

A METHOD FOR DETECTING POLLUTION IN DISSIPATIVE SYSTEMS WITH INCOMPLETE DATA.

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Abstract. Modelling environmental problems leads to mathematical systems with missing data. For instance, weather problems have generally missing initial conditions. The paper is concerned with identifying pollution terms arising in the state equation of some dissipative system with incomplete initial condition.

To this aim the so-called sentinel method is used. Here, the problem of determining a sentinel is equivalent to a null-controllability problem for which Carleman inequalities are revisited.

Résumé. La modélisation des problèmes environnementaux conduit à des systèmes mathématiques à données manquantes. C'est le cas par exemple des problèmes de météorologie où l'on ne connaît jamais la donnée initiale. Nous sommes concernés dans cet article, par l'identification des termes de pollution présents dans l'équation d'état d'un système dissipatif à donnée initiale incomplète.

Dans ce but, la méthode des sentinelles de Lions est utilisée. Ici, le problème de détermination d'une sentinelle est équivalent à un problème de contrôlabilité à zéro pour lequel on donne des estimations de type Carleman.

INTRODUCTION

Let be $T > 0$, and Ω an open subset of \mathbf{R}^d of smooth boundary $\partial\Omega$, and denote by $Q = \Omega \times]0, T[$ the space-time cylinder. We are interested with systems partially known; we consider here the state equation :

$$\left\{ \begin{array}{lll} y' - \Delta y + f(y) & = & \xi + \lambda \hat{\xi} \quad \text{in } Q, \\ y(0) & = & y^0 + \tau \hat{y}^0 \quad \text{in } \Omega, \\ y & = & 0 \quad \text{on } \Sigma_1, \\ \frac{\partial y}{\partial \nu} & = & 0 \quad \text{on } \Sigma_2, \end{array} \right. \quad (1)$$

where $y = y(x, t; \lambda, \tau)$, and where Σ_1 is a piece of the boundary $\Sigma = \partial\Omega \times]0, T[$ and $\Sigma_2 = \Sigma \setminus \Sigma_1$. We assume here that $f : \mathbf{R} \rightarrow \mathbf{R}$ is of class C^1 , the functions ξ and y^0 are known with $\xi \in L^2(Q)$ and $y^0 \in L^2(\Omega)$. But, the terms : $\lambda \hat{\xi}$ (so-called pollution term) and $\tau \hat{y}^0$ (so-called perturbation term) are unknown, $\hat{\xi}$ and \hat{y}^0 are renormalized and represent the size of pollution and perturbation

$$\|\hat{\xi}\|_{L^2(Q)} \leq 1, \quad \|\hat{y}^0\|_{L^2(\Omega)} \leq 1, \quad \text{so that the reals } \lambda, \tau \text{ are small enough.}$$

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Some non empty open subset $O \subset \Omega$, is called observatory set. The observation is y on O , for the time T . We denote by y_{obs} this observation

$$y_{obs} = m_o \in L^2(O \times (0, T)). \quad (2)$$

We suppose that (1) has a unique solution denoted by $y(\lambda, \tau) := y(x, t; \lambda, \tau)$ in some relevant space. The question is

$$\left| \begin{array}{l} \text{how to calculate the pollution term } \lambda \hat{\xi} \text{ in} \\ \text{the state equation, independently from the} \\ \text{variation } \tau \hat{y}^0 \text{ around the initial data ?} \end{array} \right. \quad (3)$$

Least squares. Question (3) is natural and leads to some developments; some answer is given by the least squares method. The method consists in considering the unknowns $\{\lambda \hat{\xi}, \tau \hat{y}^0\} = \{v, w\}$ as control variables, then the state $y(x, t; v, w)$ has to be driven as close as possible to m_o .

This comes to some optimal control problem. By this way we look for the pair (v, w) , there is then no real possibility to find v or w independently.

Sentinels. The sentinel method of Lions [12] is a particular least squares method which is adapted to the identification of parameters in ecosystems with incomplete data; many models can be found in litterature. The sentinel concept relies on the following three objects : some state equation (for instance (1)), some observation function (2), and some control function w to be determined.

Many papers use the definition of Lions in the theoretical aspect (see for example Bodart [1], Bodart-Fabre [3] [4]), as well as in the numerical one (see Bodart-Demeestere [2], Demeestere [6] and Kernevez [9]).

