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THE FILTERING PROBLEM: AN APPLICATION OF WEAK APPROXIMATIONS OF SDES

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Abstract. We present here an alternative view of the continuous time filtering problem, namely the problem is considered as a special case within the theory of weak approximations of stochastic differential equations (SDEs). The class of algorithms arising from this new perspective on the filtering problem estimate the conditional distribution of the signal by first employing an approximation result due to Picard[17] and then weakly approximating the resulting SDE. As a specific example, Lyons-Victoir cubature on Wiener space is presented. The main characteristics of these algorithms along with a convergence result for the entire class are briefly discussed.

1. Statement of the filtering problem

The continuous time filtering problem involves two components: a signal and an associated observation process. For many filtering problems, a natural mathematical model for the signal is a continuous time Markov process,

$$X = \left\{ \left(X_t^i\right)_{i=1}^k, t \geq 0 \right\}$$

that satisfies a stochastic differential equation of the form,

$$X_t = X_0 + \int_0^t V_0(X_s)ds + \sum_{j=1}^k \int_0^t V_j(X_s) \circ dW_s^j, \tag{1}$$

where W is a k-dimensional Wiener process and vector fields $\{V_i\}_{i=1}^k \in C_b^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right)$. The observation is modelled by a stochastic process,

$$Y = \left\{ \left(Y_t^i\right)_{i=1}^l, t \geq 0 \right\}$$

satisfying an evolution equation of the form,

$$Y_t = \int_0^t h(X_s)ds + B_t$$

where B is an l-dimensional Brownian motion independent of X and $h = \left(h^i\right)_{i=1}^l \in C_b^\infty\left(\mathbb{R}^d, \mathbb{R}^l\right)$. In the Let $(\mathcal{Y}_t)_{t\geq 0}$ be the filtration generated by the observation process Y,

$$\mathcal{Y}_t = \sigma\left(Y_s, \ 0 \leq s \leq t\right).$$

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2. THE APPROXIMATION

Within the continuous time framework, the filtering problem for (X, Y) involves the construction of $\pi_t(\varphi)$, where $\pi = \{\pi_t, t \geq 0\}$ is the conditional distribution of X_t given $\mathcal{Y}_t$ and φ belongs to a suitably large class of functions. If φ is square integrable with respect to the law of X_t then,

$$\pi_t(\varphi) = \mathbb{E}\left[\varphi(X_t) | \mathcal{Y}_t\right], \ \mathbb{P} - \text{almost surely}.$$

Using Girsanov's theorem, one can find a new probability measure $\tilde{\mathbb{P}}$ absolutely continuous with respect to $\mathbb{P}$ (and vice versa), so that Y is a Brownian motion under $\tilde{\mathbb{P}}$, independent of X and, almost surely,

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}, \tag{2}$$

where,

$$\rho_t(\varphi) = \tilde{\mathbb{E}}\left[\varphi(X_t)\exp\left(\sum_{i=1}^{l}\int_0^t h^i(X_s)dY_s^i - \frac{1}{2}\sum_{i=1}^{l}\int_0^t h^i(X_s)^2 ds\right)\middle| \mathcal{Y}_t\right] \tag{3}$$

and $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. The measure ρ_t is called the *unnormalised conditional distribution* of the signal. The identity (2) is called the Kallianpur-Striebel Formula.

2.1. DISCRETISATION OF OBSERVATION INPUT & APPLICATION OF PICARD's THEOREM

In the following, we will denote by $||\cdot||_p$, the L^p-norm with respect to the probability measure $\tilde{\mathbb{P}}$, $|\xi|_p = \tilde{\mathbb{E}}\left[|\xi|^p\right]^{\frac{1}{p}}$, for any random variable ξ. The laws of the families $X(x) = \{X_t(x)\}_{t\in[0,\infty)}$, $x \in \mathbb{R}^d$ and $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,\infty)}$, $x \in \mathbb{R}^d$ are not affected by the change of measure, hence, to avoid working with both $\mathbb{P}$ and $\tilde{\mathbb{P}}$ we can write,

$$(P_t\varphi)(x) = \tilde{\mathbb{E}}[\varphi(X_t(x))], \ \ (Q_t\varphi)(x) = \tilde{\mathbb{E}}[\varphi(\bar{X}_t(x))].$$

In the following, we will only consider equidistant partitions and smooth functions. The method of approximation and the results closely follow the application of the classical Euler method as described in Picard [17] and Talay [18].

