ESAIM: PROCEEDINGS, July 2007, Vol.18, 38-47
Jean-Frédéric Gerbeau & Stéphane Labbé, Editors

ON SUPRA-CONVERGENCE OF THE FINITE VOLUME METHOD FOR THE LINEAR ADVECTION PROBLEM

Frédéric Pascal[1]

Abstract. This paper investigates the supra-convergence phenomenon that one can observe in the upwind finite volume method for solving linear convection problem on a bounded domain but also in finite difference scheme with non-uniform grids. Although the scheme is no longer consistent in the finite difference sense and Lax-Richtmyer theorem not suitable, it is a well-known convergent method. In order to analyze the convergence rate, we introduce what we call a geometric corrector, which is associated with every finite volume mesh and every constant convection vector. Under a local quasi-uniformity condition and if the continuous solution is regular enough, there is a link between the convergence of the finite volume scheme and this geometric corrector : the study of this latter leads to the proof of the optimal order of convergence. We then focus our attention on an uniformly refined mesh of quadrangles and on a series of independent meshes of triangles and tetrahedrons. In these latter cases, a loss of accuracy is observed if there exists in the family of meshes a fixed straight line parallel to the convection direction.

Résumé. Nous nous intéressons au phénomène de supra-convergence que l'on observe pour le schéma volumes finis appliqué à l'équation d'advection linéaire dans un domaine borné mais aussi pour des schémas aux différences finies avec des maillages non uniformes. Bien que le schéma ne soit pas consistant au sens des différences finies et donc le théorème de Lax-Richtmyer non applicable, il est convergent. Pour mener à bien l'analyse, nous introduisons un correcteur géométrique ne dépendant que du maillage et du vecteur de convection. L'analyse de la convergence se ramène alors à l'analyse de ce correcteur si la solution est suffisamment régulière. Nous proposons l'étude numérique d'un maillage de quadrangles uniformément raffinés ainsi qu'une série de maillages de triangles et de tétraèdres indépendants les uns des autres. Dans ce dernier cas, on observe une perte dans le taux de convergence s'il persiste dans le maillage une droite fixe parallèle à la convection.

Introduction

Even for the scalar linear advection equation, to obtain optimal a priori error estimates for the finite volume scheme is still a challenging task (see for instance Merlet and Vovelle [8]). One of the main difficulties holds in the fact that the non-uniformity of unstructured mesh brings up an apparent loss of consistency (see Després [5]). In fact this loss of consistency is an artifact of standard convergence proof based on the Lax-Richtmyer theorem that states that stability and consistency are sufficient conditions for a scheme to be convergent. Actually consistency is not necessary, the scheme maintains the accuracy and the global error behaves better than the local error would indicate. This property of enhancement of the truncation error is called supra-convergence. In this paper, we introduce on two simple one-dimensional problems this phenomenon and a way to bypass it in the mathematical analysis.

[1] CMLA, ENS de Cachan, 61 Avenue du Président Wilson, 94235 Cachan

Concerning the finite volume scheme, in Bouche, Ghidaglia, Pascal [1], we construct geometric correctors that form a sequence of vectors in $\mathbb{R}^{nd}$, $\mathbf{\Gamma} = (\mathbf{\Gamma}_j)_{j=1}^N$, where N is the number of control volumes. This set of geometric correctors depends only on the mesh and on the advection vector but not on solutions to the advection equation like the class of admissible perturbations introduced by Després [4]. After having introduced the notations, we recall the stability of the scheme and apply the described analysis to the explicit finite volume scheme for solving a non-steady convection problem. We give the steps that lead to the main result that consists in Theorem 2.1 where we prove that the convergence of the finite volume scheme amounts to the study of the geometric correctors if the continuous solution is regular enough. More precisely, if this family of vectors is uniformly bounded by the mesh size h as it is the case for an arbitrary coarse conformal triangular mesh uniformly refined, the first order upwind scheme is indeed first order accurate. Finally, we present numerical results concerning the l^∞ and l^1 norms of the geometric correctors in several examples where the theoretical estimates are not yet proved.

1. Supra-Convergence

Let u be the exact solution of the equation $L(u) = F$ where L is a differential operator. Let h be the parameter of discretization that goes to zero. Let assume that the finite difference scheme can be written as

$$L_h(u_h) = F \tag{1}$$

where the discrete operator L_h is supposed to be a linear operator. We assume that the discrete right hand side is identical to F since its approximation is usually written as accurate as the expected order of convergence. Then using the terminology of the finite difference error analysis, the local error ϵ_h is obtained by applying the discrete operator to the exact solution :

$$L_h(u) = \epsilon_h + F\,. \tag{2}$$

Let us recall that the stability condition consists in bounding, independently on h, the inverse of the discrete operator i.e. in establishing the relation $\|L_h^{-1}\| \leq c$ with an appropriate norm. Therefore (see for instance [13]) Lax-Richtmyer theorem follows from the fact that the global error $e_h = u_h - u$ satisfies the relation

$$\|e_h\| \leq c\|L_h(e_h)\| \leq c\|\epsilon_h\|. \tag{3}$$

Hence if the truncation error goes to zero when h goes to zero or in an equivalent way if the scheme is consistent then the scheme is convergent at least at the same rate than the truncation error. For instance if the truncation is only $\mathcal{O}(h^{p-1})$, the global error is at least $\mathcal{O}(h^{p-1})$.