We now consider the sentinel method of Lions which is an other attempt and brings better answer to question (3), as we will explain now :

Let h_0 be some function in $L^2(O \times (0, T))$. Let on the other hand ω be some open and non empty subset of Ω . For a control function $w \in L^2(\omega \times (0, T))$, we define the functional

$$S(\lambda, \tau) = \int_0^T \int_O h_0 y(\lambda, \tau) dxdt + \int_0^T \int_\omega w y(\lambda, \tau) dxdt. \quad (4)$$

We say that S defines a sentinel for the problem (1) if there exists w such that S is insensitive (at first order) with respect the to missing terms $\tau \hat{y}^0$, which means

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \quad (5)$$

for any \hat{y}^0 where here $(0, 0)$ corresponds to $\lambda = \tau = 0$, and if w minimizes the norm $\|w\|_{L^2(\omega \times (0, T))}$.

Remark. The Lions sentinels S assume $\omega = O$. In this case, the observation and the control share the same support, and the solution $w = -h_0$ is trivial.

The definition (4) extends the one by Lions to the case where the observation and the control have different supports. This point of view (where $\omega \neq O$) has been considered for the first time by Nakoulima in [15]. Here we avoid the trivial solution.

Informations given by the sentinel. Because of (5) we can write

$$S(\lambda, \tau) \simeq S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(0, 0), \quad \text{for } \lambda, \tau \text{ small.}$$

In (4), $S(\lambda, \tau)$ is observed and using (2),

$$S(\lambda, \tau) = \int_Q (h_0 \chi_O + w \chi_\omega) m_o \, dxdt$$

so that (5) becomes

$$\lambda \frac{\partial S}{\partial \lambda}(0, 0) \simeq \int_Q (h_0 \chi_O + w \chi_\omega) (m_o - y_0) \, dxdt, \quad (6)$$

with

$$\frac{\partial S}{\partial \lambda}(0, 0) = \int_Q (h_0 \chi_O + w \chi_\omega) y_\lambda \, dxdt,$$

where here χ_O and χ_ω denote the characteristic functions of O and ω respectively.

The derivative $y_\lambda = (\partial y / \partial \lambda)(0, 0)$ only depends on $\hat{\xi}$ and other known data. Consequently, the estimates (6) contains the informations on $\lambda \hat{\xi}$ (see for details remark 1.1 below).

1. EQUIVALENT CONTROLLABILITY PROBLEM

We shall show in this section that the existence of a sentinel comes to null controllability property. We begin by transforming the insensibility condition (5).

Set

$$y_\tau = \frac{d}{d\tau} y(\lambda, \tau) \Big|_{\lambda=\tau=0}.$$

Then the function y_τ is solution of

$$\left\{ \begin{array}{ll} y'_\tau - \Delta y_\tau + f'(y_0) y_\tau & = 0 \quad \text{in } Q, \\ y_\tau(0) & = \hat{y}^0 \quad \text{in } \Omega, \\ y_\tau & = 0 \quad \text{on } \Sigma_1, \\ \frac{\partial y_\tau}{\partial \nu} & = 0 \quad \text{on } \Sigma_2, \end{array} \right. \quad (7)$$

where $y_0 = y(0, 0)$. Problem (7) is linear and has a unique solution y_τ under mild assumptions on f .

The insensibility condition (5) holds if and only if

$$\int_Q (h_0 \chi_O + w \chi_\omega) y_\tau \, dxdt = 0. \quad (8)$$

We can transform (8) by introducing the classical adjoint state. More precisely, we define the function $q = q(x, t)$ as the solution of the backward problem :

$$\left\{ \begin{array}{ll} -q' - \Delta q + f'(y_0) q & = h_0 \chi_O + w \chi_\omega \quad \text{in } Q, \\ q(T) & = 0 \quad \text{in } \Omega, \\ q & = 0 \quad \text{on } \Sigma_1, \\ \frac{\partial q}{\partial \nu} & = 0 \quad \text{on } \Sigma_2. \end{array} \right. \quad (9)$$

As for the problem (7), the problem (9) has a unique solution q (under mild assumptions on $f'(y_0)$). The function q depends on the control w that we shall determine :

Indeed, if we multiply the first equation in (9) by y_τ , and we integrate by parts over Q , we obtain

$$\int_Q (h_0 \chi_O + w \chi_\omega) y_\tau \, dx dt = \int_\Omega q(0) \hat{y}^0 \, dx \quad \forall \hat{y}^0, \quad \|\hat{y}^0\|_{L^2(\Omega)} \leq 1.$$

So, the condition (5) (or (8)) is equivalent to

$$q(0) = 0. \quad (10)$$

This is a null-controllability problem.

