

## ON STABLE DETERMINATION OF POTENTIAL BY BOUNDARY MEASUREMENTS

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**Abstract.** We give new stability estimates for the Gel'fand-Calderon inverse boundary value problem.

### 1. INTRODUCTION

Consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in D, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \partial D \in C^2, \quad v \in L^\infty(D), \quad d \geq 2. \quad (1.2)$$

We assume also that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \quad (1.3)$$

Equation (1.1) arises, in particular, in quantum mechanics, acoustics, electrodynamics. Formally, (1.1) looks as the Schrödinger equation with potential  $v$  at zero energy.

Consider the map  $\Phi$  such that

$$\frac{\partial\psi}{\partial\nu}\Big|_{\partial D} = \Phi(\psi|_{\partial D}) \quad (1.4)$$

for all sufficiently regular solutions  $\psi$  of (1.1) in  $\bar{D} = D \cup \partial D$ , where  $\nu$  is the outward normal to  $\partial D$ . The map  $\Phi$  is called the Dirichlet-to-Neumann map for equation (1.1) and is considered as boundary measurements for (1.1).

We consider the following inverse boundary value problem for equation (1.1):

**Problem 1.1.** Given  $\Phi$ , find  $v$ .

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [8], [14]). This problem can be also considered as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [19], [14]). Concerning results given in the literature on Problem 1.1 (in its Calderon or Gel'fand form) see [10], [19], [9] (note added in proof), [14], [1], [12], [13], [4], [18], [11], [15], [17], [2] and references therein.

In the present article we show that the Alessandrini stability estimates of [1] for Problem 1.1 in dimension  $d \geq 3$  (see Theorem 2.1 of the next section) admit some principle improvement. Our new stability estimates (see Theorem 2.2 of the next section) are obtained by methods developed in [15], [16], [17]. These methods

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include, in particular: (1) the  $\bar{\partial}$ - approach to inverse "scattering" at zero energy in dimension  $d \geq 3$ , going back to [3], [9], and (2) the reduction of Problem 1.1 to inverse "scattering" at zero energy, going back to [14].

The present article is organized as follows. In Section 2, we formulate and discuss old and new stability estimate for Problem 1.1. In Section 3, we recall (a) definition and some properties of the Faddeev functions and (b) formulation of the inverse "scattering" problem for the Schrödinger equation at zero energy (Problem 3.1). In Section 4, we recall formulas and equations of [14], [15] reducing Problem 1.1 to Problem 3.1. In Section 5, we recall an approximate reconstruction method of [14] for Problem 1.1. In Section 6 we prove Theorem 2.2 in the Born approximation.

## 2. STABILITY ESTIMATES

We assume for simplicity that

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \partial D \in C^2, v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \text{supp } v \subset D, d \geq 2, \quad (2.1)$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \partial^J v \in L^1(\mathbb{R}^d), |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \quad (2.2)$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^d J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}.$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \quad (2.3)$$

Let

$$\|A\| \text{ denote the norm of an operator } A : L^\infty(\partial D) \rightarrow L^\infty(\partial D). \quad (2.4)$$

We recall that if  $v_1, v_2$  are potentials satisfying (1.2), (1.3), where  $D$  is fixed, then

$$\Phi_1 - \Phi_2 \text{ is a compact operator in } L^\infty(\partial D), \quad (2.5)$$

where  $\Phi_1, \Phi_2$  are the DtN maps for  $v_1, v_2$  respectively, see [14], [15]. Note also that (2.1)  $\Rightarrow$  (1.2).

**Theorem 2.1** (variation of the result of [1]). *Let conditions (1.3), (2.1) hold for potentials  $v_1$  and  $v_2$ , where  $D$  is fixed,  $d \geq 3$ . Let  $\|v_j\|_{m,1} \leq R, j = 1, 2$ , for some  $R > 0$ . Let  $\Phi_1, \Phi_2$  denote the DtN maps for  $v_1, v_2$ , respectively. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq C_1(\ln(1 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_1}, \quad (2.6)$$

where  $C_1 = C_1(R, D, m), \alpha_1 = (m - d)/m, \|\Phi_1 - \Phi_2\|$  is defined according to (2.4).

