

AN INTRODUCTION TO FINITE VOLUME METHODS FOR HYPERBOLIC CONSERVATION LAWS

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Abstract. Hydrodynamic transport problems often take the form of systems of hyperbolic conservation laws. This minicourse intends to introduce the main notions and tools for the numerical approximation of such systems by finite volumes. The notions of consistency, stability, and approximate Riemann solvers are explained in particular. The main ingredients to go to second-order and multidimension are given.

Résumé. Les problèmes de transport hydrodynamique prennent souvent la forme de systèmes de lois de conservation hyperboliques. Ce cours vise à introduire les notions principales et les outils pour l'approximation numérique de tels systèmes par des volumes finis. En particulier, les notions de consistance, stabilité et solveurs de Riemann approchés sont expliquées. On donne aussi les arguments principaux pour passer au second ordre et aux cas multi-dimensionnels.

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1. QUASILINEAR SYSTEMS, CONSERVATION LAWS

1.1. Quasilinear systems and conservation laws

We recall here very basic definitions. The reader is referred to [6] for more theoretical results.

- Quasilinear system of partial differential equations (one dimension):

$$\partial_t U + A(U)\partial_x U = 0, \quad t > 0, x \in \mathbb{R}, \quad (1.1)$$

where $U(t, x) \in \mathbb{R}^p$, and $A(U)$ is a $p \times p$ matrix.

The system is said hyperbolic if $A(U)$ is diagonalizable for all U .

Such system undergoes the propagation of some waves, like in $U = \varphi(x - ta)$, $a \in \mathbb{R}$, which is solution to $\partial_t U + a\partial_x U = 0$.

- System of conservation laws:

$$\partial_t U + \partial_x F(U) = 0, \quad (1.2)$$

where $F(U) \in \mathbb{R}^p$. It is of the form (1.1) with $A(U) = F'(U)$.

The main features for equation (1.1) are that (see [6])

- ▷ the Cauchy problem has **bounded**, but **discontinuous**, solutions U ,
- ▷ solutions to (1.1) are **not well-defined**,
- ▷ the nonlinearity induces **nonuniqueness**.

1.2. Examples of conservation laws

- Scalar equation

$$\partial_t U + \partial_x F(U) = 0, \quad (1.3)$$

with $U \in \mathbb{R}$ and $F : \mathbb{R} \rightarrow \mathbb{R}$.

- The system of gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho(u^2/2 + e)) + \partial_x((\rho(u^2/2 + e) + p)u) = 0, \end{cases} \quad (1.4)$$

where $\rho \geq 0$, $u \in \mathbb{R}$, $e \geq 0$ and $p = (\gamma - 1)\rho e$ with $\gamma > 1$ a constant.
Here $U = (\rho, \rho u, \rho(u^2/2 + e))$.

1.3. Entropy

The notion of entropy in hyperbolic problems is used for several purposes:

- selecting admissible solutions,
- getting a priori bounds,
- proving compactness (for mathematicians only...)

▷ For the quasilinear system (1.1), i.e. $\partial_t U + A(U)\partial_x U = 0$, an **entropy** is a scalar function $\eta(U)$, such that there exist another scalar function $G(U)$, called the entropy flux, satisfying

$$G'(U) = \eta'(U)A(U). \quad (1.5)$$

▷ Interest: smooth solutions to (1.1) satisfy $\partial_t \eta(U) + \partial_x G(U) = 0$.

If η is a convex entropy, a solution U to (1.1) is said η entropy satisfying if

$$\partial_t \eta(U) + \partial_x G(U) \leq 0. \quad (1.6)$$

Under suitable boundary conditions, it implies the a priori bound

$$\frac{d}{dt} \int \eta(U) dx \leq 0. \quad (1.7)$$

1.4. Entropy: examples

- For scalar conservation laws, any convex function $\eta(U)$ is an entropy, with entropy flux defined by $G' = \eta' F'$.
- For the system of gas dynamics, an entropy is given by

$$\eta(U) = \rho \phi \left(\frac{p^{1/\gamma}}{\rho} \right), \quad G(U) = \rho u \phi \left(\frac{p^{1/\gamma}}{\rho} \right), \quad (1.8)$$

with $\phi' \leq 0$ and $\phi'' \geq 0$.

In practice, in order to find an entropy one tries to derive a new conservation law by formal multiplication of the original conservation laws by nonlinear functions of U .

1.5. Invariant domains

We say that a convex set $\mathcal{U} \subset \mathbb{R}^p$ is an *invariant domain* for (1.2) if it has the property that

$$U^0(x) \in \mathcal{U} \text{ for all } x \Rightarrow U(t, x) \in \mathcal{U} \text{ for all } x, t > 0. \quad (1.9)$$

The convexity property is with respect to the conservative variable U . For the theory, refer to [6].

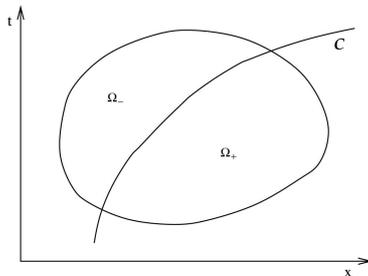
- Example: for a scalar law ($p = 1$), any closed interval is an invariant domain.
- Example: For the full gas dynamics system, the set where $e > 0$ is an invariant domain (check that this set is convex with respect to the conservative variables $(\rho, \rho u, \rho(u^2/2 + e))$).

