A LOCAL TIME-STEPPING DISCONTINUOUS GALERKIN ALGORITHM FOR THE MHD SYSTEM

CHRISTOPH ALTMANN¹, THOMAS BELAT², MICHAEL GUTNIC, PHILIPPE HELLUY,
HÉLÈNE MATHIS, ÉRIC SONNENDRÜCKER³, WILFREDO ANGULO⁴ AND JEAN-MARC
HÉRARD⁵

Abstract. This work is devoted to the simulation of the Magneto-Hydro-Dynamics (MHD) equations on unstructured meshes. The starting point is the Discontinuous Galerkin (DG) method, semi-discrete in space. For the time integration, we choose the Adams-Bashforth scheme. This scheme allows very easily local time-stepping on the smallest cells of the mesh. Finally, we present several numerical experiments.

Résumé. Ce travail est consacré à la simulation des équations de la Magnéto-Hydro-Dynamique (MHD) sur des maillages complètement déstructurés. Le point de départ est la méthode de Galerkin Discontinue (DG) semi-discrétisée en espace. Pour l'approximation en temps, nous utilisons le schéma multi-pas d'Adams-Bashforth. Ce schéma permet l'utilisation de pas de temps locaux sur les petites mailles, ce qui conduit pour certains maillages à des gains considérables de temps de calcul. Enfin, nous présentons diverses expériences numériques.

INTRODUCTION

The Magneto-Hydro-Dynamics (MHD) equation are a useful model for describing the behavior of a compressible conductive fluid. The unknowns are the fluid density ρ , the velocity $\mathbf{u} \in \mathbb{R}^3$, the internal energy e , the pressure p and the magnetic field $\mathbf{B} \in \mathbb{R}^3$. All the unknowns depend on the space variable $\mathbf{x} \in \mathbb{R}^3$ and the time variable t .

The equations read

$$\partial_t \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ Q \\ \mathbf{B} \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + (p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}) \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \\ (Q + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}) \mathbf{u} - (\mathbf{B} \cdot \mathbf{u}) \mathbf{B} \\ \mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u} \end{pmatrix} = 0, \quad Q = \rho e + \frac{\rho \mathbf{u} \cdot \mathbf{u}}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}. \quad (1)$$

The notation $\mathbf{I}$ stands for the 3×3 identity matrix. The pressure is related to the internal energy e and the density ρ by a pressure law. In this document, we shall only consider the perfect gas law with a constant

¹ Institut für Aerodynamik und Gasdynamik, Universität Stuttgart, Germany

² Institut franco-allemand de recherches de Saint-Louis, 5 rue du Général Cassagnou, 68300 Saint-Louis, France

³ Université de Strasbourg, IRMA, 7 rue Descartes, 67084 Strasbourg, France

⁴ Departamento de Física, Decanato de Ciencias y Tecnología-UCLA, Barquisimeto-Venezuela y Postgrado en Matemática de la Facultad de Ciencias-UCV, Caracas, Venezuela

⁵ EDF-DRD, Département MFEE 6 quai Watier 78400 Chatou, France

polytropic exponent γ . It reads

$$p = P(\rho, e) = (\gamma - 1)\rho e, \quad \gamma > 1. \quad (2)$$

The previous equations are supplemented by the following divergence condition on the magnetic field

$$\nabla \cdot \mathbf{B} = 0. \quad (3)$$

In this paper, we explore several aspects of the simulation of this system.

- Our method is based on the Discontinuous Galerkin approximation, which is a generalization of the finite volume methods. It is described for example in [3].
- In order to take into account the free divergence condition on the magnetic field, we adopt the divergence cleaning technique described in [7].
- Next, we will comment the choice of an entropy compatible numerical flux. In this paper we concentrate on the Rusanov flux, but we also describe in [1] other possible choices (see also [11], [4], [9]).
- In order to reduce the oscillations that are observed in discontinuous solutions, we also present a very simple entropy dissipative limiter. It can be applied to any polynomial order of the basis functions. It is based on ideas presented in [6] and included references.
- Finally, we describe a new local time-stepping algorithm based on the Adams-Bashforth time integration scheme. This method can be applied to any conservation laws system.
- The paper ends with several numerical experiments. In the 2D cases, the convergence studies are performed with real meshes produced by an automatic mesh generator and with the local time-stepping activated.