Remark 1.1. The knowledge of the optimal control w provides informations about the pollution term $\lambda \hat{\xi}$. Indeed, denote by

$$L = \frac{\partial}{\partial t} - \Delta + f'(y_0) I_d \quad (11)$$

and let $y_\lambda = \frac{\partial y}{\partial \lambda}(0, 0)$ be the solution of

$$\begin{cases} L y_\lambda = \hat{\xi} & \text{in } Q, \\ y_\lambda(0) = 0 & \text{in } \Omega, \\ y_\lambda = 0 & \text{on } \Sigma_1, \\ \frac{\partial y_\lambda}{\partial \nu} = 0 & \text{on } \Sigma_2. \end{cases} \quad (12)$$

Integrating by parts, we then obtain

$$\int_Q y_\lambda L^* q \, dx dt = \int_Q q \hat{\xi} \, dx dt$$

with

$$L^* = -\frac{\partial}{\partial t} - \Delta + f'(y_0) I_d. \quad (13)$$

So that from (9) and (6) we deduce

$$\int_Q \lambda \hat{\xi} \, dx dt = \int_Q (h_0 \chi_O + w \chi_\omega) (m_o - y_0) \, dx dt.$$

2. EXISTENCE OF A SENTINEL

We begin with some observability inequality, which will be proved in detail in the last section. Denote by

$$\mathcal{V} = \left\{ v \in \mathcal{C}^\infty(\bar{Q}) \text{ such that } : v|_{\Sigma_1} = \frac{\partial v}{\partial t}|_{\Sigma_1} = 0 \text{ and } \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0 \right\}. \quad (14)$$

Then we have :

Theorem 2.1. *Let be $u \in \mathcal{V}$, then there exists a positive constant $C = C(\Omega, \omega, O, T, f'(y_o))$ such that*

$$\int_Q \frac{1}{\theta^2} |u|^2 \, dx dt \leq C \left[\int_Q |L u|^2 \, dx dt + \int_0^T \int_\omega |u|^2 \, dx dt \right], \quad (15)$$

where $\theta \in \mathcal{C}^2(Q)$ positive with $\frac{1}{\theta}$ bounded.

According to the RHS of (15), we consider the space \mathcal{V} endowed with the bilinear form $a(.,.)$ defined by :

$$a(u, v) = \int_Q Lu Lv dxdt + \int_0^T \int_\omega u v dxdt. \tag{16}$$

Let V be the completion of \mathcal{V} with respect to the norm

$$v \mapsto \|v\|_V = \sqrt{a(v, v)}, \tag{17}$$

then, V is a Hilbert space for the scalar product $a(v, \hat{v})$ and the associated norm.

Remark 2.2. We can precise the structure of the elements of V . Indeed, let $H_\theta(Q)$ be the weighed Hilbert space defined by

$$H_\theta(Q) = \left\{ v \in L^2(Q) \text{ such that : } \int_Q \frac{1}{\theta^2} |v|^2 dxdt < \infty \right\},$$

endowed with the natural norm $\|v\|_\theta = (\int_Q \frac{1}{\theta^2} |v|^2 dxdt)^{\frac{1}{2}}$. This shows that V is imbedded continuously $\|v\|_\theta \leq C \|v\|_V$.

Now if $h_0 \in L^2(Q)$ and $\theta h_0 \in L^2(Q)$ (i.e. : $h_0 \in L^2_\theta(Q)$), then from (15) and the Cauchy-Schwartz inequality, we deduce that the linear form defined on V by

$$v \mapsto \int_Q h_0 \chi_O v dxdt$$

is continuous. Therefore, from the Lax-Milgram theorem there exists a unique u in V solution of the variational equation :

$$a(u, v) = \int_Q h_0 \chi_O v dxdt \quad \forall v \in V. \tag{18}$$

Theorem 2.3. Assume that $h_0 \in L^2_\theta(Q)$, and let u be the unique solution of (18). We set

$$\mathbf{w} = -u \chi_\omega \tag{19}$$

and

$$q = Lu. \tag{20}$$

Then, the pair (\mathbf{w}, q) is such that (9)-(10) hold (i.e there is some insensitive sentinel defined by (4)-(5)).

3. PROOF OF THEOREM 2.1

The proof for the observability inequality in theorem 2.1 will hold from Carleman estimates that we carefully show in the following results.