The property for a scheme to preserve an invariant domain is an important issue of stability. In particular, the occurrence of negative values for density or for internal energy in gas dynamics calculations leads rapidly to breakdown in the computation.

1.6. Rankine Hugoniot jump relation

It characterizes weak solutions of conservation laws with a discontinuity across a smooth curve. Consider a

FIGURE 1. Discontinuity across a smooth curve



function $U(t, x)$ that is C^1 on $\bar{\Omega}_-$ and in $\bar{\Omega}_+$, as shown on Figure 1. Then U solves $\partial_t U + \partial_x F(U) = 0$ in Ω if and only if U is a classical solution in Ω_- and Ω_+ , and the Rankine-Hugoniot jump relation

$$F(U_+) - F(U_-) = \dot{\xi}(U_+ - U_-) \quad \text{on } \mathcal{C} \cap \Omega \quad (1.10)$$

is satisfied, where the curve \mathcal{C} is parametrized by $x = \xi(t)$.

2. FINITE VOLUMES FOR ONE DIMENSIONAL CONSERVATION LAWS

2.1. First-order scheme

For the conservation law (1.2), i.e. $\partial_t U + \partial_x F(U) = 0$, a first-order **conservative finite volume scheme** is a scheme of the form

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i} (F_{i+1/2} - F_{i-1/2}) = 0, \quad (2.1)$$

where U_i^n is an approximation of the solution $U(t, x)$,

$$U_i^n \simeq \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} U(t_n, x) dx, \quad (2.2)$$

the indices n and i refer to time t_n and cell $C_i = (x_{i-1/2}, x_{i+1/2})$ of size $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, where $t_{n+1} - t_n = \Delta t$ and $\dots < x_{i-1/2} < x_{i+1/2} < \dots$ is the grid. The numerical fluxes $F_{i+1/2}$, $i \in \mathbb{Z}$, are defined by

$$F_{i+1/2} = F(U_i, U_{i+1}), \quad (2.3)$$

and $F(U_l, U_r)$, is the numerical flux function, to be chosen satisfying some **precision** and **stability** properties.

2.2. Consistency

▷ The **consistency** of a numerical method is the minimal precision property we can ask for: if the sequence U_i^n converges to a smooth function, it must be a solution to (1.1).

▷ For the **finite volume method** associated to the numerical flux $F(U_l, U_r)$, the **consistency** is defined as

$$F(U, U) = F(U), \quad (2.4)$$

(Notation: $F(U_l, U_r)$ is the numerical flux, $F(U)$ is the exact flux)

Proposition 2.1. *Assume that for all i ,*

$$U_i^n = \frac{1}{\Delta x_i} \int_{C_i} U(t_n, x) dx, \quad (2.5)$$

for some smooth solution $U(t, x)$ to (1.2), and define U_i^{n+1} by (2.1)-(2.3). If the scheme is consistent, then for all i ,

$$U_i^{n+1} = \frac{1}{\Delta x_i} \int_{C_i} U(t_{n+1}, x) dx + \Delta t \left(\frac{1}{\Delta x_i} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) \right), \quad (2.6)$$

where the conservative error $\mathcal{F}_{i+1/2}$ verifies

$$\mathcal{F}_{i+1/2} \rightarrow 0, \quad (2.7)$$

as Δt and $\sup_i \Delta x_i$ tend to 0.

The proof is given in [1].

2.3. Stability

▷ Stability properties are usually required for a finite volume scheme. They can be of two types:

- Preservation of some **invariants domains**: $U_i^n \in \mathcal{U} \forall i \Rightarrow U_i^{n+1} \in \mathcal{U} \forall i$. This means that some natural bounds are preserved during the evolution, like **nonnegativity of density**.
- Existence of a **discrete entropy inequality**

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x_i} (G_{i+1/2} - G_{i-1/2}) \leq 0 \quad (2.8)$$

for some numerical entropy flux $G_{i+1/2} = G(U_i, U_{i+1})$. It selects **suitable solutions**, and provides the **a priori bound** $\sum_i \Delta x_i \eta(U_i^n) \leq \sum_i \Delta x_i \eta(U_i^0)$.

- A variant of this property is the existence of a **semi-discrete entropy inequality**, that is valid only in the limit $\Delta t \rightarrow 0$.

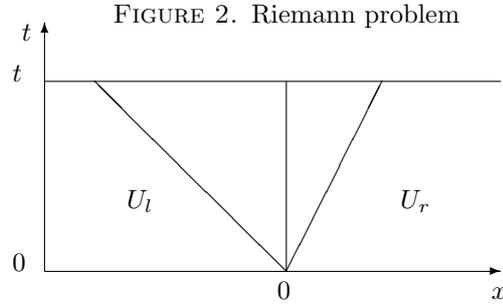
▷ *These stability properties can only hold under a CFL condition*

$$a_{i+1/2} \Delta t / \Delta x_i \leq 1, \quad (2.9)$$

where $a_{i+1/2}$ is a suitable approximation of the wave speed. This CFL condition is related to the numerical flux considered, and has to be determined by a precise analysis of it, in such a way that it implies as many stability properties as possible.

Notice that in (2.1), (2.3), the value U_i^{n+1} depends on three values at time t_n : $U_i^n, U_{i-1}^n, U_{i+1}^n$, thus in order to establish the entropy inequality or invariance property, one has merely three independent variables.

It is possible to reduce the problem to two variables U_i^n, U_{i+1}^n , and therefore to study an interface $i + 1/2$ problem. The most natural way to see this is via an **approximate Riemann solver**.