Definition 1.1. A scheme exhibits a supra-convergence phenomenon if the global error bounds behaves better than the local error would indicate and for instance if an optimal order error bounds holds despite the fact that the truncation error is of lower order.

Let us precise that this phenomenon is often observed for finite difference schemes on non-uniform grids.

1.1. **Finite difference on non-uniform grids**

First we are going to study this phenomenon with two academic test cases. Let us start with the second order diffusion equation in one dimension :

$$-u'' = f \quad \text{on} \quad [0, L] \quad \text{with} \quad u(0) = u(L) = 0 \tag{4}$$

discretized on the non-uniform grid defined on Figure 1 by the following finite difference scheme (u_0 and u_N are given and equal to the boundary value) :

$$-\frac{2u_{i-1}}{\Delta_i(\Delta_i + \Delta_{i+1})} + \frac{2u_i}{\Delta_i\Delta_{i+1}} - \frac{2u_{i+1}}{\Delta_{i+1}(\Delta_i + \Delta_{i+1})} = f(x_i) \quad \forall i \in [1, N-1]\,. \tag{5}$$

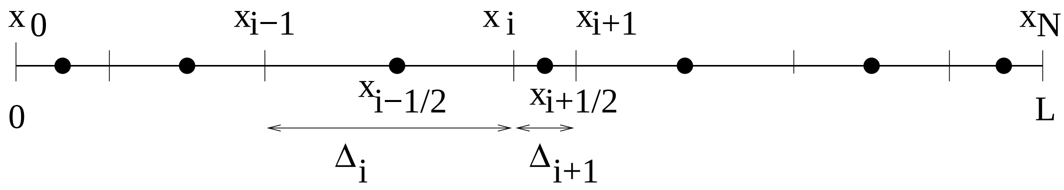

FIGURE 1. The computational domain in one dimension

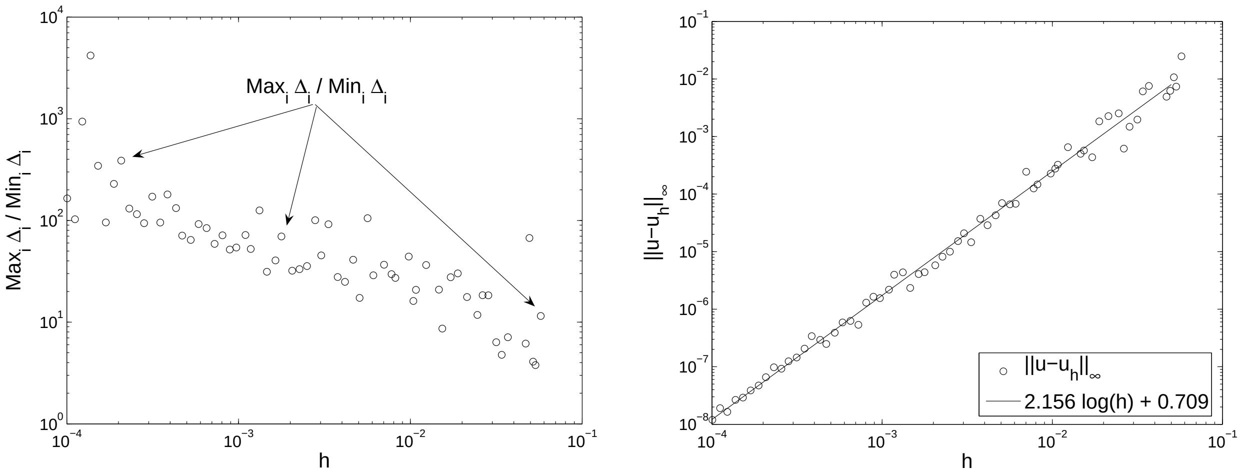

FIGURE 2. Second order problem : mesh size ratio for each computation and $\|u - u_h\|_\infty$ versus h

By replacing u_i by the exact value $u(x_i)$ and by assuming that the solution is regular enough in order to apply all the necessary Taylor's expansions, we get that the truncation error for this scheme is equal to

$$\epsilon_h^i = \frac{\Delta_i - \Delta_{i+1}}{3} u'''(x_i) + \mathcal{O}(h^2) \quad \forall i \in [1, N-1] \tag{6}$$

where $h = \max_{i=1,\ldots,N}(\Delta_i)$. Therefore, with a non-uniform grid, the local error is of order 1 and Lax-Richtmyer theorem implies that the scheme is at least first order accurate.

Now thanks to formula (5), let us compute the solution of equation (4) with the right hand side that corresponds to $u(x) = (1-x)(\arctan(10x-5) + \arctan(5))$ on a series of non-uniform grids with ratio of the extreme mesh size that lies between 5 and 4000. In Figure 2, this ratio for each computation is plotted as well as the l^∞ error versus the mesh size. The error clearly behaves like h^2. Therefore the truncation error and the Lax-Richtmyer theorem conclusion are sub-optimal.