Lemma 3.1. Let be ω_0 an open set such that $\overline{\omega_0} \subset \omega$. Then there is $\psi \in C^2(\overline{\Omega})$ such that

$$\begin{cases} \psi(x) > 0 \quad \forall x \in \Omega, \\ \psi(x) = 0 \quad \forall x \in \Gamma, \\ |\nabla \psi(x)| \neq 0 \quad \forall x \in \overline{\Omega - \omega_0}. \end{cases}$$

Proof. See Imanuvilov [8]

We now use a function ψ as given by the previous lemma, to define convenient *weight* functions. For $\lambda > 0$, we set

$$\varphi(x, t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad (21)$$

and

$$\eta(x, t) = \frac{e^{2\lambda|\psi|_\infty} - e^{\lambda\psi(x)}}{t(T-t)}. \quad (22)$$

Then

$$\nabla\varphi = \lambda\varphi\nabla\psi, \quad \nabla\eta = -\lambda\varphi\nabla\psi. \quad (23)$$

We also notice the following properties :

$$\left| \frac{\partial\varphi}{\partial t} \right| \leq T\varphi^2, \quad \left| \frac{\partial^2\varphi}{\partial t^2} \right| \leq T^2\varphi^3, \quad (24)$$

$$\left| \frac{\partial\eta}{\partial t} \right| \leq T\varphi^2, \quad \left| \frac{\partial^2\eta}{\partial t^2} \right| \leq T^2\varphi^3. \quad (25)$$

Remark 3.2. Note that η increases to $+\infty$ when $t \rightarrow T$ or $t \rightarrow 0$, but η is uniformly bounded on $\Omega \times [\delta, T - \delta]$ for any $\delta > 0$.

On the other hand, for fixed $s > 0$ the function $e^{-s\eta(x,t)}$ goes to 0 when $t \rightarrow T$ or $t \rightarrow 0$.

The following theorem states the Carleman inequalities concerning (14) :

Proposition 3.3. *There exist constants $s_0 > 0$, $\lambda_0 > 0$ and $C > 0$ depending on Ω , ω , ψ , and T , such that for all $s \geq s_0$, $\lambda \geq \lambda_0$, and for any function $u \in \mathcal{V}$ given by (14), we have*

$$\begin{aligned} & 2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |u|^2 dxdt + 4s^2\lambda \int_{\Sigma_2} \varphi \frac{\partial\eta}{\partial t} \frac{\partial\psi}{\partial\nu} e^{-2s\eta} |u|^2 d\gamma dt \\ & - 4s^3\lambda^3 \int_{\Sigma_2} \varphi^3 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} e^{-2s\eta} |u|^2 d\gamma dt - 4s^2\lambda^3 \int_{\Sigma_2} \varphi^2 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} e^{-2s\eta} |u|^2 d\gamma dt \\ & - 2s\lambda \int_{\Sigma_2} \varphi \frac{\partial\psi}{\partial\nu} e^{-2s\eta} \frac{\partial u}{\partial t} u d\gamma dt - 4s\lambda \int_{\Sigma_1} \varphi \nabla\psi e^{-2s\eta} \nabla u \frac{\partial u}{\partial\nu} d\gamma dt \\ & + 2s\lambda \int_{\Sigma} \varphi \frac{\partial\psi}{\partial\nu} e^{-2s\eta} |\nabla u|^2 d\gamma dt \\ & \leq C \left(\int_Q e^{-2s\eta} \left| \frac{\partial u}{\partial t} - \Delta u \right|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 dxdt \right). \end{aligned} \quad (26)$$

Proof. We use the method by Fursikov and Imanuvilov [7], Imanuvilov [8], Lebeau and Robbiano [10], and Puel [16] [17] (case $\Sigma_2 = \emptyset$ and $\Sigma_1 = \Sigma$).

For $s \geq s_0$ and $\lambda \geq \lambda_0$, we define

$$w(x, t) = e^{-s\eta(x,t)} u(x, t). \quad (27)$$

We easily notice that

$$w(x, 0) = w(x, T) = 0. \quad (28)$$

Calculating $g = (\partial_t - \Delta)(e^{s\eta}w)$, with notation (27), we get

$$P_1 w + P_2 w = g_s, \quad (29)$$

where

$$P_1 w = \frac{\partial w}{\partial t} + 2s\lambda\varphi \nabla\psi \nabla w + 2s\lambda^2\varphi |\nabla\psi|^2 w, \quad (30)$$

$$P_2 w = -\Delta w - s^2\lambda^2\varphi^2 |\nabla\psi|^2 w + s\frac{\partial\eta}{\partial t} w, \quad (31)$$

$$g_s = e^{-s\eta} g + s\lambda^2\varphi |\nabla\psi|^2 w - s\lambda\varphi \Delta\psi w. \quad (32)$$

Taking the L^2 norm we get :

$$\int_Q |P_1 w|^2 dxdt + \int_Q |P_2 w|^2 dxdt + 2 \int_Q P_1 w P_2 w dxdt = \int_Q |g_s|^2 dxdt. \quad (33)$$

We shall now calculate $\int_Q P_1 w P_2 w dxdt$. This will give 9 terms $I_{k,l}$.