Let y_r, $r = 1, ..., n$ be the observation process increments $y_r = Y_{\frac{(r+1)t}{n}} - Y_{\frac{rt}{n}}$ and $h_r \in C_b^\infty\left(\mathbb{R}^d\right)$, $r = 0, ..., n-1$, be the (observation dependent) functions defined by $h_r = \sum_{i=1}^{l}(h^i y_r^i - \frac{t}{2n}\left(h^i\right)^2)$. Let R_s^r, $\bar{R}_s^r : C_b^\infty\left(\mathbb{R}^d\right) \to C_b^\infty\left(\mathbb{R}^d\right)$, $r = 0, 1, ..., n$ be the following operators,

$$R_s^n\varphi(x) = P_s\varphi(x), \ \bar{R}_s^n\varphi(x) = Q_s\varphi(x) \ \text{ for } \ \varphi \in C_b^\infty\left(\mathbb{R}^d\right), \ \ x \in \mathbb{R}^d$$

and, for $r = 0, 1, ..., n-1$, and for $\varphi \in C_b^\infty\left(\mathbb{R}^d\right)$, $x \in \mathbb{R}^d$,

$$\begin{aligned} R_s^r\varphi(x) &= \tilde{\mathbb{E}}\left[\varphi(X_s(x))\exp\left(h_r(X_s(x))\right)| \mathcal{Y}_s\right] = P_s\varphi^r(x) \\ \bar{R}_s^r\varphi(x) &= \tilde{\mathbb{E}}\left[\varphi\left(\bar{X}_s(x)\right)\exp\left(h_r\left(\bar{X}_s(x)\right)\right)\middle| \mathcal{Y}_s\right] = Q_s\varphi^r(x), \end{aligned}$$

where $\varphi^r = \varphi\exp(h_r)$ and $s \in [0, 1]$.

Firstly, one approximates ρ by replacing the (continuous) observation path with a discrete version. We choose the equidistant partition $\left\{\frac{it}{n}, i = 0, 1, ...n\right\}$ of the interval $[0, t]$ and consider only the observation data

$\{y_r,\, r = 0, 1, ..., n\}$. We define the measure,

$$\begin{aligned} \rho_t^n(\varphi) &= \tilde{\mathbb{E}}\left[\varphi(X_t)\exp\left(\sum_{i=0}^{n-1} h_i\left(X_{\frac{it}{n}}\right)\right)\middle|\, \mathcal{Y}_t\right] \\ &= \tilde{\mathbb{E}}\left[R^0_{\frac{t}{n}} R^1_{\frac{t}{n}} ... R^n_{\frac{t}{n}} \varphi(X_0)\middle|\, \mathcal{Y}_t\right] \quad \text{for} \quad \varphi \in C_b\left(\mathbb{R}^d\right). \end{aligned} \tag{4}$$

Following Theorem 1 from Picard [17], for any $\varphi \in C_b^\infty(\mathbb{R}^d)$ there is a constant $c \equiv c(t, \varphi)$ such that,

$$||\rho_t(\varphi) - \rho_t^n(\varphi)||_2 \leq \frac{c}{n}$$

2.2. **WEAKLY APPROXIMATING THE RESULTING SDE**

The next step is to approximate $\prod_{i=0}^n R^i_{\frac{t}{n}}$ with $\prod_{i=0}^n \bar{R}^i_{\frac{t}{n}}$. For this we need to define an $m-$perfect family and consequently we may use functions which are parametrized by the observation path Y.