1. DIVERGENCE CLEANING MODIFICATION

The divergence free condition on the magnetic field is very important for physical reasons: it ensures that there is no magnetic charge. This condition is difficult to express on the numerical side. Therefore some authors [18], [13], [7] have suggested to extend the ideal MHD system in the following way

$$\partial_t \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ Q \\ \mathbf{B} \\ \psi \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{u} \\ (Q + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2}) \mathbf{I} - \mathbf{B} \otimes \mathbf{B} \\ \mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u} + \psi \mathbf{I} \\ c_h^2 \mathbf{B} \end{pmatrix} = 0, \quad Q = \rho e + \frac{\rho \mathbf{u} \cdot \mathbf{u}}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}. \quad (4)$$

We have added a new unknown ψ whose role is to "clean" the divergence of the solution. Actually, the divergence perturbations are convected in the computational domain at the constant velocity c_h . With adequate boundary conditions, the perturbation will be damped. The velocity c_h can be chosen arbitrarily. In practice, it has to be higher than the highest wave speed of the original MHD system.

We observe that if $\nabla \cdot \mathbf{B} = 0$ and $\psi = \text{Cst}$, then the modified system (1) is equivalent to the MHD system (4).

The two above systems can be put in a conservative form (with the space dimension $d = 3$)

$$\mathbf{w}_t + \sum_{i=1}^d \mathbf{f}^i(\mathbf{w})_{x_i} = 0 \quad (5)$$

We shall make use of the Einstein summation convention on the repeated indices and also write

$$\partial_t \mathbf{w} + \partial_i \mathbf{f}^i(\mathbf{w}) = 0 \quad (6)$$

For any vector $\mathbf{n} = (n_1, n_2, n_3) \in (R^3)$, the vector

$$\mathbf{f}(\mathbf{w}, \mathbf{n}) = \sum_{i=1}^d \mathbf{f}^i(\mathbf{w}) n_i \quad (7)$$

is called the flux vector. The flux vector is given by

$$\mathbf{f}(\mathbf{w}, \mathbf{n}) = \begin{pmatrix} \rho \mathbf{u} \cdot \mathbf{n} \\ \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u} + (p + \frac{\mathbf{B} \cdot \mathbf{B}}{2})\mathbf{n} - (\mathbf{B} \cdot \mathbf{n})\mathbf{B} \\ (Q + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2})\mathbf{u} \cdot \mathbf{n} - (\mathbf{B} \cdot \mathbf{u})(\mathbf{B} \cdot \mathbf{n}) \\ (\mathbf{u} \cdot \mathbf{n})\mathbf{B} - (\mathbf{B} \cdot \mathbf{n})\mathbf{u} + \psi \mathbf{n} \\ c_h^2 \mathbf{B} \cdot \mathbf{n} \end{pmatrix}. \quad (8)$$

The jacobian matrix of the flux vector is

$$\mathbf{A}(\mathbf{w}, \mathbf{n}) = \partial_{\mathbf{w}} \mathbf{f}^i(\mathbf{w}) n_i \quad (9)$$

We note b the normal component of the magnetic field, a the sound speed, c_A the Alfvén speed, $c_{f,s}$ the fast and slow magnetoacoustic speeds and λ_i the eigenvalues sorted by increasing order.

$$\begin{aligned} b &= \mathbf{B} \cdot \mathbf{n} \\ c_A &= \frac{b}{\sqrt{\rho}}, \\ a &= \sqrt{\frac{\gamma p}{\rho}}, \\ c_{f,s} &= \sqrt{\frac{1}{2} \left(\frac{b^2 + B^2}{\rho} + a^2 \right) \pm \sqrt{\frac{1}{4} \left(\frac{b^2 + B^2}{\rho} + a^2 \right)^2 - \frac{a^2 b^2}{\rho}}}, \\ \lambda_1 &= -c_h \\ \lambda_2 &= \mathbf{u} \cdot \mathbf{n} - c_f \quad \lambda_3 = \mathbf{u} \cdot \mathbf{n} - c_A \quad \lambda_4 = \mathbf{u} \cdot \mathbf{n} - c_s \\ \lambda_5 &= \mathbf{u} \cdot \mathbf{n} \\ \lambda_6 &= \mathbf{u} \cdot \mathbf{n} + c_s \quad \lambda_7 = \mathbf{u} \cdot \mathbf{n} + c_A \quad \lambda_8 = \mathbf{u} \cdot \mathbf{n} + c_f \\ \lambda_9 &= c_h \end{aligned} \quad (10)$$

The eigenvalues are all real and the system is always diagonalizable, thus the MHD system is hyperbolic. The expression of the eigenvectors is given in [1]. However, it is possible that several eigenvalues coincide. Then the Lax theory cannot be applied. It is indeed possible to exhibit, in some situations, several Lax solutions to the MHD system [21].