In order to organize our calculus, we denote by A and B the quantities such that A contains all the terms which can be upper bounded by

$$c(s\lambda + \lambda^2) \int_Q \varphi |\nabla w|^2 dxdt,$$

and by B all those which can be bounded by

$$c(s^2\lambda^4 + s^3\lambda^3) \int_Q \varphi^3 |w|^2 dxdt.$$

We denote by ν the outer normal on Γ . Note down that ψ cancels on Γ . We then have the following results :

$$\begin{aligned} I_{1,1} &= - \int_Q \frac{\partial w}{\partial t} \Delta w dxdt \\ &= - \int_{\Sigma} \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} d\gamma dt + 0, \end{aligned} \quad (34)$$

$$I_{1,2} = -s^2\lambda^2 \int_Q \frac{\partial w}{\partial t} \varphi^2 |\nabla\psi|^2 w dxdt = B, \quad (35)$$

$$\begin{aligned} I_{1,3} &= s \int_Q \frac{\partial w}{\partial t} \frac{\partial\eta}{\partial t} w dxdt = \frac{s}{2} \int_Q \frac{\partial\eta}{\partial t} \frac{\partial}{\partial t} (|w|^2) dxdt \\ &= -\frac{s}{2} \int_Q \frac{\partial^2\eta}{\partial t^2} |w|^2 dxdt = B. \end{aligned} \quad (36)$$

And,

$$\begin{aligned} I_{2,1} &= -2s\lambda \int_Q \varphi \nabla\psi \nabla w \Delta w dxdt \\ &= -2s\lambda \int_{\Sigma} \varphi \nabla\psi \cdot \nabla w \frac{\partial w}{\partial \nu} d\gamma dt + 2s\lambda^2 \int_Q \varphi |\nabla\psi \cdot \nabla w|^2 dxdt \\ &\quad + s\lambda \int_{\Sigma} \varphi \frac{\partial\psi}{\partial \nu} |\nabla w|^2 d\gamma dt - s\lambda^2 \int_Q \varphi |\nabla\psi|^2 |\nabla w|^2 dxdt + A, \end{aligned}$$

$$\begin{aligned}
I_{2,2} &= -2s^3\lambda^3 \int_Q \varphi^3 |\nabla\psi|^2 \nabla\psi \nabla w \cdot w = -s^3\lambda^3 \int_Q \varphi^3 |\nabla\psi|^2 \nabla\psi \nabla |w|^2 \\
&= -s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} |w|^2 + 3s^3\lambda^4 \int_Q \varphi^3 |\nabla\psi|^4 |w|^2 + B,
\end{aligned} \tag{37}$$

$$\begin{aligned}
I_{2,3} &= 2s^2\lambda \int_Q \varphi \nabla\psi \nabla w \frac{\partial\eta}{\partial t} w \, dxdt \\
&= s^2\lambda \int_\Sigma \varphi \frac{\partial\eta}{\partial t} \frac{\partial\psi}{\partial\nu} |w|^2 \, dxdt + B.
\end{aligned} \tag{38}$$

Finally :

$$\begin{aligned}
I_{3,1} &= -2s\lambda^2 \int_Q \varphi |\nabla\psi|^2 \Delta w \cdot w \, dxdt \\
&= -2s\lambda^2 \int_\Sigma \varphi |\nabla\psi|^2 \frac{\partial w}{\partial\nu} \cdot w + 2s\lambda^2 \int_Q \varphi |\nabla\psi|^2 |\nabla w|^2 + A + B,
\end{aligned} \tag{39}$$

$$I_{3,2} = -2s^3\lambda^4 \int_Q \varphi^3 |\nabla\psi|^4 |w|^2 \, dxdt, \tag{40}$$

$$I_{3,3} = 2s^2\lambda^2 \int_Q \varphi \frac{\partial\eta}{\partial t} |\nabla\psi|^2 |w|^2 \, dxdt = B. \tag{41}$$