Let us notice that the situation can be even more puzzling. The so-called first order methods can lead to a truncation error that does not go to zero. For instance, we consider the simple transport equation in one dimension

$$-u' + f = 0 \quad \text{on} \quad [0, L] \quad \text{with} \quad u(0) = 0 \tag{7}$$

for which the finite difference formula on a non-uniform grid can be with u_0 equal to the given data $u(0)$

$$-\frac{u_i - u_{i-1}}{\frac{1}{2}(\Delta_i + \Delta_{i+1})} + f(x_i) = 0 \quad \forall i \in [1, N] \,. \tag{8}$$

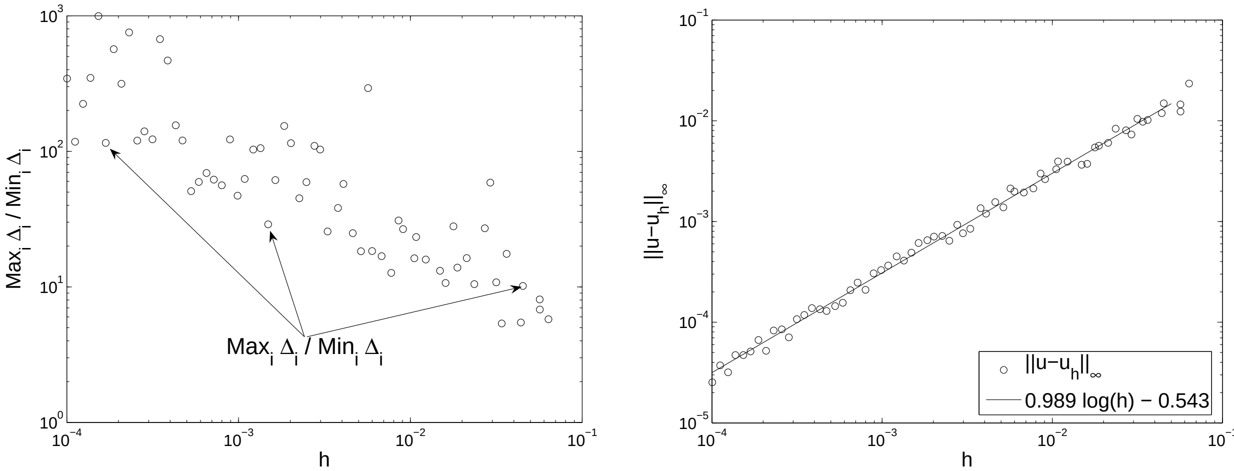

FIGURE 3. First order problem : mesh size ratio for each computation and $\|u - u_h\|_\infty$ versus h

Then the scheme is no longer consistent since the truncation error is now given by

$$\epsilon_h^i = \frac{\Delta_{i+1} - \Delta_i}{\Delta_{i+1} + \Delta_i} u'(x_i) + \mathcal{O}(h) \quad \forall i \in [1, N] \,. \tag{9}$$

However a numerical computation with $f(x) = 1$ leads to a l^∞ error of order one as it is observed on Figure 3.

1.2. Corrected analysis of the error

This phenomenon of supra-convergence discovered by Tikhonov and Samarskij [14] in the 60's was widely analyzed by Manteuffel and his co-authors in [7, 9–11] for ODE's and by Wendroff and White in [15–17] for PDE's and second order finite difference methods. If the truncation error for the finite difference scheme (1) can be rewritten in the special form

$$\epsilon_h = L_h(\gamma) + \underline{\epsilon} \tag{10}$$

where γ and $\underline{\epsilon}$ are of the optimal order $\mathcal{O}(h^p)$ and of course $L_h(\gamma) = \mathcal{O}(h^{p-1})$, then a correction of the global error by γ in the analysis leads to

$$L_h(e_h + \gamma) = -\underline{\epsilon} \,. \tag{11}$$

Now the Lax-Richtmyer analysis leads to $\|e_h + \gamma\| = \mathcal{O}(h^p)$ and finally to the optimal error estimate thanks to the assumed size of γ. This way to rewrite the truncation error in the mathematical analysis can be seen as a correction in order to cancel the leading part of the local error due to the non-uniformity of the mesh.

Let us go back to the simple diffusion equation in one dimension. If we define the second order size

$$\gamma^i = \sum_{j=1}^{i} \Delta_j^3 \frac{u'''(x_{j-1/2})}{6} \quad \forall i \in [1, N] \quad , \quad \gamma^0 = 0 \tag{12}$$

then by writing the difference with the original truncation error

$$\begin{aligned}
\underline{\epsilon}^i &= \epsilon_h^i - \left(-\frac{2\gamma^{i-1}}{\Delta_i(\Delta_i+\Delta_{i+1})} + \frac{2\gamma^i}{\Delta_i\Delta_{i+1}} - \frac{2\gamma^{i+1}}{\Delta_{i+1}(\Delta_i+\Delta_{i+1})}\right) = \epsilon_h^i + \frac{\dfrac{\gamma^{i+1}-\gamma^i}{\Delta_{i+1}} - \dfrac{\gamma^i-\gamma^{i-1}}{\Delta_i}}{\dfrac{1}{2}(\Delta_i+\Delta_{i+1})} \qquad (13)\\
&= \frac{\Delta_i^2(u'''(x_i)-u'''(x_{i-1/2})) - \Delta_{i+1}^2((u'''(x_i)-u'''(x_{i+1/2}))}{3(\Delta_i+\Delta_{i+1})} + \mathcal{O}(h^2)\,, \qquad \forall i \in [1, N-1]
\end{aligned}$$

we get the special form (10) with a corrected truncation error $\underline{\epsilon}$ of order 2 for a regular enough solution. Therefore it yields that the global error is indeed $\mathcal{O}(h^2)$.