3. APPROXIMATE RIEMANN SOLVERS

3.1. Riemann problem

The Riemann problem for

$$\partial_t U + \partial_x F(U) = 0 \quad (3.1)$$

is the Cauchy problem with Riemann initial data

$$U^0(x) = \begin{cases} U_l & \text{if } x < 0, \\ U_r & \text{if } x > 0. \end{cases} \quad (3.2)$$

Since this initial data satisfies $U^0(\lambda x) = U^0(x)$ for any $\lambda > 0$, one checks easily by homogeneity that one can expect a solution to (3.1) of the form

$$U(t, x) = R_{ex}(x/t, U_l, U_r), \quad (3.3)$$

with R_{ex} the **exact Riemann solver**.

The solution to the Riemann problem depends only on x/t , thus is constant on the lines $x/t = cst$ (see Figure 2).

3.2. Approximate Riemann solvers

Definition 3.1. An approximate Riemann solver for (3.1) is a vector function $R(x/t, U_l, U_r)$ that is an approximation of the solution to the Riemann problem, in the sense that it must satisfy the consistency relation

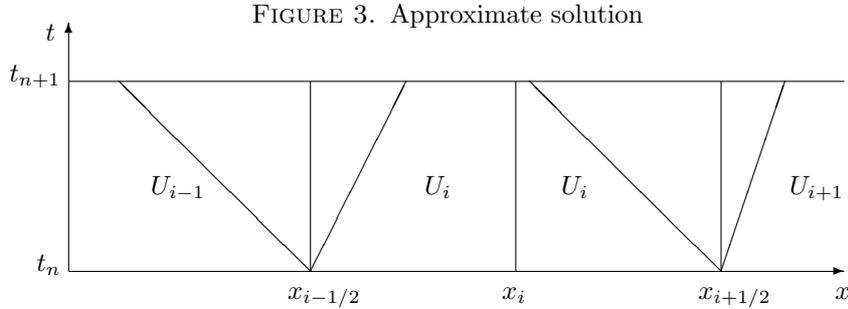
$$R(x/t, U, U) = U, \quad (3.4)$$

and the conservativity identity

$$F_l(U_l, U_r) = F_r(U_l, U_r), \quad (3.5)$$

where the left and right numerical fluxes are defined by

$$\begin{aligned} F_l(U_l, U_r) &= F(U_l) - \int_{-\infty}^0 (R(v, U_l, U_r) - U_l) dv, \\ F_r(U_l, U_r) &= F(U_r) + \int_0^{\infty} (R(v, U_l, U_r) - U_r) dv. \end{aligned} \quad (3.6)$$



The approximate Riemann solver is called dissipative with respect to a convex entropy η for (3.1) if

$$G_r(U_l, U_r) - G_l(U_l, U_r) \leq 0, \quad (3.7)$$

where

$$\begin{aligned} G_l(U_l, U_r) &= G(U_l) - \int_{-\infty}^0 \left(\eta(R(v, U_l, U_r)) - \eta(U_l) \right) dv, \\ G_r(U_l, U_r) &= G(U_r) + \int_0^{\infty} \left(\eta(R(v, U_l, U_r)) - \eta(U_r) \right) dv, \end{aligned} \quad (3.8)$$

and G is the entropy flux associated to η , $G' = \eta' F'$.

To an approximate Riemann solver we can associate a **consistent conservative scheme**, as follows.

3.3. Interpretation of an approximate solver

Take $U^n(x)$ piecewise constant over the mesh with value U_i^n in each cell C_i . We can consider that close to each interface point $x_{i+1/2}$, we have to solve a translated Riemann problem. Since (3.1) is invariant under translation in time and space, we can think of sticking together the local approximate Riemann solutions $R((x - x_{i+1/2})/(t - t_n), U_i^n, U_{i+1}^n)$, at least for times such that these solutions do not interact (see Figure 3).

There is no interaction until time t_{n+1} under a CFL condition $1/2$, in the sense that

$$\begin{aligned} x/t < -\frac{\Delta x_i}{2\Delta t} &\Rightarrow R(x/t, U_i, U_{i+1}) = U_i, \\ x/t > \frac{\Delta x_{i+1}}{2\Delta t} &\Rightarrow R(x/t, U_i, U_{i+1}) = U_{i+1}. \end{aligned} \quad (3.9)$$

Thus we define an approximate solution $U(t, x)$ for $t_n \leq t < t_{n+1}$ by

$$U(t, x) = R\left(\frac{x - x_{i+1/2}}{t - t_n}, U_i^n, U_{i+1}^n\right) \quad \text{if } x_i < x < x_{i+1}. \quad (3.10)$$

Then, we define U_i^{n+1} to be the average over C_i of this approximate solution at time $t_{n+1} - 0$. According to the definition of F_l and F_r and by using the CFL condition, we get

$$\begin{aligned}
U_i^{n+1} &= \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} U(t_{n+1} - 0, x) dx \\
&= \frac{1}{\Delta x_i} \int_0^{\Delta x_i/2} R(x/\Delta t, U_{i-1}^n, U_i^n) dx + \frac{1}{\Delta x_i} \int_{-\Delta x_i/2}^0 R(x/\Delta t, U_i^n, U_{i+1}^n) dx \\
&= U_i^n + \frac{1}{\Delta x_i} \int_0^{\Delta x_i/2} \left(R(x/\Delta t, U_{i-1}^n, U_i^n) - U_i^n \right) dx \\
&\quad + \frac{1}{\Delta x_i} \int_{-\Delta x_i/2}^0 \left(R(x/\Delta t, U_i^n, U_{i+1}^n) - U_i^n \right) dx \\
&= U_i^n - \frac{\Delta t}{\Delta x_i} [F_l(U_i^n, U_{i+1}^n) - F_r(U_{i-1}^n, U_i^n)].
\end{aligned} \tag{3.11}$$

Therefore with the conservativity and consistency assumptions (3.4), (3.5), this is a consistent conservative scheme, with numerical flux $F(U_l, U_r) = F_l(U_l, U_r) = F_r(U_l, U_r)$.