Summing all the terms, it follows :

$$\begin{aligned}
2 \int_Q P_1 w P_2 w \, dxdt &= A + B + 2s\lambda^2 \int_Q \varphi |\nabla\psi|^2 |\nabla w|^2 \, dxdt \\
&\quad + 2s^3\lambda^4 \int_Q \varphi^3 |\nabla\psi|^4 |w|^2 \, dxdt + 4s\lambda^2 \int_Q \varphi |\nabla\psi \cdot \nabla w|^2 \, dxdt \\
&\quad - 2 \int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial\nu} \, d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla\psi \cdot \nabla w \frac{\partial w}{\partial\nu} \, d\gamma dt \\
&\quad + 2s\lambda \int_\Sigma \varphi \frac{\partial\psi}{\partial\nu} |\nabla w|^2 \, d\gamma dt - 2s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} |w|^2 \, d\gamma dt \\
&\quad + 2s^2\lambda \int_\Sigma \varphi \frac{\partial\eta}{\partial t} \frac{\partial\psi}{\partial\nu} |w|^2 \, d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla\psi|^2 \frac{\partial w}{\partial\nu} \cdot w \, d\gamma dt.
\end{aligned} \tag{42}$$

But $|\nabla\psi| \neq 0$ on $\overline{\Omega - \omega_0}$, hence there is $\delta > 0$ such that

$$|\nabla\psi| \geq \delta \quad \text{on } \overline{\Omega - \omega_0}. \tag{43}$$

On the other hand

$$\int_Q |g_s|^2 \, dxdt \leq \int_Q e^{-2s\eta} |g|^2 \, dxdt + B,$$

so that

$$\begin{aligned} \int_Q |P_1 w|^2 dxdt + \int_Q |P_2 w|^2 dxdt + 2 \int_Q P_1 w P_2 w dxdt \\ \leq \int_Q e^{-2s\eta} |g|^2 dxdt + B. \end{aligned}$$

Consequently :

$$\begin{aligned} & \int_Q |P_1 w|^2 dxdt + \int_Q |P_2 w|^2 dxdt + 2s\lambda^2 \delta^2 \int_Q \varphi |\nabla w|^2 dxdt \\ & + 2s^3 \lambda^4 \delta^4 \int_Q \varphi^3 |w|^2 dxdt - 2 \int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi \cdot \nabla w \frac{\partial w}{\partial \nu} d\gamma dt \\ & + 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 d\gamma dt - 2s^3 \lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 d\gamma dt \\ & + 2s^2 \lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} |w|^2 d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} \cdot w d\gamma dt + A + B \\ & \leq \int_Q e^{-2s\eta} |g|^2 dxdt + B + 2s\lambda^2 \delta^2 \int_0^T \int_{\omega_0} \varphi |\nabla w|^2 dxdt + 2s^3 \lambda^4 \delta^4 \int_0^T \int_{\omega_0} \varphi^3 |w|^2 dxdt. \end{aligned}$$

We can eliminate A and B by choosing s and λ large enough. And we observe that :

$$\begin{aligned} & \int_0^T \int_{\omega} \varphi \theta^2 |\nabla w|^2 dxdt \\ & \leq C \left(\int_Q \varphi P_2 w \theta^2 w dxdt + \int_0^T \int_{\omega} \varphi^{\frac{1}{2}} \theta^2 |\nabla w| \varphi^{\frac{1}{2}} w dxdt + s^2 \lambda^2 \int_0^T \int_{\omega} \varphi^3 w^2 dxdt \right), \end{aligned}$$

for $\theta \in \mathcal{D}(\omega)$ such that $0 \leq \theta \leq 1$ and $\theta(x) = 1$ on ω_0 gives

$$2s\lambda^2 \delta^2 \int_0^T \int_{\omega_0} \varphi |\nabla w|^2 dxdt \leq \frac{1}{2} \int_0^T \int_{\Omega} |P_2 w|^2 dxdt + cs^3 \lambda^4 \int_0^T \int_{\omega} \varphi^3 |w|^2 dxdt, \quad (44)$$

Now, we should write the inequality below in terms of the solution u , since

$$|w|^2 = e^{-2s\eta} |u|^2.$$

So,

$$\begin{aligned}
& \frac{1}{2} \int_Q |P_1 w|^2 dxdt + \frac{1}{2} \int_Q |P_2 w|^2 dxdt \\
& + 2s\lambda^2 \int_Q \varphi |\nabla w|^2 dxdt + 2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |u|^2 dxdt \\
& + 2s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 d\gamma dt - 2s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 d\gamma dt \\
& - 2 \int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi \cdot \nabla w \frac{\partial w}{\partial \nu} d\gamma dt \\
& + 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} \cdot w d\gamma dt \\
& \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 dxdt \right).
\end{aligned}$$

Now from

$$\nabla u = e^{s\eta} (\nabla w - s\lambda \varphi \nabla \psi w),$$

we deduce

$$\int_Q \varphi e^{-2s\eta} |\nabla u|^2 dxdt \leq C \left(\int_Q \varphi |\nabla w|^2 dxdt + s^2\lambda^2 \int_Q \varphi^3 |w|^2 dxdt \right).$$