Let $C_b^{Y,\infty}(\mathbb{R}^d)$ be the set of measurable functions, $f : \mathbb{R}^d \times C\left([0,T], \mathbb{R}^l\right) \to \mathbb{R}$ with the following properties:

i. for any $y \in C\left([0,T], \mathbb{R}^l\right)$ the function $x \to f(x, y)$ belongs to $C_b^\infty(\mathbb{R}^d)$.
ii. for any multi-index $\alpha \in \mathcal{A}$, any $x \in \mathbb{R}^d$ and $p \geq 1$, $|D_\alpha f(x, Y)|_p < \infty$.
iii. for any multi-index $\alpha \in \mathcal{A}$ and $p \geq 1$, $|||D_\alpha f|||_{p,\infty} = \sup_{x\in\mathbb{R}^d} |D_\alpha f(x, Y)|_p < \infty$.

For $f \in C_b^{Y,\infty}(\mathbb{R}^d)$ we define the norm $|||f|||_p^m = \sum\limits_{\alpha\in\mathcal{A}(m)} |||D_\alpha f|||_{p,\infty}$. We note that if $f : \mathbb{R}^d \times C\left([0,T], \mathbb{R}^l\right) \to \mathbb{R}$ is constant in the y variable, then $||D_\alpha f(x, Y)||_p = |D_\alpha f(x, Y)|$ and $|||f|||_p^m = ||f||_\infty + \left\|\nabla^1 f\right\|_\infty + ... \|\nabla^m f\|_\infty$.

Then in the filtering context, for any $m \in \mathbb{N}$, the family $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,\infty)}$ where $x \in \mathbb{R}^d$, is said to be *m-perfect* for the process X if for any $f \in C_b^{Y,\infty}(\mathbb{R}^d)$,

$$\left|\left|\left|Q_t f - \tilde{\mathbb{E}}[f_t^m|Y]\right|\right|\right|_{p,\infty} \leq C \sum_{i=m+1}^{M} t^{i/2}\, |||f|||_p^i, \tag{5}$$

for some constants $C > 0$ and $M \geq m + 1$, where f_t^m is the truncation,

$$f_t^m(x) := \varphi(x) + \sum_{\alpha\in\mathcal{A}_0(m)} f_{\alpha,\varphi}(x) I_\alpha(t)$$

and

$$\mathcal{A}_0(m) = \{\alpha \in \mathcal{A}_0 : \|\alpha\| \leq m\}$$

where $\mathcal{A}_0 = \mathcal{A}\backslash\{\emptyset\}$ and $\mathcal{A}$ is the set of multi-indices,

$$\mathcal{A} = \{\emptyset\} \cup \bigcup_{m=1}^{\infty} \{0, 1, \ldots, k\}^m$$

with k being the dimension of the Brownian Motion introduced above and we have the norm $\|\cdot\|$ defined on $\mathcal{A}$ by $\|\alpha\| = |\alpha| + card\{1 \leq j \leq |\alpha| : \; i_j = 0\}$ where,

$$|\emptyset| = 0, \; |\alpha| = r \; if \; \alpha = (i_1, \ldots, i_r) \in \{0, 1, \ldots, k\}^r \; \text{for } r \in \mathbb{N}$$

The notation $\mathcal{A}_0(m)$ is due to Kusuoka[6].

Finally let us define now the measures,

$$\bar{\rho}_t^n(\varphi) = \tilde{\mathbb{E}}\left[\bar{R}^0_{\frac{t}{n}}\bar{R}^1_{\frac{t}{n}}...\bar{R}^n_{\frac{t}{n}}\varphi(X_0)\Big|\mathcal{Y}_t\right] \text{ for } \varphi \in C_b\left(\mathbb{R}^d\right).$$

and let $\bar{\pi}_t^n = \bar{\rho}_t^n / \bar{\rho}_t^n(1)$ be its normalized version.