Now for the transport equation, let us define

$$\gamma^i = -\frac{1}{2}\Delta_{i+1}u'(x_{i+1/2}) \qquad (14)$$

then we get in a similar way $\underline{\epsilon}^i$ and a first order scheme :

$$\underline{\epsilon}^i = \epsilon_h^i + \frac{\gamma^i - \gamma^{i-1}}{\frac{1}{2}(\Delta_i+\Delta_{i+1})} = \frac{\Delta_{i+1}(u'(x_i)-u'(x_{i+1/2})) + \Delta_i(u'(x_{i-1/2})-u'(x_i))}{\Delta_{i+1}+\Delta_i} + \mathcal{O}(h)\,. \qquad (15)$$

2. The finite volume case

We consider the scalar linear convection equation

$$\frac{\partial u}{\partial t} + (\mathbf{a}\cdot\nabla)u = 0 \text{ in } \Omega \quad \text{with} \quad u(x,0) = \phi(x) \text{ in } \Omega \quad \text{and} \quad u(x,t) = \psi(x,t) \text{ on } \partial\Omega^- \times [0,\infty[\qquad (16)$$

where $\mathbf{a}$ is a non-zero vector, Ω a bounded polygonal domain in $\mathbb{R}^{nd}$, $\partial\Omega^-$ the inflow boundary equal to $\{x \in \partial\Omega, \mathbf{a}\cdot\mathbf{n}(x) < 0\}$ with $\mathbf{n}$ the unit outward normal to Ω, ϕ and ψ given functions satisfying compatibility conditions. Let $\mathcal{T} = \{K_j :\ j = 1,\ldots,N\}$ be a partition of the domain Ω in polyhedral volumes K_j that forms a triangulation of Ω and such that the hyper-face between two adjacent volumes is included in an hyper-plane.

Let $j \in [1,N]$. If K_j has no hyper-face on the boundary $\partial\Omega$, we denote by $\mathcal{N}(j)$ the set of indices $k \in [1,N]$ such that $K_k \cap K_j$ has $(nd-1)$ positive measure. If the boundary of the volume K_j meets the boundary $\partial\Omega$, then we denote by $\mathcal{N}_0(j)$ the set of indices $k \in [1,N]$ such that $K_k \cap K_j$ has $(nd-1)$ positive measure and we complete this set into $\mathcal{N}(j)$ by negative integers numbering the hyper-faces of K_j which are on the boundary $\partial\Omega$. We denote the set of these negative integers by $\mathcal{N}_b(j)$. In both cases we have $\mathcal{N}(j) = \mathcal{N}_0(j) \cup \mathcal{N}_b(j)$, since $\mathcal{N}_b(j)$ is empty when K_j has no hyper-face on $\partial\Omega$.

Let $k \in \mathcal{N}(j)$. If $k \in \mathcal{N}_0(j)$, we denote by $\mathbf{n}_{j,k}$ the unit normal on $K_j \cap K_k$ which points out from K_j and by $\mathbf{N}_{j,k}$ the product $\mathbf{N}_{j,k} = |K_j \cap K_k|\mathbf{n}_{j,k}$ where $|K_j \cap K_k|$ denotes the $(nd-1)$ positive measure of $K_j \cap K_k$. If $k \in \mathcal{N}_b(j)$, we denote by K_k the symmetric of K_j with respect to the hyper-face $K_j \cap \partial\Omega$ and keep the same notations as above. We shall use in what follows the partition of $\mathcal{N}(j)$:

$$\mathcal{N}(j) = \mathcal{N}^+(j) \cup \mathcal{N}^-(j) \cup \mathcal{N}^0(j)\,, \qquad (17)$$

where $\mathcal{N}^+(j) = \{k \in \mathcal{N}(j), \mathbf{a}\cdot\mathbf{n}_{j,k} > 0\}$, $\mathcal{N}^-(j) = \{k \in \mathcal{N}(j), \mathbf{a}\cdot\mathbf{n}_{j,k} < 0\}$ and $\mathcal{N}^0(j) = \{k \in \mathcal{N}(j), \mathbf{a}\cdot\mathbf{n}_{j,k} = 0\}$. Similar definitions are extended to $\mathcal{N}_0^\varepsilon(j)$ and $\mathcal{N}_b^\varepsilon(j)$, $\varepsilon \in \{0,+,-\}$. Finally, let g_j be the centroid of K_j and $g_{j,k}$ be the centroid of $K_j \cap K_k$.

Since we are interested in convergence results, we consider families of triangulation $\mathcal{T}^h$ indexed by the number $h = \max_{K_j\in\mathcal{T}^h} h_j$ where h_j is the diameter of the volume K_j. By definition, we have $|K_j| \leq h^{nd}$ and

$|K_j \cap K_k| \leq h^{nd-1}$ for all $K_j, K_k \in \mathcal{T}^h$. We also assume that there exist $h_0 > 0$ and $\kappa_1, \kappa_2 > 0$ such that

$$\frac{h_j^{nd}}{|K_j|} \leq \kappa_1 \quad \text{and} \quad \sharp\mathcal{N}(j) \leq \kappa_2\,, \quad \forall K_j \in \mathcal{T}^h\,, \quad \forall h < h_0\,. \tag{18}$$