We then use the explicit form of $P_1 w$ and $P_2 w$, and get

$$\frac{1}{s} \int_Q \frac{1}{\varphi} \left| \frac{\partial w}{\partial t} \right|^2 dxdt \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt \right),$$

and

$$\frac{1}{s} \int_Q \frac{1}{\varphi} |\Delta w|^2 dxdt \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt \right).$$

We sum up to finally have

$$\begin{aligned}
& 2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |u|^2 dxdt + 4s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 d\gamma dt \\
& - 4s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 d\gamma dt - 4s^2\lambda^3 \int_\Sigma \varphi^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 d\gamma dt \\
& + \int_\Sigma \left(2s \frac{\partial \eta}{\partial t} - 4s\lambda^2 \varphi |\nabla \psi|^2 - 4s^2\lambda^2 \varphi^2 |\nabla \psi|^2 \right) e^{-2s\eta} \frac{\partial u}{\partial \nu} u \\
& - 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} \frac{\partial u}{\partial t} u d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi e^{-2s\eta} \nabla u \frac{\partial u}{\partial \nu} d\gamma dt \\
& + 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |\nabla u|^2 d\gamma dt - 2 \int_\Sigma e^{-2s\eta} \frac{\partial u}{\partial t} \frac{\partial u}{\partial \nu} d\gamma dt \\
& \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 dxdt \right).
\end{aligned}$$

Using the fact that $u = \frac{\partial u}{\partial t} \Big|_{\Sigma_1} = 0$, and $\frac{\partial u}{\partial \nu} \Big|_{\Sigma_2} = 0$, we obtain (26).

Now we proceed as the following. We define

$$\tilde{\varphi}(x, t) = \frac{e^{-\lambda\psi(x)}}{t(T-t)}, \quad (45)$$

and

$$\tilde{\eta}(x, t) = \frac{e^{2\lambda|\psi|_\infty} - e^{-\lambda\psi(x)}}{t(T-t)}. \quad (46)$$

Then

$$\nabla\tilde{\varphi} = -\lambda\tilde{\varphi}\nabla\psi, \quad \nabla\tilde{\eta} = \lambda\tilde{\varphi}\nabla\psi; \quad (47)$$

we still have the properties (24) and (25).

For $s \geq s_0$ and $\lambda \geq \lambda_0$, we define

$$\tilde{w}(x, t) = e^{-s\tilde{\eta}(x,t)}u(x, t). \quad (48)$$

We easily notice that

$$\tilde{w}(x, 0) = \tilde{w}(x, T) = 0. \quad (49)$$

Calculating $\tilde{P}\tilde{w} = e^{-s\tilde{\eta}}g = e^{-s\tilde{\eta}}[(\partial_t - \Delta)(e^{s\tilde{\eta}}\tilde{w})]$, using notation (48), we set

$$\tilde{P}_1\tilde{w} + \tilde{P}_2\tilde{w} = \tilde{g}_s, \quad (50)$$

where this time

$$\tilde{P}_1\tilde{w} = \frac{\partial\tilde{w}}{\partial t} - 2s\lambda\tilde{\varphi}\nabla\psi\nabla\tilde{w} + 2s\lambda^2\tilde{\varphi}|\nabla\psi|^2\tilde{w}, \quad (51)$$

$$\tilde{P}_2\tilde{w} = -\Delta\tilde{w} - s^2\lambda^2\tilde{\varphi}^2|\nabla\psi|^2\tilde{w} + s\frac{\partial\tilde{\eta}}{\partial t}\tilde{w}, \quad (52)$$

$$\tilde{g}_s = e^{-s\tilde{\eta}}g + s\lambda^2\tilde{\varphi}|\nabla\psi|^2\tilde{w} + s\lambda\tilde{\varphi}\Delta\psi\tilde{w}. \quad (53)$$

We easily deduce the following result :

Proposition 3.4. *There exists $s_0 > 0$, $\lambda_0 > 0$ and C a positive constant depending on Ω , ω , ψ , and T , such that for all $s \geq s_0$, $\lambda \geq \lambda_0$, and for any fuction u of (14), we have*

