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STRONG PSEUDOMONOTONICITY, SHARP EFFICIENCY AND STABILITY FOR PARAMETRIC VECTOR EQUILIBRIA

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Abstract. We investigate Hőlder type estimates of solutions to parametric vector equilibrium problems. Our results rely on the notion of strong pseudomonotonicity of the bifunctions defining problems. When applied to vector optimization problems, the strong pseudomonotonicity introduced in the present paper implies the uniform (with the same constant) sharpness of solutions.

Introduction

Let X and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y with nonempty interior, $\text{int}\mathcal{K} \neq \emptyset$. Let $C \subset X$ be a subset of X and let $F : C \times C \to Y$ be a mapping (bifunction) satisfying the condition

$$F(x,x) \geq 0 \quad \text{for} \quad x \in C.$$

Our goal here is to investigate two types of vector equilibrium problems:

- **•:** *weak vector equilibrium problems* $(WVEP)$: find $x \in C$ such that

$$F(x,y) \not< 0 \quad \text{for all} \quad y \in C, \tag{1}$$

  i.e.,

$$-F(x,y) \notin \text{int}\mathcal{K} \quad \text{for all} \quad y \in C,$$

- **•:** *vector equilibrium problems* (VEP): find $x \in C$ such that

$$F(x,y) \not\leq 0 \quad \text{for all} \quad y \in C, \quad F(x,y) \neq 0, \tag{2}$$

  i.e.,

$$-F(x,y) \notin \mathcal{K} \quad \text{for all} \quad y \in C, \quad F(x,y) \neq 0.$$

By using the usual concepts of efficiency we can reformulate both $(WVEP)$ and (VEP) in the following way.
Let $\varphi : C \rightrightarrows Y$ be a set-valued mapping which assigns to each $x \in C$ the set of efficient points of $F(x,C)$, i.e. $\varphi(x) = E(F(x,C))$, where for any subset $A \subset Y$, $y \in E(A)$, if $y \in A$ and $(A - y) \cap (-\mathcal{K}) = \{0\}$. Then

$$x_0 \in C \quad \text{solves (VEP)} \quad \Leftrightarrow 0 \in \varphi(x_0).$$

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Let $\psi : C \rightrightarrows Y$ be a set-valued mapping defined as $\psi(x) = WE(F(x, C))$, where for any subset $A \subset Y$, $y \in WE(A)$, if $y \in A$ and $(A - y) \cap (-\text{int}\mathcal{K}) = \{\emptyset\}$. Then

$$x_0 \in C \text{ solves (WVEP)} \quad \Leftrightarrow 0 \in \psi(x_0).$$

Vector equilibrium problems provide a unifying framework for investigating a large variety of problems of variational analysis such as vector variational inequalities, vector optimization problems, variational inequalitites, optimization problems. (see e.g. [10, 20]). In particular, $(WVEP)$ generalize vector variational inequalitites (VVI) introduced in [12].

Let $f : X \to L(X, Y)$ be a mapping which assigns to an element $x \in X$ a linear continuous operator acting from X into Y. If

$$F(x, y) = \langle f(x), y - x \rangle,$$

then $(WVEP)$ gives rise to *vector variational inequality* (VVI): find $x \in C$ such that

$$\langle f(x), y - x \rangle \not< 0 \quad \text{for all} \quad y \in C.$$

For any mapping $\phi : X \to Y$, by letting

$$F(x, y) = \phi(y) - \phi(x)$$

in $(WVEP)$ and (VEP) we obtain, respectively,

- **•:** *vector optimization problem* $(WVOP)$: find $x \in C$ such that

$$\phi(y) - \phi(x) \notin \text{int}\mathcal{K} \text{ for all } y \in C,$$

- **•:** *(weak) vector optimization problem* (VOP) : find $x \in C$ such that

$$\phi(y) - \phi(x) \notin \mathcal{K} \text{ for all } y \in C, \quad \phi(y) \neq \phi(x).$$

Some more relations between (VVI) and (VOP) are given by Yang and Gog [27].

Our goal here is to investigate Hőlder properties of solutions to parametric (VEP) and $(WVEP)$ problems.