Theorem 1. *Let $\bar{X}(x)$ be an m-perfect family that satisfies (5). Then there is a constant $c_1 \equiv c_1(t,m,p) > 0$ such that for any $\varphi \in C_b^\infty(\mathbb{R}^d)$ we have,*

$$||\rho_t^n(\varphi) - \bar{\rho}_t^n(\varphi)||_p \le c_1 n^{-(m-1)/2}||\varphi||_M.$$

Corollary 2. *Let $\bar{X}(x)$ be an m-perfect family that satisfies (5) with $m \ge 3$ and assume that X_0 has all moments finite. Then for any $\varphi \in C_b^\infty(\mathbb{R}^d)$ there is a constant $c_2 \equiv c_2(t,m,p,\varphi) > 0$ such that,*

$$||\pi_t^n(\varphi) - \bar{\pi}_t^n(\varphi)||_p \le \frac{c_2(\varphi)}{n}.$$

3. EXAMPLE: LYONS-VICTOIR CUBATURE ON WIENER SPACE

In the following example, the family of processes $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,1]}$, where $x \in \mathbb{R}^d$, corresponds to the Lyons-Victoir approximation (see [13]) and is m-perfect for any $m \in \mathbb{N}$. The example involves a set of l finite variation paths, $\omega_1,\ldots,\omega_l \in C_0^0([0,1],\mathbb{R}^k)$, for some $l \in \mathbb{N}$, together with some weights $\lambda_1,\ldots,\lambda_l \in \mathbb{R}^+$ such that $\sum_{j=1}^{l}\lambda_j = 1$. These paths are said to define a *cubature formula on Wiener Space of degree m* if, for any $\alpha \in \mathcal{A}_0(m)$,

$$\mathbb{E}[I_\alpha(1)] = \sum_{j=1}^{l}\lambda_j I_\alpha^{\omega_j}(1)$$

where,

$$I^{\omega_j}_{(i_1,\ldots,i_r)}(1) := \int_0^1\int_0^{s_0}\cdots(\int_0^{s_{r-2}} d\omega_j^{i_1}(s_{r-1}))\cdots d\omega_j^{i_{r-1}}(s_1)d\omega_j^{i_r}(s_0).$$

From the scaling properties of the Brownian motion we can deduce, for $t \ge 0$,

$$\mathbb{E}[I_\alpha(t)] = \sum_{j=1}^{l}\lambda_j I_\alpha^{\omega_{t,j}}(t)$$

where $\omega_{t,1},\ldots,\omega_{t,l} \in C_0^0([0,t],\mathbb{R}^k)$ is defined by $\omega_{t,j}(s) = \sqrt{t}\omega_j\left(\frac{s}{t}\right),\ s \in [0,t]$. In other words, the expectation of the iterated Stratonovich integrals $I_\alpha(t)$ defined,

$$I_{(i_1,\ldots,i_r)}(t) := \int_0^t\int_0^{s_0}\cdots(\int_0^{s_{r-2}} 1\circ dW^{i_1}_{s_{r-1}})\circ\cdots\circ dW^{i_{r-1}}_{s_1}\circ dW^{i_r}_{s_0}.$$

is the same under the Wiener measure as it is under the measure,

$$\mathbb{Q}_t := \sum_{j=1}^{l}\lambda_j\delta_{\omega_{t,j}}.$$

If we choose $\bar{X}$ to satisfy the evolution equation (1) but with the driving Brownian motion replaced by the paths $\omega_{t,1},\ldots,\omega_{t,l}$ defined above then the family of processes, $\{\overline{X}_t(x)\}_{t\in[0,1]}$, with corresponding operator