The standard upwind finite volume scheme takes into account the direction where the information comes from and consists in finding at each time step an approximation of the mean value of the exact solution on each control volume $u_j^n \approx \frac{1}{|K_j|}\int_{K_j} u(x,t^n)dx$ such that for all $j \in [1,N]$ and for all $n \geq 0$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t_n} + \frac{1}{|K_j|}\left(\sum_{k\in\mathcal{N}^+(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}u_j^n + \sum_{k\in\mathcal{N}_0^-(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}u_k^n + \sum_{k\in\mathcal{N}_b^-(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}\psi(g_{j,k},t^n)\right) = 0\,. \tag{19}$$

With $\mathcal{L}^n$ the operator from $\mathbb{R}^N$ to $\mathbb{R}^N$ defined by

$$\mathcal{L}_j^n\left(\{\xi_k\}_{k=1}^N\right) = \xi_j - \frac{\Delta t_n}{|K_j|}\left(\sum_{k\in\mathcal{N}^+(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}\xi_j + \sum_{k\in\mathcal{N}_0^-(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}\xi_k\right) \tag{20}$$

we can rewrite formula (19) in a more concise form ($\mathcal{L}_{bdy}^n$ takes into account the boundary terms)

$$\frac{1}{\Delta t_n}\left(u_j^{n+1} - \mathcal{L}_j^n(\{u_k^n\}_{k=1}^N)\right) + \mathcal{L}_{bdy}^n = 0\,. \tag{21}$$

In a way similar to the first section, the truncation error is obtained by replacing in this relation u_j^n by the value of the exact solution at the center of gravity $u(g_j,t^n)$ and it satisfies

$$\frac{1}{\Delta t_n}\left(u(g_j,t^{n+1}) - \mathcal{L}_j^n(\{u(g_k,t^n)\}_{k=1}^N)\right) + \mathcal{L}_{bdy}^n = \epsilon_j^n\,. \tag{22}$$

Therefore the global error $e_j^n = u_j^n - u(g_j,t^n)$ fulfills

$$\frac{1}{\Delta t_n}\left(e_j^{n+1} - \mathcal{L}_j^n(\{e_k^n\}_{k=1}^N)\right) = -\epsilon_j^n \tag{23}$$

and under the CFL condition $\Delta t_n \leq \min_j \tau_j$ with $\tau_j = |K_j|/\sum_{k\in\mathcal{N}^+(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}$, we get (see [1]) the following stability inequalities

$$\begin{cases} \|\mathcal{L}^n(\xi)\|_p \leq \|\xi\|_p \quad \forall\xi\in\mathbb{R}^N \qquad \text{with} \quad ||\xi||_p = \left(\displaystyle\sum_{j=1}^N |K_j||\xi_j|^p\right)^{1/p} \quad \text{and} \quad ||\xi||_\infty = \displaystyle\max_{1\leq j\leq N}|\xi_i| \\ \|e^n\|_p \leq ||e^0||_p + \displaystyle\sum_{i=0}^{n-1}\Delta t_i||\epsilon^i||_p \end{cases} \tag{24}$$

2.1. **Consistency of the finite volume scheme**

Now let us study the local error on the volume K_j. The error can be divided into two terms $\epsilon_j^n = G_j^n + I_j^n$. The first term concerns the centered part of the finite volume scheme and is of order 1 :

$$G_j^n = \frac{u(g_j,t^{n+1}) - u(g_j,t^n)}{\Delta t^n} + \frac{1}{|K_j|}\sum_{k\in\mathcal{N}(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}u(g_{j,k},t^n) \tag{25}$$

The second part is due to the upwinding nature of the method :

$$I_j^n = \frac{1}{|K_j|}\left(\sum_{k\in\mathcal{N}^+(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}\left(u(g_j,t^n)-u(g_{j,k},t^n)\right) + \sum_{k\in\mathcal{N}_0^-(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}\left(u(g_k,t^n)-u(g_{j,k},t^n)\right)\right) \tag{26}$$

and a Taylor's expansion only leads to a bounded term. Therefore in the terminology of the finite difference analysis the scheme is inconsistent :

2.2. Introduction of the geometric corrector

In [1], we introduce what we call a geometric corrector i.e. a sequence of vectors in $\mathbb{R}^{nd}$, $\{\mathbf{\Gamma}_j\}_{j=1}^N$ that depends only on the mesh and the vector $\mathbf{a}$ and that satisfies the following $N\times N$ system of vector equations :

$$\sum_{k\in\mathcal{N}^+(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}(\mathbf{\Gamma}_j - g_{j,k}+g_j) + \sum_{k\in\mathcal{N}_0^-(j)} \mathbf{a}\cdot\mathbf{N}_{j,k}(\mathbf{\Gamma}_k - g_{j,k}+g_k) \;=\; 0\,. \tag{27}$$

The role played by the natural boundary condition is an essential feature in the existence of this sequence of vectors. In a way similar to the first section, we propose to tackle the lack of consistency by correcting the error e^n with the corrector $\gamma^n=\{\gamma_j^n\}_{j=1}^N$ defined by $\gamma_j^n = -\mathbf{\Gamma}_j\cdot\nabla u(g_j,t^n)$. One gets $\underline{e}_j = e_j+\gamma_j$ so that

$$\frac{1}{\Delta t_n}\left(\underline{e}_j^{n+1} - \mathcal{L}_j^n(\underline{e}^n)\right) = -\underline{\epsilon}_j \tag{28}$$

where the corrected truncation error has the form

$$\underline{\epsilon}_j^n = -\frac{1}{\Delta t^n}\Big(\gamma_j^{n+1} - \mathcal{L}_j^n(\gamma^n)\Big) + \epsilon_j^n\,. \tag{29}$$