3.4. Entropy

If the approximate Riemann solver is dissipative with respect to a convex entropy η , then a **discrete entropy inequality** holds according to

$$\begin{aligned}
\eta(U_i^{n+1}) &\leq \frac{1}{\Delta x_i} \int_0^{\Delta x_i/2} \eta \left(R(x/\Delta t, U_{i-1}^n, U_i^n) \right) dx \\
&\quad + \frac{1}{\Delta x_i} \int_{-\Delta x_i/2}^0 \eta \left(R(x/\Delta t, U_i^n, U_{i+1}^n) \right) dx \\
&= \eta(U_i^n) - \frac{\Delta t}{\Delta x_i} [G_l(U_i^n, U_{i+1}^n) - G_r(U_{i-1}^n, U_i^n)].
\end{aligned} \tag{3.12}$$

Under the entropy assumption $G_r \leq G_l$ this gives

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x_i} [G(U_i^n, U_{i+1}^n) - G(U_{i-1}^n, U_i^n)] \leq 0, \tag{3.13}$$

for any numerical entropy flux function $G(U_l, U_r)$ such that $G_r(U_l, U_r) \leq G(U_l, U_r) \leq G_l(U_l, U_r)$.

3.5. Invariant domains

The **invariant domains** are also analyzed easily for approximate Riemann solvers.

Proposition 3.2. *Assume that R is an approximate Riemann solver that preserves a convex invariant domain \mathcal{U} , in the sense that*

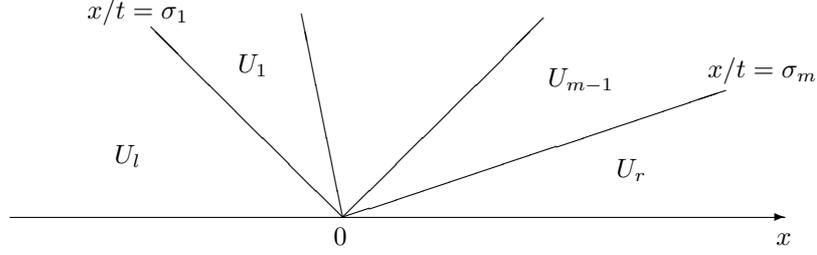
$$U_l, U_r \in \mathcal{U} \quad \Rightarrow \quad R(x/t, U_l, U_r) \in \mathcal{U} \text{ for any value of } x/t. \tag{3.14}$$

Then the numerical scheme associated to R also preserves \mathcal{U} .

This is easily seen by the convex averaging formula

$$U_i^{n+1} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} U(t_{n+1} - 0, x) dx, \tag{3.15}$$

FIGURE 4. A simple solver



where $U(t_{n+1} - 0, x)$ is obtained by sticking the local solvers

$$R\left(\frac{x - x_{i+1/2}}{t - t_n}, U_i, U_{i+1}\right). \quad (3.16)$$

3.6. Simple solvers

A **simple solver** is an approximate Riemann solver consisting in a set of finitely many simple discontinuities. This means that there exists a finite number $m \geq 1$ of speeds $\sigma_0 = -\infty < \sigma_1 < \dots < \sigma_m < \sigma_{m+1} = +\infty$, and intermediate states $U_0 = U_l, U_1, \dots, U_{m-1}, U_m = U_r$ (depending on U_l and U_r), such that as shown on Figure 4,

$$R(x/t, U_l, U_r) = U_k \quad \text{if} \quad \sigma_k < x/t < \sigma_{k+1}. \quad (3.17)$$

Then the conservativity identity becomes

$$\sum_{k=1}^m \sigma_k (U_k - U_{k-1}) = F(U_r) - F(U_l), \quad (3.18)$$

and the entropy inequality becomes

$$\sum_{k=1}^m \sigma_k (\eta(U_k) - \eta(U_{k-1})) \geq G(U_r) - G(U_l). \quad (3.19)$$

Conservativity thus enables to define the intermediate fluxes F_k , $k = 0, \dots, m$, by

$$F_k - F_{k-1} = \sigma_k (U_k - U_{k-1}), \quad F_0 = F(U_l), \quad F_m = F(U_r), \quad (3.20)$$

which is a kind of generalization of the Rankine-Hugoniot relation. The numerical flux is then given by

$$F(U_l, U_r) = F_k, \quad \text{where } k \text{ is such that } \sigma_k \leq 0 \leq \sigma_{k+1}, \quad (3.21)$$

or equivalently by

$$\begin{aligned} F(U_l, U_r) &= F(U_l) + \sum_{\sigma_k < 0} \sigma_k (U_k - U_{k-1}) \\ &= F(U_r) - \sum_{\sigma_k > 0} \sigma_k (U_k - U_{k-1}). \end{aligned} \quad (3.22)$$