Theorem 2.1 follows from formulas (3.9)-(3.11), (4.1) (of Sections 3 and 4).

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \text{ for any } m > d \text{ even if } m \text{ is very great.} \quad (2.7)$$

**Theorem 2.2.** *Let the assumptions of Theorem 2.1 hold. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq C_2(\ln(1 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_2}, \quad (2.8)$$

where  $C_2 = C_2(R, D, m), \alpha_2 = m - d, \|\Phi_1 - \Phi_2\|$  is defined according to (2.4).

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \rightarrow +\infty \text{ as } m \rightarrow +\infty, \quad (2.9)$$

in contrast with (2.7).

In the Born approximation, that is in the linear approximation near zero potential, Theorem 2.2 is proved in Section 6.

For sufficiently small  $R$  in dimension  $d = 3$ , Theorem 2.2 follows from (3.9) (of Section 3) and results of [15], [17]. The scheme of our proof for this case is, actually, similar to the scheme of our proof for the case of the Born approximation. The main difference is that instead of the inverse Fourier transform (used in Section 6) we use now the zero-energy inverse "backscattering" transform of [17]. We plan to give this "nonlinear" "small-norm" proof in a separate article.

In the general case, the proof of Theorem 2.2 is not completed yet. However, except restrictions in time, we see no difficulties for completing this proof by methods of [15], [16], [17].

### 3. FADDEEV FUNCTIONS

We consider the Faddeev functions  $G$ ,  $\psi$  and  $h$  (see [6], [7], [9], [15]):

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (3.1)$$

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \quad (3.2)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \Sigma$ ,

$$\Sigma = \{k \in \mathbb{C}^d, \quad k^2 = k_1^2 + \dots + k_d^2 = 0\}; \quad (3.3)$$

$$h(k, l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (3.4)$$

where  $(k, l) \in \Theta$ ,

$$\Theta = \{k \in \Sigma, \quad l \in \Sigma : \operatorname{Im} k = \operatorname{Im} l\}. \quad (3.5)$$

We recall that:

$$\Delta G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \Sigma; \quad (3.6)$$

formula (3.2) at fixed  $k$  is considered as an equation for

$$\psi = e^{ikx} \mu(x, k), \quad (3.7)$$

where  $\mu$  is sought in  $L^\infty(\mathbb{R}^d)$ ; as a corollary of (3.2),  $\psi$  satisfies (1.1);  $h$  of (3.4) is a generalized "scattering" amplitude in the complex domain at zero energy.

Note that, actually,  $G$ ,  $\psi$ ,  $h$  of (3.1)-(3.5) are zero energy restrictions of functions introduced by Faddeev as extensions to the complex domain of some functions of the classical scattering theory for the Schrödinger equation at positive energies. In addition,  $G$ ,  $\psi$ ,  $h$  in their zero energy restriction were considered for the first time in [3]. The Faddeev functions  $G$ ,  $\psi$ ,  $h$  were, actually, rediscovered in [3].

We recall also that, under the assumptions of Theorem 2.1

$$\mu(x, k) \rightarrow 1 \quad \text{as } |\operatorname{Im} k| \rightarrow \infty \quad (\text{uniformly in } x) \quad (3.8)$$

and, for any  $c > 1$ ,

$$|\mu(x, k)| < c \quad \text{for } |\operatorname{Im} k| \geq \rho_1(R, D, m, c), \quad (3.9)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \Sigma$ ;

$$\hat{v}(p) = \lim_{\substack{(k, l) \in \Theta, \quad k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \rightarrow \infty}} h(k, l) \quad \text{for any } p \in \mathbb{R}^d, \quad (3.10)$$

$$|\hat{v}(p) - h(k, l)| \leq \frac{C_3(D, m)R^2}{\rho} \quad \text{for } (k, l) \in \Theta, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho \geq \rho_2(R, D, m), \quad (3.11)$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (3.12)$$

Results of the type (3.8), (3.9) go back to [3]. Results of the type (3.10), (3.11) (with less precise right-hand side in (3.11)) go back to [9]. Estimates (3.8), (3.11) are related also with some important  $L_2$ -estimate going back to [19] on the Green function  $g$  of (3.1).