Let us assume the local quasi-uniformity of the mesh i.e. that there is a positive constant κ_3 such that

$$\frac{1}{\kappa_3}|K_k| \le |K_j| \le \kappa_3|K_k|\,,\quad \forall h<h_0,\quad \forall K_j\in\mathcal{T}^h,\quad \forall k\in\mathcal{N}(j)\,, \tag{30}$$

thus we obtain (see [1]) the following general result where the required smoothness is due to the use of second order Taylor expansions :

Theorem 2.1. *Let ϕ and ψ be arbitrary smooth functions satisfying compatibility conditions, u be the smooth solution to* (16). *Let assume that the discretizations of the boundary and the initial data satisfy the natural estimations $\|\{u_j^0-\varphi(g_j)\}_{j=1}^N\|_p \le \kappa_4 h$ and $|u_k^n - \psi(g_{j,k},t^n)| \le \kappa_5 h^2$ for all j, for all k in $\mathcal{N}_b^-(j)$ and for all t^n. For every $p\in[1,+\infty]$, if there exists C_p such that the geometric corrector $\mathbf{\Gamma}$ satisfies the estimate*

$$||\mathbf{\Gamma}||_p \le C_p h^\delta \tag{31}$$

for some $\delta\in]0,1]$ then the error for the upwind finite volume scheme satisfies the estimate

$$\|\{u_j^n - U_j^n\}_{j=1}^N\|_p \le C_p' h^\delta \qquad \textit{with} \qquad U_j^n = \frac{1}{|K_j|}\int_{K_j} u(x,t^n)dx \tag{32}$$

2.3. Study of the geometric corrector

The next step of the analysis consists in studying the geometric corrector. In [1], we proved the optimal estimate $||\mathbf{\Gamma}||_p \le C_p h$ for a given arbitrary and unstructured triangular conformal mesh globally refined by dividing each triangle of this mesh into 4 congruent triangles. We are going to study some other situations.

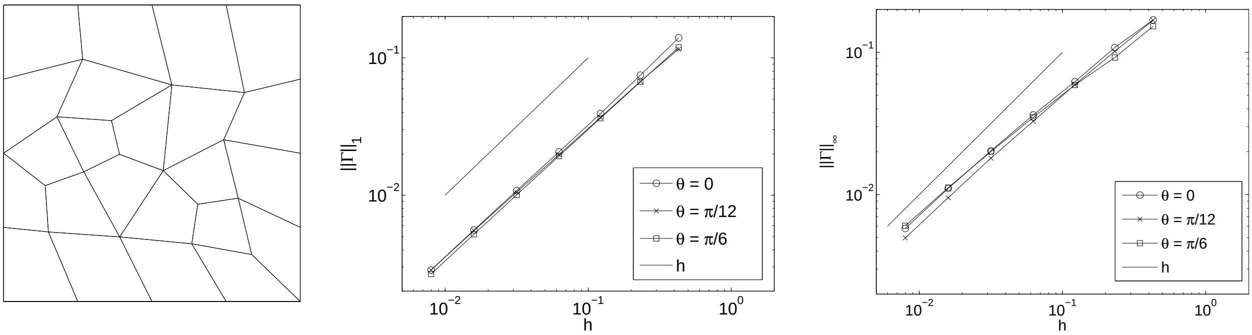

FIGURE 4. Coarse mesh and the estimates $\|\mathbf{\Gamma}\|_1$, $\|\mathbf{\Gamma}\|_\infty$ versus h

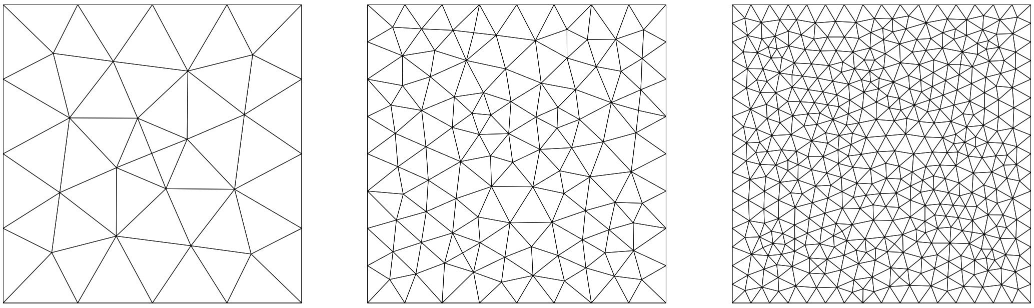

FIGURE 5. A series of 3 independent meshes

First let us consider a coarse mesh of convex quadrilaterals like the one displayed in Figure 4 and which is uniformly refined by dividing in four each quadrangle by joining mid-sides. Here the mesh size is divided by two at each refinement but contrary to the triangular element, the quality of the mesh slightly decreases with the level of refinement. Since we are able to compute exactly $\mathbf{\Gamma}$, numerical simulations allow us to plot in Figure 4 the l^1 and l^∞ norm of the corrector for 6 refinements and with three different convection vectors, one being parallel to the side of the domain. For both norm, we can observe the one order size of the corrector. From Theorem 2.1, the finite volume scheme is again first order accurate : a theoretical proof is available in [3]. The proof is in the broad outline identical to that given for the triangles. The main difference lies in the estimation of the correctors for one quadrangle divided in ℓ^2 sub-quadrangles. The number of configurations is larger and the situation is much more complex since sub-quadrangles Q with $\sharp\mathcal{N}^+(Q) = 1$, with $\sharp\mathcal{N}^+(Q) = 2$ and with $\sharp\mathcal{N}^+(Q) = 3$ depend on the shape of the quadrangle and the orientation of the convection vector.