Stability results for parametric (scalar) variational inequalities are given e.g. in [1, 9, 15, 19, 22, 25, 28]. The stability questions for parametric variational inequalities are investigated on different levels of generality and by applying different tools. In [9, 19, 28] the stability of solution is investigated via properties of metric projection on closed convex sets. In [9, 19, 28] the crucial assumptions are related to strong monotonicity and/or strong pseudomonotonicity which imply the uniqueness of solutions. The question of Hőlder stability for parametric (scalar) equilibrium problems is addressed directly in [4]. The results presented in [4] rely on strong pseudomonotonicity of real-valued bifunctions which also implies the uniqueness of solutions.

Our main attempt is to obtain Hőlder estimates for solutions to parametric vector equilibria problems and to relax the uniqueness assumption. To this aim we introduce the definition of strong pseudomonotonicity for bifunctions F taking values in partially ordered normed space Y. In general, our definition does not imply the uniqueness of solutions to vector equilibrium problems. When applied to scalar equilibrium problems our definition reduces to the strong pseudomonotonicity of real-valued bifunctions F (with $Y = R$) as used in investigations of Hőlder stability of solutions to parametric (scalar) set-valued variational inequalitites (VI) in [19], and (scalar) equilibrium problems [4]. For other variants of pseudomonotonicity see [24].

The organization of the paper is as follows. In Section 3 we discuss the strong pseudomonotonicity for bifunctions $F : C \times C \to Y$. In Section 4 we prove our main results on Hőlder continuity of solutions to parametric (VEP). In Section 5 we specialize the obtained results for vector optimization problems. In a similar way one can apply the results of Section 4 to vector variational inequalities, variational inequalities, and optimization problems, obtaining in these way Hőlder stability results in these classes of problems (see also [4, 9, 19, 28]).

1. Monotonicity concepts

In this section we introduce basic monotonicity concepts for bifunctions, namely monotonicity, pseudomonotonicity and quasimonotonicity as defined in [5,6]. These concepts generalize monotonicity, pseudomonotonicity and quasimonotonicity for functions used in investigations of variational inequalities and equilibrium problems e.g. in [8,11,13,21] and in deriving characterizations of generalized convexity e.g. in [17,18].

For any elements $x, y \in Y$ we will use the following notations:

$$\begin{array}{ll} x < y \Leftrightarrow y - x \in \text{int}\mathcal{K} & x \leq y \Leftrightarrow y - x \in \mathcal{K} \\ x \not< y \Leftrightarrow y - x \notin \text{int}\mathcal{K} & x \not\leq y \Leftrightarrow y - x \notin \mathcal{K} \\ x \not> y \Leftrightarrow y \not< x & x \not\geq y \Leftrightarrow y \not\leq x. \end{array}$$

We say that F is:

- *monotone* if for all $x, y \in C$
$$F(y,x) + F(x,y) \not> 0,$$
- *pseudomonotone on* C if for all $x, y \in C$
$$F(x,y) \not< 0 \rightarrow F(y,x) \not> 0,$$
- *strictly pseudomonotone on* C if for all $x, y \in C$
$$F(x,y) \not< 0 \rightarrow F(y,x) < 0,$$
- *quasimonotone on* C if for all $x, y \in C$
$$F(x,y) > 0 \rightarrow F(y,x) \leq 0$$

Monotonicity implies pseudomonotonicity which, in turn, implies quasimonotonicity. These concepts and their variants were used in [2,5,6,14] in proving the existence of solutions to vector equilibrium problems $(WVEP)$ and (VEP).

2. Strong pseudomonotonicity

For any element $z \in Y$ the relation $z \not> \alpha B_Y$, means that $z \not> \alpha b$ i.e., $z - \alpha b \notin \text{int}\mathcal{K}$ for all $b \in B_Y$. By the symmetry of balls, the latter condition yields that $(z + \alpha B_Y) \cap \text{int}\mathcal{K} = \emptyset$ or $z \cap [\alpha B_Y + \text{int}\mathcal{K}] = \emptyset$. Analogously, the relation $z \not\geq \alpha B_Y$ means that $z \cap [\alpha B_Y + \mathcal{K}] = \emptyset$. Moreover, for any sufficiently small $\varepsilon > 0$

$$z \cap [\alpha B_Y + \text{int}\mathcal{K}] = \emptyset \Leftrightarrow z \cap [(\alpha - \varepsilon)B_Y + \mathcal{K}] = \emptyset. \tag{3}$$

Indeed, if $z \notin [\alpha B_Y + \text{int}\mathcal{K}]$, then $z \notin \mathcal{K}$ and there exists $0 < \beta \leq \alpha$ such that $z \cap [\beta B_Y + \mathcal{K}] = \emptyset$.