2. DISCONTINUOUS GALERKIN APPROXIMATION

The Discontinuous Galerkin (DG) approximation technique is a generalization of the finite volume approach in order to achieve higher order. It is well suited to hyperbolic systems of conservation laws. Its application to the MHD equations is studied for example by Barth in [3]. In order to make the presentation simpler, we first present the space semi-discrete version of the scheme. The time integration will be studied later on.

We are interested in an approximation of the following system

$$\partial_t \mathbf{w} + \partial_i \mathbf{f}^i = 0 \quad (11)$$

stated in the whole space $\mathbb{R}^d$ (the boundary conditions problematic is not addressed in this document). Let us consider a mesh $\mathcal{T}$ of $\mathbb{R}^d$ made of cells K satisfying

- (1) $\forall K \in \mathcal{T}$, K is an open set;
- (2) $\forall (K, L) \in \mathcal{T} \times \mathcal{T}$ $K \cap L = \emptyset$;
- (3) $\bigcup_{K \in \mathcal{T}} \overline{K} = \mathbb{R}^d$.

The thickness of the mesh can be measured by the following parameter

$$h = \sup_{K \in \mathcal{T}} \frac{|K|}{|\partial K|}, \quad (12)$$

where $|K|$ denoted the volume of the cell K and $|\partial K|$ the surface of the cell boundary ∂K .

We are looking for an approximation of the solution $\mathbf{w}$ that is polynomial in each cell K . More precisely, let $P^k(K)$ be a linear space of polynomials of degree $\leq k$ defined on the cell K . We denote by $\mathbf{P}^k(K) = (P^k(K))^m$ the corresponding vector space. The approximation is thus discontinuous at the cell boundaries ∂K (and it justifies the name of the method). The approximation space is then

$$\mathcal{E}_h = \left\{ \mathbf{w} \in (L^2(\mathbb{R}^d))^m, \forall K \in \mathcal{T}, \mathbf{w}|_K \in \mathbf{P}^k(K) \right\} \quad (13)$$

The test functions $\mathbf{v}$ are taken in this vector space $\mathcal{E}_h$. We multiply the conservation laws by $\mathbf{v}$, integrate on a cell K and sum over all the cells. This leads naturally to the introduction of the following form

$$B(\mathbf{w}, \mathbf{v}) = \sum_{K \in \mathcal{T}} \int_{\partial K} \mathbf{f}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) \mathbf{v}_L - \int_K \mathbf{f}^i \partial_i \mathbf{v}. \quad (14)$$

The form is linear with respect to $\mathbf{v}$ and would be bilinear if the conservation system were linear.

It is necessary to introduce the numerical flux $\mathbf{f}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) \mathbf{v}_L$ because the solution $\mathbf{w}$ and the test function $\mathbf{v}$ may be discontinuous at the cell interfaces ∂K .

The approximation consists then in finding an element $\mathbf{w}$ in $C^1([0, T], \mathcal{E}_h)$ such that for all elements $\mathbf{v}$ in $\mathcal{E}_h$

$$\int_{\mathbf{x} \in \mathbb{R}^3} \partial_t \mathbf{w} \cdot \mathbf{v} + B(\mathbf{w}, \mathbf{v}) = 0. \quad (15)$$

The numerical flux has to satisfy

$$\begin{aligned} \mathbf{f}(\mathbf{w}, \mathbf{w}, \mathbf{n}) &= \mathbf{f} \cdot \mathbf{n} = \mathbf{f}^i \mathbf{n}_i \quad (\text{consistency}) \\ \mathbf{f}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) &= -\mathbf{f}(\mathbf{w}_R, \mathbf{w}_L, -\mathbf{n}) \quad (\text{conservation}) \end{aligned} \quad (16)$$