Since the analysis we described previously is valid for arbitrary types of meshes, we perform some tests with a sequence of independent unstructured meshes where the mesh size decreases, in order to validate from a numerical point of view the estimate (31). With this end in view we consider a square with several meshes (some of them are plotted in Figure 5) composed from 48 triangles to 56528 triangles computed with the software Gmsh [6]. In the present case, if we take two consecutive grids, one is not the refinement of the other one (by dividing, for instance, each triangle into four congruent sub-triangles), but the mesh size is reduced.

In Figures 6 and 7 respectively for a convection vector with an angle of $\frac{\pi}{4}$ with the horizontal axes and for a horizontal convection vector, we present numerical results that lead us to conjecture that the first order estimate holds true in the case of independent refined meshes if the advection vector is not parallel to a side of the polygonal domain or to a fixed straight line in the domain. On the other hand, on the basis of numerical evidence, we conjecture that the best estimate should be in $h^{1/2}$ at least for the l^∞ norm. This behavior is similar to the loss of accuracy proved in Peterson [12]. More precisely, for the mesh and the subtle alignment

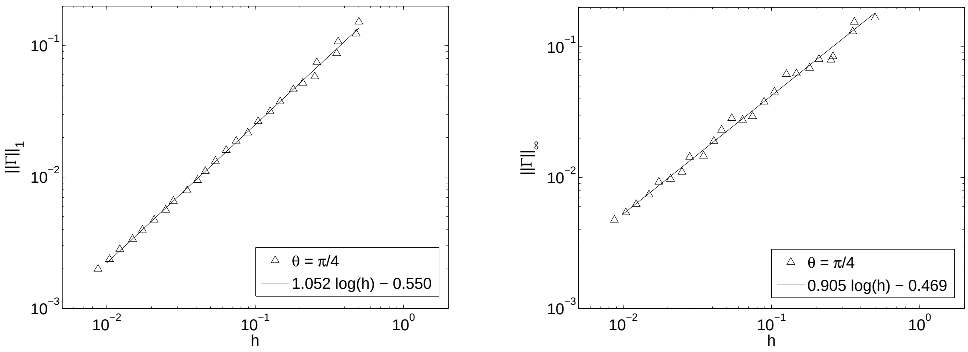

FIGURE 6. For $\mathbf{a} = (1, 1)$, $\|\mathbf{\Gamma}\|_1$ and $\|\mathbf{\Gamma}\|_\infty$ versus h

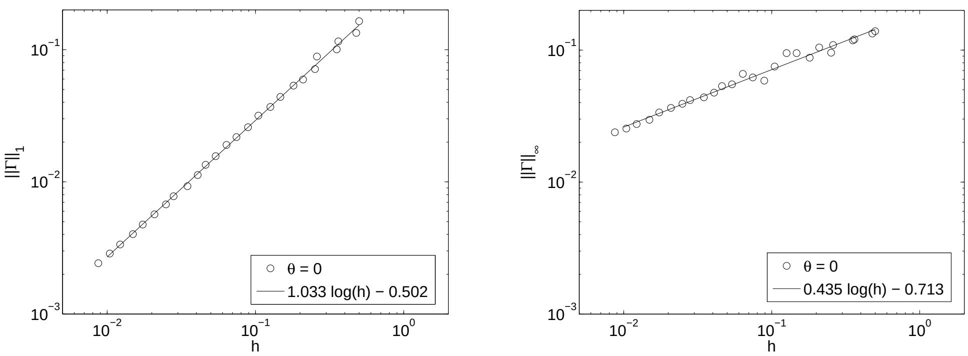

FIGURE 7. For $\mathbf{a} = (1, 0)$, $\|\mathbf{\Gamma}\|_1$ and $\|\mathbf{\Gamma}\|_\infty$ versus h

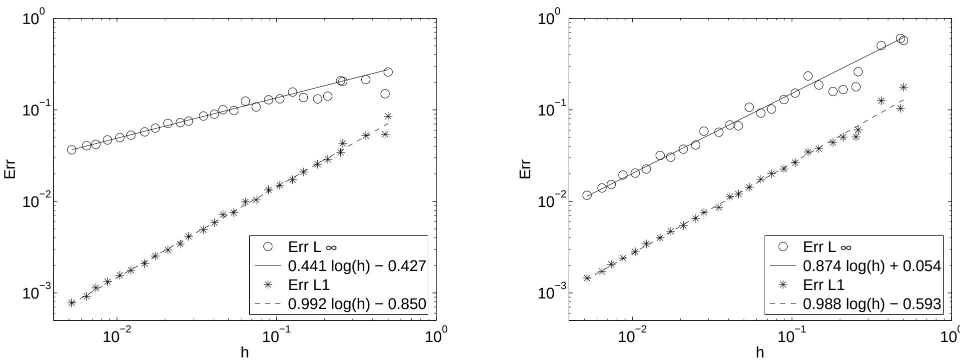

FIGURE 8. $\|u - u_h\|_1$ and $\|u - u_h\|_\infty$ versus h for $\mathbf{a} = (1, 0)$ and $\mathbf{a} = (1, 1)$