We say that F is:

- *strongly monotone on* C if there exists $\alpha > 0$ such that for any pair of distinct points $x, y \in C$ we have that
$$F(x,y) + F(y,x) \not> \alpha \|y - x\|^2 B_Y,$$
- *strongly pseudomonotone* if there exists $\beta > 0$ such that for any pair of distinct points $x, y \in C$ we have that
$$F(x,y) \not< 0 \rightarrow F(y,x) \not> \alpha \|y - x\|^2 B_Y.$$

In view of (3), the strong monotonicity holds iff there exists $\alpha > 0$ such that for any pair of distinct points $x, y \in C$

$$F(x,y) + F(y,x) \ngeq \alpha\|y-x\|^2 B_Y,$$

and the strong pseudomonotonicity holds iff there exists $\alpha > 0$ such that for any pair of distinct points $x, y \in C$

$$F(x,y) \nless 0 \to F(y,x) \ngeq \alpha\|y-x\|^2 B_Y.$$

Moreover, by (3), the relation $F(y,x) \cap (\alpha B_Y + \text{int}\mathcal{K}) = \emptyset$ implies that $F(y,x) \notin \mathcal{K}$ which shows that the strong pseudomonotonicity implies that for any $x, y \in C$, $x \neq y$,

$$F(x,y) \nless 0 \to F(y,x) \ngeq 0$$

which in turn implies that F is pseudomonotone.

Clearly, the strong pseudomonotonicity of F implies that for any small $\varepsilon > 0$ and any pair of distinct points $x, y \in C$ we have

$$F(x,y) \nleq 0 \to F(y,x) \ngeq (\alpha - \varepsilon)\|y-x\|^2 B_Y. \tag{4}$$

The converse implication does not hold in general. However, the following holds true.

Proposition 2.1. *If $F(y,x) = -F(x,y)$, then the strong pseudomonotonicity holds if and only if (4) holds.*

Proof. Take any distinct $x, y \in C$ such that $F(x,y) \ngtr 0$. By the strong pseudomonotonicity, $F(y,x) \ngeq 0$ and hence $F(x,y) \nleq 0$. □

This means, for instance, that in the case of vector variational inequalities and vector optimization problems and variational inequalitites the strong pseudomonotonicity is equivalent to (4).

Example 2.2.

1.: Consider the bifunction $F : C \times C \to Y$ defined as $F(x,y) = \phi(y) - \phi(x)$, where $\phi : X \to Y$ is a mapping from X into Y. Note that $F(x,y) = -F(y,x)$ and (VEP) reduces to (VOP). By Proposition 2.1, F is strongly pseudomonotone iff (4) holds, i.e. there exists $\alpha > 0$ such that for any $x, y \in C$

$$\phi(y) - \phi(x) \nleq 0 \to \phi(y) - \phi(x) \nleq \alpha\|y-x\|^2 B_Y. \tag{5}$$

Let $x_0 \in C$ be a solution to (VOP), i.e., $\phi(y) - \phi(x_0) \nleq 0$ for any $y \in C$. Then the strong pseudomonotonicity of F yields that

$$(\phi(y) - \phi(x_0)) \cap (\alpha\|y-x\|^2 B_Y - \mathcal{K}) = \emptyset,$$

which means that x_0 is a sharp solution to (VOP) as defined in [7,16]. Moreover, each solution x_0 is sharp with the same constant $\alpha > 0$.

