1}^{out} \right\}, \tag{9}$$

problem (7) becomes

$$\begin{cases} \dfrac{dN_j(t)}{dt} = \hat{L}_j\, N_j(t)\,,\ t \geq 0\,, \\ N_j(0) = N_{j,0}. \end{cases} \tag{10}$$

Remark 3.6. We note that problem (10) has a unique solution. In fact, the boundary operator $\beta\Lambda$ has norm smaller than one and applying the results of [7] we can conclude that the operator $\hat{L}_j$ generates a strongly continuous semigroup of contractions and we can write the solution of (10). □

Moreover, if $N(t) = B\,n(t) \in R(B)$, then $n(t) = B^{-1}\,N(t)$, where $n(t)$ is the solution of (5) which we know from [7]. By applying Lemma 3.4 we have:

$$B\,n^{in} = B\,\hat{\Lambda}\,n^{out} = \beta\,\hat{\Lambda}\,B\,n^{out}, \tag{11}$$

If $N^{in} = Bn^{in}$ and $N^{out} = Bn^{out}$, we get from (11):

$$N^{in} = \beta\,\hat{\Lambda}N^{out},$$

where now $\|\beta\hat{\Lambda}\| < 1$. In fact, we have:

$$\|\beta\hat{\Lambda}f\|_{in} = \sum_{j=0}^{\infty}\|\beta\Lambda f_j\|_{-} < \sum_{j=0}^{\infty}\|f_j\|_{+} = \|f\|_{out}.$$

Applying B to problem (5), we get:

$$\begin{cases} \dfrac{dN(t)}{dt} = B\,L\,n(t)\,,\ t \geq 0\,, \\ N(0) = B\,n_0. \end{cases} \tag{12}$$

Hence, defining the operator $\hat{L}$, which acts formally like L, as follows:

$$\begin{gathered} \hat{L}\,f = L\,f \\ D(\hat{L}) = \left\{ f \in Y,\ v\frac{\partial f}{\partial x} \in Y,\ |v|\sigma f \in Y,\ f^{in} \in Y^{in},\ f^{out} \in Y^{out},\ f^{in} = \beta\hat{\Lambda}f^{out} \right\}, \end{gathered} \tag{13}$$

we can prove the following Lemma

Lemma 3.7. *The operators B, L and $\hat{L}$ verify the following relation*

$$B\,L = \hat{L}\,B.$$

Proof: We easily compute:

$$B\,L = \begin{pmatrix} \beta^0 L_0 & 0 & 0 & \dots \\ 0 & \beta^1 L_1 & 0 & \dots \\ 0 & 0 & \beta^2 L_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} L_0\beta^0 & 0 & 0 & \dots \\ 0 & L_1\beta^1 0 & 0 & \dots \\ 0 & 0 & L_2\beta^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \hat{L}\,B.$$

□

Remark 3.8. We note that B does not commute with $\hat{L}$, in fact if $f \in D(\hat{L})$ then $B\,f \notin D(\hat{L})$. □

Thanks to Lemma 3.7, from (12) we obtain:

$$\begin{cases} \dfrac{dN(t)}{dt} = \hat{L}\, N(t)\,,\ t \geq 0 \\ N(0) = B\, n_0 = N_0. \end{cases} \tag{14}$$

Since $\|\beta\hat{\Lambda}\| < 1$, the operator $\hat{L}$ generates a strongly continuous semigroup of contractions. Thus, the problem (14) has a unique solution (see [7]) given by:

$$N(t) = \exp(\hat{L}t)N_0 = \exp(\hat{L}t)B\, n_0\,, \tag{15}$$

where $\exp(\hat{L}t)$ is the C_0-semigroup generated by $\hat{L}$.

Finally, by defining the operator $Z(t)$ as follows:

$$Z(t) = \exp(\hat{L}t)B, \tag{16}$$

we can prove that:

Lemma 3.9. *The family $Z(t)$ with $t \geq 0$ defined by (16) is a B-bounded semigroup.*

Proof: By construction, we have:

$$n(t) = B^{-1}\exp(\hat{L}t)B\, n_0 = B^{-1}Z(t)n_0\,. \tag{17}$$

Moreover, by integrating (14) with respect to t, it follows also that:

$$N(t) = N_0 + \int_0^t \hat{L}\, N(s)\, ds\,,$$

and considering (15) we get:

$$\exp(\hat{L}t)B\, n_0 = B\, n_0 + \int_0^t \hat{L}\exp(\hat{L}s)B\, n_0\, ds.$$

By Lemma 3.7 and thanks to the fact that $\exp(\hat{L}t)$ and $\hat{L}$ commute (see [5] and [9]) , we finally obtain:

$$\begin{aligned} Z(t)n_0 &= Bn_0 + \int_0^t \exp(\hat{L}s)\, \hat{L}\, B\, n_0\, ds \\ &= B\, n_0 + \int_0^t \exp(\hat{L}s)B\, L\, n_0\, ds \\ &= B\, n_0 + \int_0^t Z(s)L\, n_0\, ds. \end{aligned} \tag{18}$$

Formula (18) gives us the implicit expression of $Z(t)$ and proves that it is a B-semigroup. □

Moreover, relations (17) and (18) lead to the following theorem which concludes our study.

Theorem 3.10. *The evolution problem (5) has a unique solution given by:*

$$n(t) = B^{-1}Z(t)n_0 = n_0 + B^{-1}\int_0^t Z(s)L\, n_0\, ds \tag{19}$$

where $Z(t) = exp(\hat{L}t)B$ is the B-semigroup generated by the operators $\hat{L}$ and B, respectively defined by (13) and (6).

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