The simplest example is the centered flux

$$\mathbf{f}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) = \frac{1}{2} (\mathbf{f}^i(\mathbf{w}_L) + \mathbf{f}^i(\mathbf{w}_R)) \mathbf{n}_i \quad (17)$$

But this choice leads to oscillations in discontinuous solutions, even if a proper time integration gives a linearly stable scheme. Below we propose another numerical flux that leads to a better approximation. The approximation can be stated in a more precise way. For this, we consider a basis $(\mathbf{e}_{K,i})$ of the space $\mathbf{P}^k(K)$. As a convention, we extend these functions by zero outside K . We then take $\mathbf{v} = \mathbf{e}_{K,i}$ in the weak form (15). We

obtain that for all the cells L

$$\begin{aligned} w'_{L,j}(t) \int_L \mathbf{e}_{L,j} \cdot \mathbf{e}_{L,i} + \sum_{R \in V(L)} \int_{\partial L \cap \partial R} \mathbf{f}(w_{L,j} \mathbf{e}_{L,j}, w_{R,j} \mathbf{e}_{R,j}, \mathbf{n}_{L,R}) \cdot \mathbf{e}_{L,i} \\ - \int_L \mathbf{f}^k \cdot \partial_k \mathbf{e}_{L,i} = 0. \end{aligned} \quad (18)$$

In this formula, we have used the Einstein summation convention. We also denote by $\mathbf{n}_{L,R}$ the normal vector oriented from cell L to cell R along the boundary ∂L of the cell L (we take as a convention that the Left cell is on the side of $-\mathbf{n}_{L,R}$ and the Right cell on the side of $\mathbf{n}_{L,R}$). The set of the neighboring cells R to the cell L is $V(L)$.

The term $\int_L \mathbf{e}_{L,j} \cdot \mathbf{e}_{L,i}$ corresponds to a mass matrix term that can be inverted once at the beginning of the computation. If the chosen basis on $\mathbf{P}^k(K)$ is orthonormal, the mass matrix is diagonal. All the integrals on the cells or their boundaries are computed with a Gauss integration, which we do not describe here in order to avoid heavy notations.

Then, the approximation system is transformed into a first order differential equations system and can be solved by any standard integration algorithm (as Runge-Kutta, Adams, *etc.*)

A stable and very simple numerical flux is the Rusanov flux, which reads

$$\begin{aligned} \lambda_{\max} &= \max_{i=1 \dots 9} \max(|\lambda_i(\mathbf{w}_L)|, |\lambda_i(\mathbf{w}_R)|), \\ \mathbf{f}(\mathbf{w}_L, \mathbf{w}_R, \mathbf{n}) &= \frac{\mathbf{f}(\mathbf{w}_L) + \mathbf{f}(\mathbf{w}_R)}{2} \cdot \mathbf{n} - \frac{\lambda_{\max}}{2} (\mathbf{w}_R - \mathbf{w}_L). \end{aligned} \quad (19)$$

where we have noted $\lambda_j(\mathbf{w})$ the wave speeds at the state $\mathbf{w}$ (the eigenvalues of $\mathbf{A}(\mathbf{w}, \mathbf{n})$). By taking all the components of $\mathbf{v}$ to 1 in the Galerkin weak formulation, we see that the integral of $\mathbf{w}$ over the whole space is constant with respect to time, thanks to the conservation property of the flux. It is also possible to state a discrete entropy dissipation property for the Rusanov scheme¹. Thanks to the Lax-Wendroff theorem, this property ensures that the scheme converges towards an entropy solution (when it converges).

We have also tested other numerical fluxes. They are described in [1]. Under some conditions, the DG approximation satisfies a discrete entropy principle, which is stated for example in [3] and recalled in [1].

3. TIME INTEGRATION

We now address the problem of the time approximation. The Runge-Kutta method is a standard approach that is not described here. We will concentrate on an Adams approach, which has some advantages (the possibility to use very easily different time steps) and some drawbacks (a sometimes more limiting CFL condition).

3.1. Adams time integration

In order to obtain the Adams scheme, we suppose that the solution is approximated at some times

$$t^0 < t^1 < \dots < t^n < \dots \quad (20)$$

by a sequence of elements

$$\mathbf{w}^n(\cdot) \simeq \mathbf{w}(t^n, \cdot) \in \mathcal{E}_h \quad (21)$$

¹ with a slightly modified definition of $\lambda_{\max}$. See [3].