2.: Let $F : C \times C \to Y$ be given as $F(x,y) = \langle f(x), y - x\rangle$, where $f : X \to L(X,Y)$ is a mapping from X into the space of all continuous linear operators from X into Y. Then $(WVEP)$ reduces to (VVI) of finding all $x \in C$ such that $\langle f(x), y - x\rangle \nless 0$ for all $y \in C$ and F is strongly pseudomonotone iff the exists $\alpha > 0$ such that for any $x, y \in C$

$$\langle f(x), y-x\rangle \nless 0 \to \langle f(y), y-x\rangle \nless \alpha\|x-y\|^2 B_Y.$$

3.: Let $F : C \times C \to R$. Then $(WVEP)$ reduces to the equilibrium problem (EP) of finding $x \in C$ such that $f(x,y) \geq 0$ for all $y \in C$ and F is strongly pseudomonotone iff there exists $\alpha > 0$ such that for all $x, y \in C$

$$F(x,y) \geq 0 \to F(y,x) \leq -\alpha\|x-y\|^2,$$

as defined in [4].

4.: Let $F : C \times C \to R$ be of the form $F(x, y) = \langle f(x), y - x\rangle$, where $f : X \to X$. Then $(WVEP)$ reduces to (VI) of finding $x \in C$ such that $\langle f(x), y - x\rangle \geq 0$ for all $y \in C$ and F is strongly pseudomonotone iff there exists $\alpha > 0$ such that

$$\langle f(x), y - x\rangle \geq 0 \to \langle f(y), y - x\rangle \geq \alpha\|x - y\|^2$$

as defined e.g. in [24,26].

It is easy to prove that in the case of (EP) the strong pseudomonotonicity implies the uniqueness of solutions. In general, the solution set to (VEP) satisfying the strong pseudomonotonicity need not be unique.

3. Applications to stability of (VEP)

Let U be a normed space with the open unit ball B_U. Consider the parametric vector equilibrium problem $(VEP)_u$ of finding all $x \in C(u)$ such that

$$F(u, x, y) \not\leq 0 \quad \text{for all} \quad y \in C(u), F(u, x, y) \neq 0,$$

where $F : U \times X \times X \to Y$ is a bifunction and $C : U \rightrightarrows X$ is a set-valued mapping. In this section we investigate the behaviour of the set-valued mapping $S : U \rightrightarrows X$ defined as

$$S(u) = \{x \in C(u) \ : \quad F(u, x, y) \not\leq 0 \text{ for all} y \in C(u), \quad F(u, x, y) \neq 0\}$$

with $S(u_0) = S_0$. We recall that the set-valued mapping C is *Hőlder* of order $m > 0$ around u_0 if there exist constants $t_c > 0$, $L_c > 0$ such that $C(u) \subset C(u') + L_c\|u - u'\|^m B_Y$ for $u, u' \in u_0 + t_c B_U$. Moreover, C is *upper Hőlder* of order $m > 0$ at u_0 if there exist constants $t_c > 0$, $L_c > 0$ such that $C(u) \subset C(u_0) + L_c\|u - u_0\|^m B_Y$ for $u \in u_0 + t_c B_U$, and C is *lower Hőlder* of order $m > 0$ at u_0 if there exist constants $t_c > 0$, $L_c > 0$ such that $C(u_0) \subset C(u) + L_c\|u - u_0\|B_Y$ for $u \in u_0 + t_c B_U$. If C is Hőlder, or upper Hőlder, or lower Hőlder of order $m = 1$, then C is called Lipschitz, upper Lipschitz, or lower Lipschitz, respectively.

We say that F is Lipschitz around $\{u_0\} \times X \times X$ if there exists a neighbourhood U_0 of u_0 such that F is Lipschitz on $U \times X \times X$, i.e., there exists a constant $L_f > 0$ such that

$$F(u, x, y) - F(u', x', y') \in l_f(\|u - u'\| + \|x - x'\| + \|y - y'\|)B_Y$$

for any $u, u' \in U_0$, $x, x', y, y' \in X$.