For more information on properties of the Faddeev functions  $G$ ,  $\psi$ ,  $h$ , see [9], citeNo2, [17] and references therein.

In the next section we recall that Problem 1.1 (of Introduction) admits a reduction to the following inverse "scattering" problem:

**Problem 3.1.** Given  $h$  on  $\Theta$ , find  $v$  on  $\mathbb{R}^d$ .

#### 4. REDUCTION OF [14], [15]

Let conditions (1.2), (1.3) hold for potentials  $v_1$  and  $v_2$ , where  $D$  is fixed. Let  $\Phi_i$ ,  $\psi_i$ ,  $h_i$  denote the DtN map  $\Phi$  and the Faddeev functions  $\psi$ ,  $h$  for  $v = v_i$ ,  $i = 1, 2$ . Let also  $\Phi_i(x, y)$  denote the Schwartz kernel  $\Phi(x, y)$  of the integral operator  $\Phi$  for  $v = v_i$ ,  $i = 1, 2$ . Then (see [15] for details):

$$h_2(k, l) - h_1(k, l) = \left(\frac{1}{2\pi}\right)^d \int \int_{\partial D \partial D} \psi_1(x, -l) (\Phi_2 - \Phi_1)(x, y) \psi_2(y, k) dy dx, \quad (4.1)$$

where  $(k, l) \in \Theta$ ;

$$\psi_2(x, k) = \psi_1(x, k) + \int_{\partial D} A(x, y, k) \psi_2(y, k) dy, \quad x \in \partial D, \quad (4.2a)$$

$$A(x, y, k) = \int_{\partial D} R_1(x, z, k) (\Phi_2 - \Phi_1)(z, y) dz, \quad x, y \in \partial D, \quad (4.2b)$$

$$R_1(x, y, k) = G(x - y, k) + \int_{\mathbb{R}^d} G(x - z, k) v_1(z) R_1(z, y, k) dz, \quad x, y \in \mathbb{R}^d, \quad (4.3)$$

where  $k \in \Sigma$ . Note that: (4.1) is an explicit formula, (4.2a) is considered as an equation for finding  $\psi_2$  on  $\partial D$  from  $\psi_1$  on  $\partial D$  and  $A$  on  $\partial D \times \partial D$  for each fixed  $k$ , (4.2b) is an explicit formula, (4.3) is an equation for finding  $R_1$  from  $G$  and  $v_1$ , where  $G$  is the function of (3.1).

Note that formulas and equations (4.1)-(4.3) for  $v_1 \equiv 0$  were given in [14] (see also [9] (Note added in proof), [12], [13]). In this case  $h_1 \equiv 0$ ,  $\psi_1 = e^{ikx}$ ,  $R_1 = G(x - y, k)$ . Formulas and equations (4.1)-(4.3) for the general case were given in [15].

Formulas and equations (4.1)-(4.3) with fixed background potential  $v_1$  reduce Problem 1.1 (of Introduction) to Problem 3.1 (of Section 3).

#### 5. RECONSTRUCTION OF [14] IN THE BORN APPROXIMATION

In the Born approximation, that is in the linear approximation near zero potential, we have that

$$h(k, l) \approx \hat{v}(k - l), \quad (5.1)$$

$$h(k, l) \approx \left(\frac{1}{2\pi}\right)^d \int \int_{\partial D \partial D} e^{-ilx} (\Phi - \Phi_0)(x, y) e^{iky} dx dy, \quad (5.2)$$

where  $(k, l) \in \Theta$ ,  $\hat{v}$  is defined by (3.12),  $\Phi_0$  denotes the DtN map for  $v \equiv 0$ .

Formulas (5.1), (5.2) follow from (3.1)-(3.4) and (4.1). Formulas (5.1), (5.2) imply, in particular, that

$$\hat{v}(p) \approx \left(\frac{1}{2\pi}\right)^d \int \int_{\partial D \bar{\partial} D} e^{-il(p)x} (\Phi - \Phi_0)(x, y) e^{ik(p)y} dx dy, \quad (5.3)$$

$$k(p) = \frac{p}{2} + i\frac{|p|}{2}\gamma(p), \quad l(p) = -\frac{p}{2} + i\frac{|p|}{2}\gamma(p), \quad p \in \mathbb{R}^d, \quad (5.4a)$$

where  $\gamma(p)$  is a piecewise continuous function of  $p \in \mathbb{R}^d$  with values in  $\mathbb{S}^{d-1}$  and such that

$$\gamma(p)p = 0, \quad p \in \mathbb{R}^d. \quad (5.4b)$$

One can see that formula (5.3) gives a reconstruction method for Problem 1.1,  $d \geq 2$ , in the Born approximation.