3.7. HLL solver

The most simple approximate Riemann solver is the well-known HLL solver,

$$R(x/t, U_l, U_r) = \begin{cases} U_l & \text{if } x/t < c_1, \\ \frac{c_2 U_r - F(U_r)}{c_2 - c_1} + \frac{-c_1 U_l + F(U_l)}{c_2 - c_1} & \text{if } c_1 < x/t < c_2, \\ U_r & \text{if } c_2 < x/t, \end{cases} \quad (3.23)$$

where $c_1 < c_2$ are two parameters. The HLL numerical flux is given by

$$F(U_l, U_r) = \begin{cases} F(U_l) & \text{if } 0 < c_1, \\ \frac{c_2 F(U_l) - c_1 F(U_r)}{c_2 - c_1} + \frac{c_1 c_2}{c_2 - c_1} (U_r - U_l) & \text{if } c_1 < 0 < c_2, \\ F(U_r) & \text{if } c_2 < 0. \end{cases} \quad (3.24)$$

- In case $c_2 = -c_1$ one gets the Lax-Friedrichs scheme.
- In case $c_1 > 0$ one gets the upwind scheme $F(U_l, U_r) = F(U_l)$.

Properties:

- One can prove entropy inequalities and invariant domains properties if $c_1 \leq \lambda_j(U) \leq c_2$, where $\lambda_j(U)$ are the eigenvalues of $F'(U)$.

The choice for the best accuracy is

$$c_1 = \inf_{U=U_l, U_r} \inf_j \lambda_j(U), \quad c_2 = \sup_{U=U_l, U_r} \sup_j \lambda_j(U). \quad (3.25)$$

- The HLL solver works for arbitrary nonlinearity F
- It well resolves the waves with maximal and minimal speeds $\lambda_1(U)$ and $\lambda_p(U)$, but puts a lot of diffusion on intermediate waves.

Generalization with three waves for gas dynamics: HLLC scheme [1]. All the waves are well-resolved, and

- Density and internal energy remain nonnegative,
- Entropy inequalities hold.

More generally, approximate Riemann solvers can be built via relaxation approximations, see [1].

4. SECOND-ORDER SCHEMES

4.1. Reconstruction operator

A classical method to go to second-order accuracy: use first-order fluxes and a reconstruction operator.

Definition 4.1. A second-order reconstruction is an operator which to a sequence U_i associates values $U_{i+1/2-}$, $U_{i+1/2+}$ for $i \in \mathbb{Z}$, in such a way that it is conservative,

$$\frac{U_{i-1/2+} + U_{i+1/2-}}{2} = U_i, \quad (4.1)$$

and it is second-order in the sense that whenever for all i ,

$$U_i = \frac{1}{\Delta x_i} \int_{C_i} U(x) dx, \quad (4.2)$$

for some smooth function $U(x)$, then

$$U_{i+1/2-} = U(x_{i+1/2}) + O(h^2), \quad U_{i+1/2+} = U(x_{i+1/2}) + O(h^2). \quad (4.3)$$

The second-order scheme is then defined as

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i} (F_{i+1/2} - F_{i-1/2}) = 0, \quad (4.4)$$

$$F_{i+1/2} = F(U_{i+1/2-}^n, U_{i+1/2+}^n), \quad (4.5)$$

where $F(U_l, U_r)$ is a given consistent numerical flux, and where $(U_{i+1/2\pm}^n)_{i \in \mathbb{Z}}$ are obtained from the reconstruction of $(U_i^n)_{i \in \mathbb{Z}}$.

4.2. Accuracy

Proposition 4.2 (Second-order accuracy). *Assume that the numerical flux is Lipschitz continuous and consistent. If for all i ,*

$$U_i^n = \frac{1}{\Delta x_i} \int_{C_i} U(t_n, x) dx, \quad (4.6)$$

for some smooth solution $U(t, x)$ to (3.1), then U_i^{n+1} satisfies for all i

$$U_i^{n+1} = \frac{1}{\Delta x_i} \int_{C_i} U(t_{n+1}, x) dx + \Delta t \left(\frac{1}{\Delta x_i} (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) \right), \quad (4.7)$$

where

$$\mathcal{F}_{i+1/2} = O(\Delta t) + O(h^2), \quad (4.8)$$

as Δt and $h = \sup_i \Delta x_i$ tend to 0.

The proof is given in [1].

4.3. Stability

The reconstruction is said to preserve a convex invariant domain \mathcal{U} if

$$U_i \in \mathcal{U} \text{ for all } i \Rightarrow U_{i+1/2\pm} \in \mathcal{U} \text{ for all } i. \quad (4.9)$$

Proposition 4.3. *If under a CFL condition the numerical flux preserves a convex invariant domain \mathcal{U} , and if the reconstruction also preserves this invariant domain, then under the half original CFL condition, the second-order scheme also preserves this invariant domain.*

The proof is given in [1] and relies on an interpretation of the scheme via a piecewise constant function by half cell (see Figure 5).

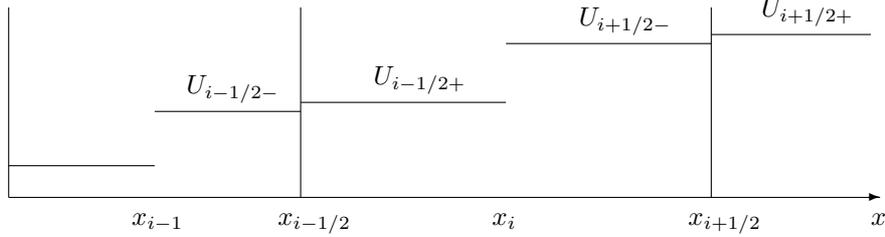
4.4. Examples of reconstruction operators

The second-order reconstruction must also be nonoscillatory in some sense, see [2], [3].