$(\overline{Q}_t\varphi)(x) := \mathbb{E}_{\mathbb{Q}_t}[\varphi(\overline{X}_t(x))]$, is $m-$perfect. More precisely, there exists a constant $c_6 > 0$ such that for $\varphi \in C_b^{V,m+2}(\mathbb{R}^d)$,

$$\sup_x \left|\overline{Q}_t\varphi(x) - \mathbb{E}[\varphi_t^m(x)]\right| \leq c_3 \sum_{i=m+1}^{m+2} t^{i/2} \left\|\varphi\right\|_{V,i}$$

As a particular example, if $(\lambda_j, \omega_{t,j})$ are chosen such that for $l = 2^k$ the paths are $\omega_{t,j} : t \mapsto t(1, z_j^1, .., z_j^k)$ for $j = 1, \ldots, 2^k$ with points $z_j \in \{-1, 1\}^k$ and weights $\lambda_j = 2^{-k}$, we obtain a cubature formula of degree 3 and a corresponding 3-perfect family.

4. THE GENERAL THEORY

We begin giving the general definition of an m-perfect family. In the following, we define a class of approximations of X expressed in terms of certain families of stochastic processes, $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,\infty)}$ for $x \in \mathbb{R}^d$, which are explicitly solvable. In particular, we can explicitly compute the operator,

$$(Q_t\varphi)(x) = \mathbb{E}[\varphi(\bar{X}_t(x))]. \tag{6}$$

The semigroup $P_T = \mathbb{E}[\varphi(X_T(x))]$ will then be approximated by $Q_{h_n}^m Q_{h_{n-1}}^m \ldots Q_{h_1}^m$ where $\{h_j := t_j - t_{j-1}\}_{j=1}^n$ and $\pi_n = \{t_j := (\frac{j}{n})^\gamma T\}_{j=0}^n$ for $n \in \mathbb{N}$, is a sufficiently fine partition of the interval $[0, T]$. In particular $h_j \in [0, 1)$ for $j = 1, ..., n$.

So let $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,\infty)}$, where $x \in \mathbb{R}^d$, be a family of progressively measurable stochastic processes such that, $\lim_{y\to x_0} \bar{X}_{t_0}(y) = \bar{X}_{t_0}(x_0)$ $\mathbb{P}-$almost surely, for any $t_0 \geq 0$ and $x_0 \in \mathbb{R}^d$. As a result, the operator Q_t defined in (6) has the property that $Q_t\varphi \in C_b(\mathbb{R}^d)$ for any $\varphi \in C_b(\mathbb{R}^d)$. In particular, $Q_t : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is a Markov operator.

Definition 3. *For $m \in \mathbb{N}$, the family $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,\infty)}$ where $x \in \mathbb{R}^d$, is said to be **m-perfect** for the process X if there exist constants $C > 0$ and $M \geq m + 1$ such that for $\varphi \in C_b^{V,M}(\mathbb{R}^d)$,*

$$\sup_{x\in\mathbb{R}^d} |Q_t\varphi(x) - \mathbb{E}[\varphi_t^m(x)]| \leq C \sum_{i=m+1}^{M} t^{i/2} \left\|\varphi\right\|_{V,i}. \tag{7}$$

by defining

Let us define the function,

$$\Upsilon^p(n) = \begin{cases} n^{-\frac{1}{2}\min(\gamma p,(m-1))} & \text{if } \gamma p \neq m-1 \\ n^{-(m-1)/2}\ln n & \text{for } \gamma p = m-1 \end{cases}$$

In the following,

$$\mathcal{E}^{\gamma,n}(\varphi) := \left\|P_T\varphi - Q_{h_n}^m Q_{h_{n-1}}^m \ldots Q_{h_1}^m \varphi\right\|_\infty$$

for $\gamma \in \mathbb{R}$, $n \in \mathbb{N}$.