Theorem 3.1. *Let U, X, and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that $S(u) \neq \emptyset$ for all u in some neighbourhood of u_0. If*

- ***(i):*** *F is Lipschitz around $\{u_0\} \times X \times X$,*
- ***(ii):*** *C is Lipschitz around u_0,*
- ***(iii):*** *there exists a neighbourhood U_0 of u_0 such that for each $u \in U_0$ and each $x \in C(u)$ there is $x_0 \in S(u)$ such that $F(u, x_0, x) \in \mathcal{K}$,*
- ***(iv):*** *$F(u, \cdot, \cdot)$ are uniformly strongly pseudomonotone on $C(u)$ for $u \in U_0$, i.e., there is $\beta > 0$ such that for any $u \in U_0$ and any $x, x_0 \in C(u)$, $x \neq x_0$,*

$$F(u, x_0, x) \not\leq 0 \to F(u, x, x_0) \notin \beta\|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping S is Hőlder of order $\frac{1}{2}$ around u_0.

Proof. Take any $u, u' \in U_1 \cap U_0$ and $x_0 \in S(u)$. By (ii), there exists $z \in C(u')$ such that

$$\|x_0 - z\| \leq L_c\|u - u'\|.$$

By (iii), there exists $z_0 \in S(u')$ such that
$$F(u', z_0, z) \in \mathcal{K}.$$
Again, by (ii), there exists $x \in C(u)$ such that
$$\|z_0 - x\| \le L_c \|u - u'\|$$
By Lipschitzness of F,
$$\|F(u, x, x_0) - F(u', z_0, z)\| \le L_f(\|x_0 - z\| + \|x - z_0\| + \|u - u'\|) \le L_f(2L_c + 1)\|u - u'\|.$$
By this,
$$F(u, x, x_0) \in F(u', z_0, z) + L_f(2L_c + 1)\|u - u'\| B_Y \subset L_f(2L_c + 1)\|u - u'\| B_Y + \mathcal{K}. \tag{6}$$
On the other hand, by (iv),
$$F(u, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}. \tag{7}$$
Combining (6) and (7) we obtain
$$\|x - x_0\|^2 \le \frac{L_f(2L_c + 1)}{\beta} \|u - u'\|$$
and finally, assuming that $\|u - u'\| < 1$ we get
$$\|x_0 - z_0\| \le \|x_0 - x\| + \|x - z_0\| \le \left(L_c + \sqrt{\frac{L_f(2L_c + 1)}{\beta}} \right) \|u - u'\|^{\frac{1}{2}}$$

Theorem 3.2. *Let U, X, and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that $S(u) \neq \emptyset$ for all u in some neighbourhood of u_0. If*

- ***(i):*** *F is Lipschitz around $\{u_0\} \times X \times X$,*
- ***(ii):*** *C is Lipschitz around u_0,*
- ***(iii):*** *for each $u \in U_0$ and each $x \in C(u)$ there is $x_0 \in S(u)$ such that $F(u, x_0, x) \in \mathcal{K}$,*
- ***(iv):*** *$F(u_0, \cdot, \cdot)$ is strongly pseudomonotone on C, i.e., there is $\beta > 0$ such that for each $x, x_0 \in C$, $x \neq x_0$,*

$$F(u_0, x_0, x) \not\le 0 \to F(u_0, x, x_0) \notin \beta \|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping S is lower Hőlder of order $\frac{1}{2}$ around u_0.

Proof. Take any $u \in U_1 \cap U_0$ and $x_0 \in S(u_0)$. By (ii), there exists $z \in C(u)$ such that
$$\|x_0 - z\| \le L_c \|u_0 - u\|.$$
By (iii), there exists $z_0 \in S(u)$ such that
$$F(u, z_0, z) \in \mathcal{K}.$$
Again, by (ii), there exists $x \in C$ such that
$$\|z_0 - x\| \le L_c \|u_0 - u\|$$
By Lipschitzness of F,
$$\|F(u_0, x, x_0) - F(u, z_0, z)\| \le L_f(\|x_0 - z\| + \|x - z_0\| + \|u - u_0\|) \le L_f(2L_c + 1)\|u_0 - u\|.$$
By this,
$$F(u_0, x, x_0) \in F(u, z_0, z) + L_f(2L_c + 1)\|u_0 - u\| B_Y \subset L_f(2L_c + 1)\|u_0 - u\| B_Y + \mathcal{K}. \tag{8}$$