- The **minmod** reconstruction for scalar functions is defined as

$$U_{i-1/2+} = U_i - \frac{\Delta x_i}{2} DU_i, \quad U_{i+1/2-} = U_i + \frac{\Delta x_i}{2} DU_i, \quad (4.10)$$

FIGURE 5. Interpretation of second-order scheme via half cell values



with

$$DU_i = \text{minmod} \left(\frac{U_i - U_{i-1}}{(\Delta x_{i-1} + \Delta x_i)/2}, \frac{U_{i+1} - U_i}{(\Delta x_i + \Delta x_{i+1})/2} \right), \quad (4.11)$$

and

$$\text{minmod}(x, y) = \begin{cases} \min(x, y) & \text{if } x, y \geq 0, \\ \max(x, y) & \text{if } x, y \leq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.12)$$

- For the gas dynamics system, the conservative variable is $U \equiv (\rho, \rho u, \rho u^2/2 + \rho e)$. Given the values U_i , we would like to find second-order accurate values $U_{i+1/2\pm}$ such that

$$\frac{U_{i-1/2+} + U_{i+1/2-}}{2} = U_i. \quad (4.13)$$

Knowing that the U_i have positive densities and positive internal energies $\rho_i > 0$, $e_i > 0$, we require the same properties for the reconstructed values, $\rho_{i+1/2\pm} > 0$, $e_{i+1/2\pm} > 0$.

The first step of the construction is to write a parametrization of all possible values $U_{i-1/2+}$, $U_{i+1/2-}$ satisfying (4.13). An algebraic computation gives that they must be of the form

$$\begin{aligned} \rho_{i-1/2+} &= \rho_i - \frac{\Delta x_i}{2} D\rho_i, & \rho_{i+1/2-} &= \rho_i + \frac{\Delta x_i}{2} D\rho_i \\ u_{i-1/2+} &= u_i - \frac{\rho_{i+1/2-}}{\rho_i} \frac{\Delta x_i}{2} Du_i, & u_{i+1/2-} &= u_i + \frac{\rho_{i-1/2+}}{\rho_i} \frac{\Delta x_i}{2} Du_i \\ \rho_{i-1/2+} e_{i-1/2+} &= \rho_i \tilde{e}_i - \frac{\Delta x_i}{2} D(\rho e)_i, & \rho_{i+1/2-} e_{i+1/2-} &= \rho_i \tilde{e}_i + \frac{\Delta x_i}{2} D(\rho e)_i \end{aligned} \quad (4.14)$$

with $\tilde{e}_i = e_i - \frac{\rho_{i-1/2+} \rho_{i+1/2-}}{\rho_i^2} \frac{\Delta x_i^2}{8} Du_i^2$.

We observe then that the second-order accuracy means that $D\rho_i$, Du_i , $D(\rho e)_i$ must be consistent with the derivatives of ρ , u , ρe respectively. Concerning the positivity conditions, they can be expressed as

$$\frac{\Delta x_i}{2} |D\rho_i| < \rho_i, \quad \frac{\Delta x_i^2}{8} Du_i^2 < \frac{\rho_i}{\rho_{i-1/2+} \rho_{i+1/2-}} \left(\rho_i e_i - \frac{\Delta x_i}{2} |D(\rho e)_i| \right). \quad (4.15)$$

Therefore, the computation of the slopes can be done as follows: we first compute $D\rho_i$, Du_i , $D(\rho e)_i$ by the minmod slope formula, where the U_i are replaced by ρ_i , u_i , $\rho_i e_i$ respectively, and then we eventually diminish the absolute value of Du_i so as to satisfy the second inequality in (4.15). This computation gives consistent values of $D\rho_i$, Du_i , $D(\rho e)_i$.

Thus the reconstruction is second-order accurate, and preserves the nonnegativity of density and internal energy.

4.5. Second-order accuracy in time

The second-order accuracy in time is usually recovered by the Heun method, which reads as follows. The second-order method in x can be written as

$$U^{n+1} = U^n + \Delta t \Phi(U^n), \quad (4.16)$$

where $U = (U_i)_{i \in \mathbb{Z}}$, and Φ is a nonlinear operator depending on the mesh. Then the second-order scheme in time and space is

$$\begin{aligned} \tilde{U}^{n+1} &= U^n + \Delta t \Phi(U^n), \\ \tilde{U}^{n+2} &= \tilde{U}^{n+1} + \Delta t \Phi(\tilde{U}^{n+1}), \\ U^{n+1} &= \frac{U^n + \tilde{U}^{n+2}}{2}. \end{aligned} \quad (4.17)$$

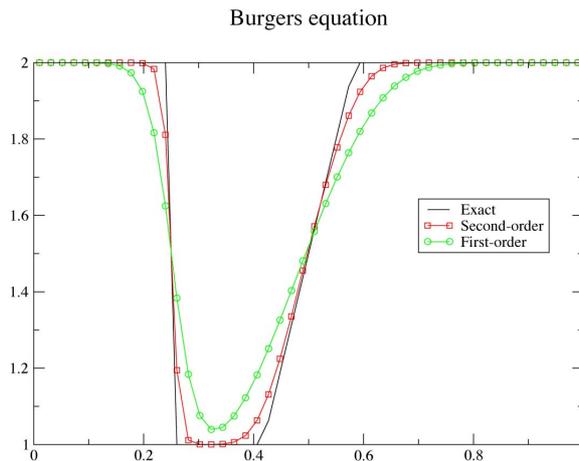
It is easy to see that if the numerical flux does not depend explicitly on Δt , this procedure gives a fully second-order scheme in the sense that we get an error in $O(\Delta t^2) + O(h^2)$. The invariant domains are also preserved because of the average, without any further limitation on the CFL.