Theorem 4. *Let $T, \gamma > 0$ and $\pi_n = \{t_j = (\frac{j}{n})^\gamma T\}_{j=0}^n$ be a partition of the interval $[0, T]$ where $n \in \mathbb{N}$ is such that $\{h_j = t_j - t_{j-1}\}_{j=1}^n \subseteq (0, 1]$. Then for any m-perfect family $\{\overline{X}_t(x)\}_{t\in[0,T]}$ with corresponding operator $Q = \{Q_t\}_{t\in(0,1]}$ we have, for $\varphi \in C_b^p(\mathbb{R}^d)$ where $p = 1, ..., m$,*

$$\mathcal{E}^{\gamma,n}(\varphi) \leq c_4\Upsilon^p(n)\left\|\varphi\right\|_p + \left\|P_{h_1}\varphi - Q_{h_1}^m\varphi\right\|_\infty \tag{8}$$

for some constant $c_4 > 0$*. In particular, if* $\gamma \geq \frac{m-1}{p}$ *then,*

$$\mathcal{E}^{\gamma,n}(\varphi) \leq \frac{c_4}{n^{\frac{m-1}{2}}} \|\varphi\|_p + \left\|P_{h_1}\varphi - Q^m_{h_1}\varphi\right\|_\infty$$

We observe that the rate of convergence is the controlled by the maximum between $\Upsilon(n)$ and the rate at which $\left\|P_{h_1}\varphi - Q^m_{h_1}\varphi\right\|_\infty$ convergences to 0. We define $\bar{\Upsilon}^{k_1,k_2}(n) = \Upsilon^{k_1}(n) + n^{-\frac{\gamma k_2}{2}}$. Hence we have the following corollary:

Corollary 5.
(i) For any $\varphi \in C^M_b(\mathbb{R}^d)$*,*

$$\mathcal{E}^{\gamma,n}(\varphi) \leq c_5 \bar{\Upsilon}^{m+1,m+1}(n) \|\varphi\|_M .$$

for some constant $c_5 > 0$*. In particular, if* $\gamma \geq 1$*, then* $\mathcal{E}^{\gamma,n}(\varphi) \leq \frac{c_5}{n^{\frac{m-1}{2}}} \|\varphi\|_M$*.*
(ii) For any $\varphi \in C^1_b(\mathbb{R}^d)$*,*

$$\mathcal{E}^{\gamma,n}(\varphi) \leq c_7 \bar{\Upsilon}^{1,1}(n) \|\varphi\|_1$$

for some constant $c_7 > 0$*, if there exists a constant* $c_6 > 0$ *independent of* t *such that,*

$$\sup_{x\in\mathbb{R}^d} \left|\bar{X}_t(x) - x\right| \leq c_6\sqrt{t}. \tag{9}$$

In particular, if $\gamma \geq m-1$*, then* $\mathcal{E}^{\gamma,n}(\varphi) \leq \frac{c_7}{n^{\frac{m-1}{2}}} \|\varphi\|_1$*.*
(iii) For any $\varphi \in C^l_b(\mathbb{R}^d)$ *where* $1 < l < M$*,*

$$\mathcal{E}^{\gamma,n}(\varphi) \leq c_{10} \bar{\Upsilon}^{l,c_9}(n) \|\varphi\|_l$$

for some constant $c_{10} > 0$*, if there exist two constants* $c_8, c_9 > 0$ *independent of* t *such that,*

$$\|P_t\varphi - Q^m_t\varphi\|_\infty \leq c_8 t^{\frac{c_9}{2}} \|\varphi\|_l . \tag{10}$$

In particular, if $\gamma \geq m-1$*, then* $\mathcal{E}^{\gamma,n}(\varphi) \leq \frac{c_{10}}{n^{\frac{m-1}{2}}} \|\varphi\|_l$ *.*

Remark We deduce that there is a payoff between the rate of convergence and the coarseness of the norm employed: the finer the norm the slower the rate of convergence. Hence intermediate results such as part *(iii)* of Corollaries 5 may prove useful in subsequent applications. The additional constraint (10) holds, for example, for the Lyons-Victoir method, as a cubature formula of degree m is also a cubature formula of degree m' for $m' \leq m$.

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