On the other hand, by (iv),

$$F(u_0, x, x_0) \notin \beta\|x - x_0\|^2 B_Y + \mathcal{K}. \tag{9}$$

Combining (8) and (9) we obtain

$$\|x - x_0\|^2 \leq \frac{L_f(2L_c + 1)}{\beta}\|u_0 - u\|$$

and finally

$$\|x_0 - z_0\| \leq \|x_0 - x\| + \|x - z_0\| \leq \left(L_c + \sqrt{\frac{L_f(2L_c + 1)}{\beta}}\right)\|u_0 - u\|^{\frac{1}{2}}$$

Theorem 3.3. *Let U, X, and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that $S(u_0) \neq \emptyset$. If*

- ***(i):*** *F is Lipschitz around $\{u_0\} \times X \times X$,*
- ***(ii):*** *C is Lipschitz around u_0,*
- ***(iii):*** *for each $x \in C(u_0) = C$ there is $x_0 \in S_0$ such that $F(u_0, x_0, x) \in \mathcal{K}$,*
- ***(iv):*** *there exists a neighbourhood U_0 of u_0 such that all $F(u, \cdot, \cdot)$ are uniformly strongly pseudomonotone on $C(u)$ for $u \in U_0$, i.e., there is $\beta > 0$ such that for all $u \in U_0$ and any $x, x_0 \in C(u)$, $x \neq x_0$,*

$$F(u, x_0, x) \not\leq 0 \rightarrow F(u, x, x_0) \notin \beta\|x - x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping S is upper Hőlder of order $\frac{1}{2}$ around u_0.

Proof. Take any $u \in U_1 \cap U_0$ and $x_0 \in S(u)$. By (ii), there exists $z \in C$ such that

$$\|x_0 - z\| \leq L_c\|u - u_0\|.$$

By (iii), there exists $z_0 \in S_0$ such that

$$F(u_0, z_0, z) \in \mathcal{K}.$$

Again, by (ii), there exists $x \in C(u)$ such that

$$\|z_0 - x\| \leq L_c\|u - u_0\|$$

By Lipschitzness of F,

$$\|F(u, x, x_0) - F(u_0, z_0, z)\| \leq L_f(\|x_0 - z\| + \|x - z_0\| + \|u - u_0\|) \leq L_f(2L_c + 1)\|u - u_0\|.$$

By this,

$$F(u, x, x_0) \in F(u_0, z_0, z) + L_f(2L_c + 1)\|u - u_0\|B_Y \subset L_f(2L_c + 1)\|u - u_0\|B_Y + \mathcal{K}. \tag{10}$$

On the other hand, by (iv),

$$F(u, x, x_0) \notin \beta\|x - x_0\|^2 B_Y + \mathcal{K}. \tag{11}$$

Combining (10) and (11) we obtain

$$\|x - x_0\|^2 \leq \frac{L_f(2L_c + 1)}{\beta}\|u - u_0\|$$

and finally

$$\|x_0 - z_0\| \leq \|x_0 - x\| + \|x - z_0\| \leq \left(L_c + \sqrt{\frac{L_f(2L_c + 1)}{\beta}}\right)\|u - u_0\|^{\frac{1}{2}}$$

Remark 3.4. Let us note that in Theorems 3.1 and 3.3 the condition (iv) can be replaced by the weaker condition

(iv)': there exists a neighbourhood U_0 of u_0 such that $F(u,\cdot,\cdot)$ are uniformly strongly pseudomonotone on $C(u)$ for $u \in U_0$, i.e., there is $\beta > 0$ such that for all $u \in U_0$ each $x_0 \in S(u)$ and all $x \in C(u)$, $x \neq x_0$,

$$F(u,x_0,x) \not\leq 0 \rightarrow F(u,x,x_0) \notin \beta\|x-x_0\|^2 B_Y + \mathcal{K}$$

Remark 3.5. In the above theorems we have assumed that the solution sets $S(u)$ are nonempty in a neighbourhood of u_0, and/or $S(u_0) \neq \emptyset$. This is not our aim here to discuss this topic. The conditions guarantying the existence of solutions to $(WVEP)$ and (VEP) are given for instance in [2,3,6,10]. In particular, in [6] the conditions for the existence of solutions to $(WVEP)$ are formulated for pseudomonotone and for quasimonotone bifunctions.