4.6. Test 1: Burgers equation

The first numerical illustration we propose is for Burgers equation

$$\partial_t U + \partial_x (U^2/2) = 0, \quad (4.18)$$

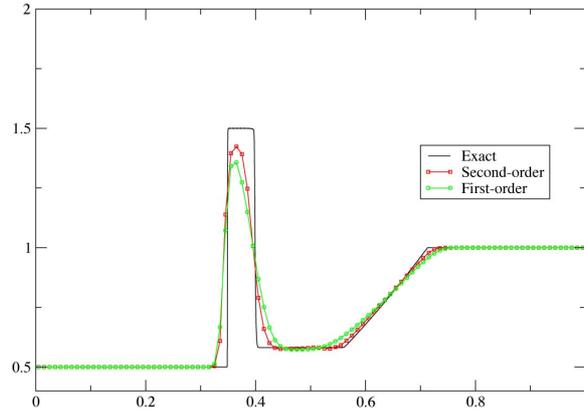
with $U \in \mathbb{R}$. The exact solution, in continuous line on figure, contains a shock and a rarefaction. The numerical solutions obtained with the HLL scheme at first and second-order are shown.



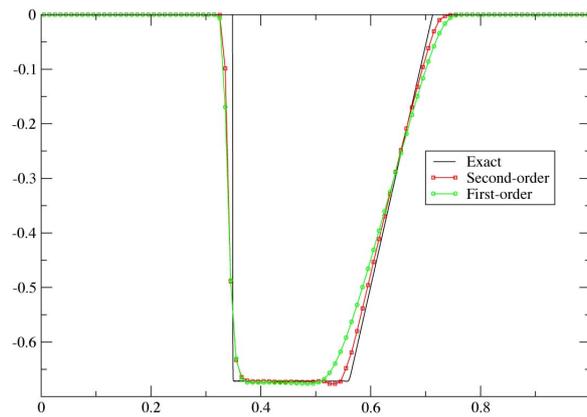
4.7. Test 2: Gas dynamics

The second numerical illustration is on the gas dynamics system (1.4). It is for a Riemann problem with a shock, a contact discontinuity and a rarefaction wave. First and second-order computations obtained with the HLLC scheme are shown on figures.

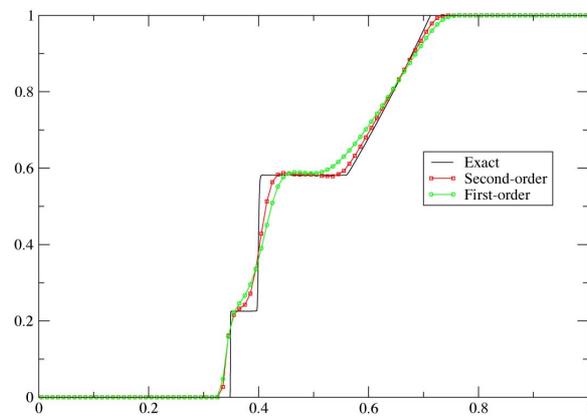
Gas dynamics, Density



Gas dynamics, Velocity



Gas dynamics, Internal energy

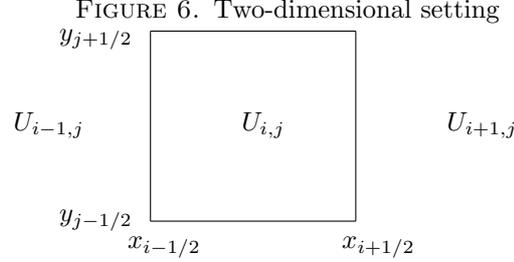


5. MULTIDIMENSIONAL PROBLEMS

5.1. Bidimensional systems of conservation laws

Bidimensional systems of conservation laws are of the form

$$\partial_t U + \partial_1(F_1(U)) + \partial_2(F_2(U)) = 0, \quad (5.1)$$



where the space variable is $x = (x_1, x_2) \in \mathbb{R}^2$, and the partial derivatives ∂_1, ∂_2 refer to these variables. The unknown is $U(t, x) \in \mathbb{R}^p$. More generally for a *two-dimensional quasilinear system*

$$\partial_t U + A_1(U) \partial_1 U + A_2(U) \partial_2 U = 0, \quad (5.2)$$

we can consider planar solutions, i.e. solutions of the form $U(t, x) = U(t, \zeta)$ with $\zeta = x \cdot n$ and $n = (n^1, n^2)$ is any unit vector in \mathbb{R}^2 , which leads to

$$\partial_t U + A_n(U) \partial_\zeta U = 0, \quad (5.3)$$

with

$$A_n(U) = n^1 A_1(U) + n^2 A_2(U). \quad (5.4)$$

The notions introduced previously can be applied to the one-dimensional quasilinear system (5.3), and by this way one defines hyperbolicity, entropies, and other notions for (5.2) by requiring the one-dimensional properties for all directions n . In particular, the system (5.2) is hyperbolic if for any unit vector n in \mathbb{R}^2 and any U , $A_n(U)$ is diagonalizable.