An approximate reconstruction method based on (5.1), (5.2) for Problem 1.1 in dimension  $d \geq 2$  was proposed for the first time in [14].

In the next section we show that, in the Born approximation, Theorem 2.2 (of Section 2) follows, actually, from (5.3).

## 6. PROOF OF THEOREM 2.2 IN THE BORN APPROXIMATION

We have that

$$v_1(x) - v_2(x) = \left( \int_{|p| < \rho} + \int_{|p| > \rho} \right) e^{-ipx} (\hat{v}_1(p) - \hat{v}_2(p)) dp, \quad (6.1)$$

$$|v_1(x) - v_2(x)| \leq I_1(\rho) + I_2(\rho), \quad (6.2a)$$

$$I_1(\rho) = \int_{|p| < \rho} |\hat{v}_1(p) - \hat{v}_2(p)| dp, \quad (6.2b)$$

$$I_2(\rho) = \int_{|p| > \rho} |\hat{v}_1(p) - \hat{v}_2(p)| dp, \quad (6.2c)$$

where  $x \in \mathbb{R}^d$ ,  $\rho > 0$ .

The assumptions  $\|v_j\|_{m,1} \leq R$ ,  $j = 1, 2$ , imply that

$$|\hat{v}_1(p) - \hat{v}_2(p)| \leq \frac{C_4(d, m)R}{(1 + |p|)^m}, \quad p \in \mathbb{R}^d. \quad (6.3)$$

Using (5.3) we obtain that

$$|\hat{v}_1(p) - \hat{v}_2(p)| \leq C_5(D) e^{L\rho} \|\Phi_1 - \Phi_2\|, \quad |p| \leq \rho, \quad C_5(D) = \left(\frac{1}{2\pi}\right)^d \int_{\partial D} dx, \quad L = \max_{x \in D} |x|, \quad (6.4)$$

where  $\|\cdot\|$  is defined according to (2.4).

Formulas (6.2b), (6.4) imply that

$$I_1(\rho) \leq C_6(D) \rho^d e^{L\rho} \|\Phi_1 - \Phi_2\| \leq C_6(D) e^{L_1\rho} \|\Phi_1 - \Phi_2\|, \quad \rho > 0, \quad C_6(D) = C_5(D) \int_{\theta \in \mathbb{S}^{d-1}} d\theta, \quad L_1 = L + d. \quad (6.5)$$

Formulas (6.2c), (6.3) imply that

$$I_2(\rho) \leq C_7(d, m) R \rho^{-(m-d)}, \quad \rho > 0. \quad (6.6)$$

Let

$$\alpha \in ]0, 1[ \text{ be fixed, } \|\Phi_1 - \Phi_2\| = \delta, \quad \rho = \lambda \ln(1 + \delta^{-1}), \quad \lambda = \frac{1 - \alpha}{L_1}. \quad (6.7)$$

Then, due to (6.5), (6.6),

$$I_1(\rho) \leq C_6 \delta (1 + \delta^{-1})^{\lambda L_1} = C_6 (1 + \delta)^{1 - \alpha} \delta^\alpha, \quad I_2(\rho) \leq C_7 R (\lambda \ln(1 + \delta^{-1}))^{-(m-d)}. \quad (6.8)$$

Estimate (2.8) for  $\delta = \|\Phi_1 - \Phi_2\| \leq 1/2$  follows from (6.2a), (6.8). Estimate (2.8) in the general case (with modified  $C_2$ ) follows from (2.8) for  $\delta \leq 1/2$  and the assumptions that  $\|v_j\|_{L^1(\mathbb{R}^d)} \leq R$ ,  $j = 1, 2$ .

Thus, in the Born approximation Theorem 2.2 is proved. (This proof is valid also for  $d = 2$ ).

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