In a similar way the Hőlder stability results can be proved for parametric problems $(WVEP)_u$ of finding $x \in C(u)$ such that

$$F(u,x,y) \not< 0 \quad \text{for all} \quad y \in C(u), \tag{12}$$

For instance Theorem 3.1 takes the following form.

Theorem 3.6. *Let U, X, and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that $S(u) \neq \emptyset$ for all u in some neighbourhood of u_0. If*

(i): *F is Lipschitz around $\{u_0\} \times X \times X$,*

(ii): *C is Lipschitz around u_0,*

(iii): *there exists a neighbourhood U_0 of u_0 such that for each $u \in U_0$ and each $x \in C(u)$ there is $x_0 \in S(u)$ such that $F(u,x_0,x) \in int\mathcal{K}$,*

(iv): *$F(u,\cdot,\cdot)$ is uniformly strongly pseudomonotone on $C(u)$, $u \in U_0$, i.e., there is $\beta > 0$ such that for each $x, x_0 \in C(u)$*

$$F(u,x_0,x) \not< 0 \rightarrow F(u,x,x_0) \notin \beta\|x-x_0\|^2 B_Y + \mathcal{K}$$

then the solution set-valued mapping S is Hőlder of order $\frac{1}{2}$ around u_0.

Theorems 3.2 and 3.3 can be rephrased in a similar way.

4. Special case: vector optimization problems

As pointed out in Introduction, vector equilibrium problems (VEP) with $F(x,y) = \phi(y) - \phi(x)$, where $\phi : X \rightarrow Y$ is a mapping from X into Y, reduce to vector optimization problems (VOP) of finding all $x \in C$ such that $\phi(x) \in E(\phi(C))$. Now we apply the results of the previous section to parametric vector optimization $(VOP)_u$,

$$(VOP)_u \quad \begin{array}{ll} \mathcal{K}-\min & \phi(u,x) \\ x \in C(u) & \end{array}$$

of finding $S(u) = \{x \in C(u) \;\; \phi(u,x) \in E(\phi(u,C(u))\}$, with $S(u_0) = S_0$.

In the case of vector optimization problems the condition (iii) of Theorem 3.1 is rephrased as follows: there exists a neighbourhood U_0 of u_0 such that for $u \in U_0$ and each $x \in C(u)$ there exists $x_0 \in S(u)$ such that $\phi(u,x) - \phi(u,x_0) \in \mathcal{K}$. In other words, there exists a neighbourhood U_0 of u_0 such that the domination property (DP) holds for $(VOP)_u$, $u \in U_0$ (see [23]).

As shown in Example 2.2.1, in the case of vector optimization problems (VOP) , the strong pseudomonotonicity implies that all the solutions to (VOP) are sharp as defined in [7,16] with the same constant β.

Theorem 4.1. *Let U, X, and Y be normed vector spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Suppose that $S(u) \neq \emptyset$ for all u in some neighbourhood of u_0. If*

(i): *ϕ is Lipschitz around $\{u_0\} \times X$,*
(ii): *C is Lipschitz around u_0,*
(iii): *there exists a neighbourhood U_0 of u_0 such that (DP) holds for all $(VOP)_u$ with $u \in U_0$,*
(iv): *the solution sets $S(u)$, $u \in U_0$, are uniformly sharp, i.e., there is $\beta > 0$ such that for each $x \in C(u)$ and $x_0 \in S(u)$, $x \neq x_0$,*

$$(\phi(u,x) - \phi(u,x_0)) \cap (\beta \|x - x_0\|^2 B_Y - \mathcal{K}) = \emptyset,$$

then the solution set-valued mapping S is Hőlder of order $\frac{1}{2}$ around u_0.

In a similar way Theorem 3.2 and Theorem 3.3 can be rewritten for $(VOP)_u$. In [7] weak sharp solutions to vector optimization are defined and the Hőlder type continuities of weak sharp solutions to parametric problems are investigated. The generalization of this approach to (VEP) will be given elsewhere.

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