By an integration in time of (15) we see that the semi-discrete solution satisfies

$$\begin{aligned}
& (w_{L,j}(t^{n+1}) - w_{L,j}(t^n)) \int_L \mathbf{e}_{L,j} \cdot \mathbf{e}_{L,i} + \\
& \int_{t=t^n}^{t^{n+1}} \sum_{R \in V(L)} \int_{\partial L \cap \partial R} \mathbf{f}(w_{L,j} \mathbf{e}_{L,j}, w_{R,j} \mathbf{e}_{R,j}, \mathbf{n}_{L,R}) \cdot \mathbf{e}_{L,i} \\
& - \int_{t=t^n}^{t^{n+1}} \int_L \mathbf{f}^k \cdot \partial_k \mathbf{e}_{L,i} = 0.
\end{aligned} \tag{22}$$

In order to approximate the time integration, we first set

$$\begin{aligned}
F_{L/R,i}(t) &= \int_{\partial L \cap \partial R} \mathbf{f}(w_{L,j}(t) \mathbf{e}_{L,j}, w_{R,j}(t) \mathbf{e}_{R,j}, \mathbf{n}_{L,R}) \cdot \mathbf{e}_{L,i} \\
S_{L,i}(t) &= \int_L \mathbf{f}(w_{L,j}(t) \mathbf{e}_{L,j})^k \cdot \partial_k \mathbf{e}_{L,i}
\end{aligned} \tag{23}$$

in such a way that the weak form can also be written

$$\begin{aligned}
& (w_{L,j}(t^{n+1}) - w_{L,j}(t^n)) \int_L \mathbf{e}_{L,j} \cdot \mathbf{e}_{L,i} + \sum_{R \in V(L)} \int_{t=t^n}^{t^{n+1}} F_{L/R,i}(t) \\
& - \int_{t=t^n}^{t^{n+1}} S_{L,i}(t) = 0.
\end{aligned} \tag{24}$$

We also define the discrete flux and source term at time t^n

$$\begin{aligned}
F_{L/R,i}^n &= \int_{\partial L \cap \partial R} \mathbf{f}(w_{L,j}^n \mathbf{e}_{L,j}, w_{R,j}^n \mathbf{e}_{R,j}, \mathbf{n}_{L,R}) \cdot \mathbf{e}_{L,i} \\
S_{L,i}^n &= \int_L \mathbf{f}(w_{L,j}^n \mathbf{e}_{L,j})^k \cdot \partial_k \mathbf{e}_{L,i}
\end{aligned} \tag{25}$$

We then construct the time interpolation polynomials $\tilde{F}_{L/R,i}(t)$ and $\tilde{S}_{L,i}(t)$ of $F_{L/R,i}^n$ and $S_{L,i}^n$ by using $r+1$ interpolation points. In other words, the interpolation polynomials are defined by

$$\tilde{F}_{L/R,i}(t^{n-l}) = F_{L/R,i}^{n-l} \quad \text{and} \quad \tilde{S}_{L,i}(t^{n-l}) = S_{L,i}^{n-l}, \quad l = 0 \dots r \tag{26}$$

The time integration of the boundary terms and the source terms are then obtained by an exact integration of $\tilde{F}_{L/R,i}(t)$ and $\tilde{S}_{L,i}(t)$ on the interval $[t^n, t^{n+1}]$

$$\begin{aligned}
& \int_{t=t^n}^{t^{n+1}} \int_{\partial L \cap \partial R} \mathbf{f}(w_{L,j} \mathbf{e}_{L,j}, w_{R,j} \mathbf{e}_{R,j}, \mathbf{n}_{L,R}) \cdot \mathbf{e}_{L,i} \simeq \int_{t=t^n}^{t^{n+1}} \tilde{F}_{L/R,i}(t) dt, \\
& \int_{t=t^n}^{t^{n+1}} \int_L \mathbf{f}^k \cdot \partial_k \mathbf{e}_{L,i} \simeq \int_{t=t^n}^{t^{n+1}} \tilde{S}_{L,i}(t) dt.
\end{aligned} \tag{27}$$

The method requires to store the flux terms on the cell edges and the source terms in the cells at the $r+1$ previous times. It is also necessary to initialize the scheme, for example by r steps of a Runge-Kutta algorithm.