$$\begin{aligned} & 2s^3\lambda^4 \int_Q \tilde{\varphi}^3 e^{-2s\tilde{\eta}} |u|^2 dxdt - 4s^2\lambda \int_{\Sigma_2} \tilde{\varphi} \frac{\partial\tilde{\eta}}{\partial t} \frac{\partial\psi}{\partial\nu} e^{-2s\tilde{\eta}} |u|^2 d\gamma dt \\ & + 4s^3\lambda^3 \int_{\Sigma_2} \tilde{\varphi}^3 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} e^{-2s\tilde{\eta}} |u|^2 d\gamma dt + 4s^2\lambda^3 \int_{\Sigma_2} \tilde{\varphi}^2 |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu} e^{-2s\tilde{\eta}} |u|^2 d\gamma dt \\ & + 2s\lambda \int_{\Sigma_2} \tilde{\varphi} \frac{\partial\psi}{\partial\nu} e^{-2s\tilde{\eta}} \frac{\partial u}{\partial t} d\gamma dt + 4s\lambda \int_{\Sigma_1} \tilde{\varphi} \nabla\psi e^{-2s\tilde{\eta}} \nabla u \frac{\partial u}{\partial\nu} d\gamma dt \\ & - 2s\lambda \int_{\Sigma} \tilde{\varphi} \frac{\partial\psi}{\partial\nu} e^{-2s\tilde{\eta}} |\nabla u|^2 d\gamma dt \\ & \leq C \left(\int_Q e^{-2s\tilde{\eta}} \left| \frac{\partial u}{\partial t} - \Delta u \right|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \tilde{\varphi}^3 e^{-2s\tilde{\eta}} |u|^2 dxdt \right). \end{aligned} \quad (54)$$

Proof. The proof is similar to the one of Proposition 3.3, so we let it to the reader.

Finally, we give below the conclusion to theorem 2.1.

We obtain from Proposition 3.3 and Proposition 3.4 the following observability inequality :

Corollary 3.5. *There is a positive constant $C = C(\Omega, \omega, \psi, T)$ such that we have*

$$\int_Q \frac{1}{\theta^2} |u|^2 dxdt \leq C \left[\int_Q \left| \frac{\partial u}{\partial t} - \Delta u \right|^2 dxdt + \int_0^T \int_\omega |u|^2 dxdt \right], \quad (55)$$

where $\frac{1}{\theta^2} = \varphi^3 e^{-2s\eta} + \tilde{\varphi}^3 e^{-2s\tilde{\eta}}$ is a bounded weight function.

Proof. Summing the terms in (26) and (54) we get the following :

$$\begin{aligned} & 2s^3 \lambda^4 \int_Q \left(\varphi^3 e^{-2s\eta} + \tilde{\varphi}^3 e^{-2s\tilde{\eta}} \right) |u|^2 dxdt \\ & - 4s^2 \lambda \int_{\Sigma_2} \left(\varphi \frac{\partial \eta}{\partial t} e^{-2s\eta} - \tilde{\varphi} \frac{\partial \tilde{\eta}}{\partial t} e^{-2s\tilde{\eta}} \right) \frac{\partial \psi}{\partial \nu} |u|^2 d\gamma dt \\ & - 4s^3 \lambda^3 \int_{\Sigma_2} \left(\varphi^3 e^{-2s\eta} - \tilde{\varphi}^3 e^{-2s\tilde{\eta}} \right) |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |u|^2 d\gamma dt \\ & - 4s^2 \lambda^3 \int_{\Sigma_2} \left(\varphi^2 e^{-2s\eta} - \tilde{\varphi}^2 e^{-2s\tilde{\eta}} \right) |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |u|^2 d\gamma dt \\ & + 2s\lambda \int_{\Sigma_2} \left(\varphi e^{-2s\eta} - \tilde{\varphi} e^{-2s\tilde{\eta}} \right) \frac{\partial \psi}{\partial \nu} \frac{\partial u}{\partial t} u d\gamma dt \\ & - 4s\lambda \int_{\Sigma_1} \left(\varphi e^{-2s\eta} - \tilde{\varphi} e^{-2s\tilde{\eta}} \right) \nabla \psi \nabla u \frac{\partial u}{\partial \nu} d\gamma dt \\ & + 2s\lambda \int_{\Sigma} \left(\varphi e^{-2s\eta} - \tilde{\varphi} e^{-2s\tilde{\eta}} \right) \frac{\partial \psi}{\partial \nu} |\nabla u|^2 d\gamma dt \\ & \leq C \left[\int_Q \left(e^{-2s\eta} + e^{-2s\tilde{\eta}} \right) \left| \frac{\partial u}{\partial t} - \Delta u \right|^2 dxdt \right. \\ & \left. + s^3 \lambda^4 \int_0^T \int_\omega \left(\varphi^3 e^{-2s\eta} + \tilde{\varphi}^3 e^{-2s\tilde{\eta}} \right) |u|^2 dxdt \right]. \end{aligned}$$

Now, it suffices to notice that $\varphi = \tilde{\varphi}$ and $\eta = \tilde{\eta}$ on Σ .

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