with the direction of transport proposed by Peterson, we proved in [2] that the l^∞ norm of the geometric corrector behaves like $\mathcal{O}(h^{1/2})$ and the l^1 norm is of order one. In Figure 8, we plot the error of the finite volume approximation of (16) at time $t = 2$ with the following boundary condition $\psi(x, y, t) = (x + y)^2$. We clearly observe that the error behaves like the corrector does.

Finally, we compute for the three dimensional domain consisting in $[0, 1]^3 \setminus [0, 0.5]^3$ several independent meshes from 84 tetrahedrons for the coarsest one to 256872 tetrahedrons for the finest one. We use four

(θ,φ)	$\|\mathbf{\Gamma}\|_1$	$\|\mathbf{\Gamma}\|_2$	$\|\mathbf{\Gamma}\|_\infty$
$(\frac{\pi}{4},\frac{\pi}{4})$	0.9662	0.9650	0.7808
$(\frac{\pi}{6},\frac{\pi}{6})$	0.9599	0.9540	0.7798
$(0,0)$	0.9846	0.9037	0.4587
$(\frac{\pi}{4},\frac{\pi}{2})$	0.9810	0.8986	0.4537

Table 1. 3-dimensional results : convergence rate estimated by data fitting

advection vectors of the form $\mathbf{a} = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi))$. In Table 1 the slopes computed by data fitting for the l^1, l^2 and l^∞ estimates of the geometric corrector versus mesh size are gathered. For the first values, it behaves like the mesh size h. In the last cases, the advection vectors are parallel to one of the faces of the cube and like in two dimension the slope is almost 1 for the l^1 norm, but only 1/2 for the l^∞ norm.

References

[1] D. Bouche, J.-M. Ghidaglia, and F. Pascal. Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation. *SIAM J. Numer. Anal.*, 43(2):578–603 (electronic), 2005.

[2] D. Bouche, J.-M. Ghidaglia, and F. Pascal. An optimal a priori error analysis of the finite volume method for linear convection problems. In F. Benkhaldoun, D. Ouazar, and S. Raghay, editors, *Finite volumes for complex applications IV, Problems and perspectives*, pages 225–236. Hermes Science publishing, London, 2005.

[3] D. Bouche, J.-M. Ghidaglia, and F. Pascal. Error estimate with the geometric corrector for the finite volume method : the quadrangle case. *Submitted*, 2006.

[4] B. Després. An explicit a priori estimate for a finite volume approximation of linear advection on non-cartesian grids. *SIAM J. Numer. Anal.*, 42(2):484–504, 2004.

[5] B. Després. Lax theorem and finite volume schemes. *Math. Comp.*, 73(247):1203–1234 (electronic), 2004.

[6] C. Geuzaine and J.-F. Remacle. Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities. http://www.geuz.org/gmsh/, 1997-2003.

[7] H.-O. Kreiss, T.A. Manteuffel, B. Swartz, B. Wendroff, and A. B. White, Jr. Supra-convergent schemes on irregular grids. *Math. Comput.*, 47:537–554, 1986.

[8] B. Merlet and J. Vovelle. Error estimate for the finite volume scheme applied to the advection equation. *http://hal.ccsd.cnrs.fr/ccsd-00008660*, 2005.

[9] T. A. Manteuffel and A. B. White, Jr. The numerical solution of second order boundary value problems on nonuniform meshes. *Math. Comput.*, 47:511–535, 1986.

[10] T. A. Manteuffel and A. B. White, Jr. On the efficient numerical solution of systems of second order boundary value problems. *SIAM J. Numer. Anal.*, 23:996–1006, 1986.

[11] T. A. Manteuffel and A. B. White, Jr. A calculus of difference schemes for the solution of boundary-value problems on irregular meshes. *SIAM J. Numer. Anal.*, 29(5):1321–1346, 1992.

[12] T. E. Peterson. A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation. *SIAM J. Numer. Anal.*, 28(1):133–140, 1991.

[13] J. Strikwerda. *Finite difference schemes and partial differential equations. 2nd ed.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). xii, 435 p., 2004.

[14] A.N. Tikhonov and A.A. Samarskij. Homogeneous difference schemes on non-uniform nets. *U.S.S.R. Comput. Math. Math. Phys.*, 1963:927–953, 1964.

[15] B. Wendroff. Supraconvergence in two dimensions. Technical Report LA-UR 95-3068, Los Alamos National Laboratory, 1995.

[16] B. Wendroff and A. B. White, Jr. Some supraconvergent schemes for hyperbolic equations on irregular grids. In *Nonlinear hyperbolic equations—theory, computation methods, and applications (Aachen, 1988)*, volume 24 of *Notes Numer. Fluid Mech.*, pages 671–677. Vieweg, Braunschweig, 1989.

[17] B. Wendroff and A. B. White, Jr. A supraconvergent scheme for nonlinear hyperbolic systems. *Comput. Math. Appl.*, 18(8):761–767, 1989.