5.2. Schemes by interface

Let us consider a two-dimensional system of conservation laws

$$\partial_t U + \partial_1(F_1(U)) + \partial_2(F_2(U)) = 0, \quad (5.5)$$

and a two-dimensional cartesian mesh made of cells

$$C_{ij} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}), \quad (5.6)$$

with lengths $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, $\Delta y_j = y_{j+1/2} - y_{j-1/2}$. The finite volume formula *by interface* then reads

$$U_{ij}^{n+1} - U_{ij}^n + \frac{\Delta t}{\Delta x_i} (F_{i+1/2,j}^1 - F_{i-1/2,j}^1) + \frac{\Delta t}{\Delta y_j} (F_{i,j+1/2}^2 - F_{i,j-1/2}^2) = 0. \quad (5.7)$$

The interface fluxes are computed as in one dimension

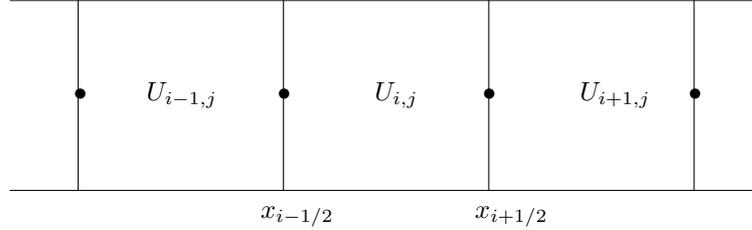
$$F_{i+1/2,j}^1 = F^1(U_{i,j}, U_{i+1,j}), \quad F_{i,j+1/2}^2 = F^2(U_{i,j}, U_{i,j+1}), \quad (5.8)$$

where $F^1(U_l, U_r)$, $F^2(U_l, U_r)$ are one-dimensional numerical fluxes corresponding to the nonlinearities $F^1(U)$ and $F^2(U)$ (see Figure 6).

Proposition 5.1. *If the one-dimensional fluxes let invariant a domain \mathcal{U} under a CFL condition*

$$\Delta t a(U_i, U_{i+1}) \leq \Delta x_i, \quad (5.9)$$

FIGURE 7. Reconstruction by line



then the two-dimensional scheme also lets this domain invariant, under the half CFL condition

$$\Delta t a(U_{i,j}, U_{i+1,j}) \leq \frac{1}{2} \Delta x_i, \quad \Delta t a(U_{i,j}, U_{i,j+1}) \leq \frac{1}{2} \Delta y_j. \quad (5.10)$$

A similar result is true for entropy inequalities.

5.3. Second-order accuracy

The reconstruction method extends easily to several dimensions. The interface fluxes are computed at reconstructed values

$$F_{i+1/2,j}^1 = F^1(U_{i+1/2-,j}, U_{i+1/2+,j}), \quad F_{i,j+1/2}^2 = F^2(U_{i,j+1/2-}, U_{i,j+1/2+}), \quad (5.11)$$

where $(U_{i+1/2\pm,j})_{i \in \mathbb{Z}}$ is obtained by a one-dimensional reconstruction of $(U_{i,j})_{i \in \mathbb{Z}}$ at fixed j (reconstruction by line, see Figure 7), and $(U_{i,j+1/2\pm})_{j \in \mathbb{Z}}$ is obtained by a one-dimensional reconstruction of $(U_{i,j})_{j \in \mathbb{Z}}$ at fixed i (reconstruction by column).

Proposition 5.2. *If the one-dimensional fluxes let invariant a domain \mathcal{U} under a CFL condition*

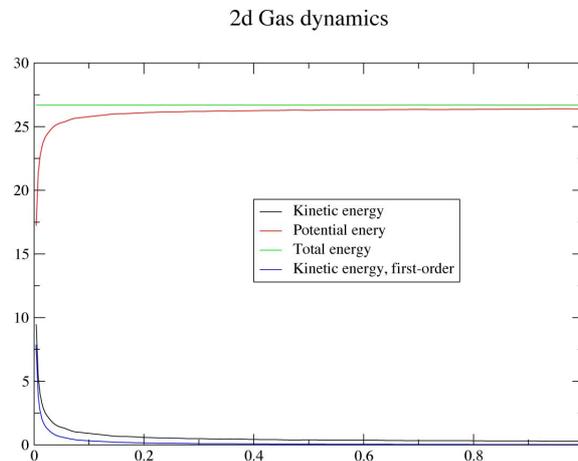
$$\Delta t a(U_i, U_{i+1}) \leq \Delta x_i, \quad (5.12)$$

and if the reconstruction also lets \mathcal{U} invariant, then the two-dimensional second-order scheme also lets this domain invariant, under the 1/4 CFL condition

$$\Delta t a(U_{i,j}, U_{i+1,j}) \leq \frac{1}{4} \Delta x_i, \quad \Delta t a(U_{i,j}, U_{i,j+1}) \leq \frac{1}{4} \Delta y_j. \quad (5.13)$$

5.4. Test 3: Gas dynamics, random velocity

This test case for gas dynamics in two dimensions is intended to asses the stability of the schemes. The initial data has constant density, vanishing internal energy, and randomly distributed velocity. The solution quickly develops small vacuum areas everywhere, and then kinetic energy is transferred to potential energy. We use the HLLC scheme at second order, which has the property to ensure proper computation of vacuum, see [1]. The energies with respect to time are plotted on figure.



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Most of the material in this presentation can be found in more details in reference [1]. The motivation here is to give basic tools for solving generic systems of conservation laws.