3.2. The local time-stepping algorithm

An advantage of the Adams approach is that it is quite easy to adapt it to the case where the time steps are different from one cell to one another. This is useful when small cells are mixed with big cells in order to reduce the computational cost. First, we attribute to each cell what we call a CFL level, which is only based on a geometric criterion. We define

$$\begin{aligned} h_K &= \frac{|K|}{|\partial K|} \\ h_{\min} &= \min_{K \in \mathcal{T}} h_K \\ h_{\max} &= \max_{K \in \mathcal{T}} h_K \end{aligned} \quad (28)$$

With a standard one-step time method, the time step is fixed by the smallest cell

$$\Delta t = CFL \times \frac{h_{\min}}{\lambda_{\max}} \quad (29)$$

where $\lambda_{\max}$ is the highest wave speed in the mesh. We shall say that a cell K is of level n (and we note $\text{level}(K) = n$) if

$$2^n h_{\min} \leq h_K < 2^{n+1} h_{\min} \quad (30)$$

In this way, the smallest cells are of level $n = 0$ and the biggest cells are of level

$$N = \left\lceil \log_2 \left(\frac{h_{\max}}{h_{\min}} \right) \right\rceil \quad (31)$$

We also define a level for the edge $L/R = \partial L \cap \partial R$

$$\text{level}(L/R) = \min(\text{level}(L), \text{level}(R)). \quad (32)$$

Let Δt be the time step associated to the biggest cells (we call it the macro time step)

$$\Delta t = CFL \times \frac{h_{\min}}{\lambda_{\max}} \times 2^N \quad (33)$$

According to the previous definitions, this time step satisfies

$$\Delta t \leq CFL \times \frac{h_{\max}}{\lambda_{\max}} \quad (34)$$

The time loop algorithm is then the following

```

for  $i = 1$  to  $2^N$  do
  let  $j$  be the biggest integer such that  $2^j$  divides  $i$ 
  for all the edges  $L/R$  of level  $\leq j$  do
    compute the integral of the flux term  $F_{L/R}$  on a time interval of length  $\Delta t 2^{j-N}$  and distribute it to the
    two neighboring cells
  end for
  for all the cells  $L$  of level  $\leq j$  do
    compute the integral of the source term  $S_L$  on a time interval of length  $\Delta t 2^{j-N}$  and distribute it to the
    corresponding cell.
  end for
  Update only the cells of level  $\leq j$ 
end for

```

With this algorithm, the fluxes are computed more times on the small edges than on the big edges but are always distributed on the two sides of the edge in order to keep a conservative scheme. The time integration is always performed by the Adams approach: the interpolation polynomial is calculated from the r more recent fluxes or sources evaluations. At the end of a macro time step of size Δt (when $i = 2^N$ in the algorithm) all the cells are updated together.

If the number of small cells is small and if the other cells have almost the same size, the gain is almost of 2^N . In order to illustrate this assertion we give some examples for one and two dimension problems.

- *1D test case:* We consider the transport of a sinusoidal function through the computational domain. A transformation is applied to the mesh which consists in mapping x to x^2 . We obtain thus 8 levels of CFL. We use a second order scheme with a $CFL = 0.2$ and $t_{final} = 1$ s. Without the local time-stepping approach, the computation lasts 51 s. while it takes 2.9 s. using the algorithm.
- *2D test case:* The second test case is a simple Sod shock tube test applied to compressible Euler equations. The solution is one-dimensional, but we solve it on a two-dimensional mesh. The mesh is plotted on Figures 1 and 2. A second order scheme is applied with $CFL = 0.3$ and $t_{final} = 0.5$. While the computation lasts 112.81 s with the standard time-stepping, it lasts 34.29 s using the local time-stepping approach.

Our approach is classical (see for example [16], [2]), stable and very simple. However it can lead to a small loss of accuracy due to a lack of consistency [19]. It is possible to recover consistency by a very simple fix described in [12] and [15]. The fix consists in always computing numerical fluxes from left and right cell values taken at the same time step.

Let us mention that we observed rather constraining CFL condition for the Adams-Bashforth time integration algorithm. Typically, for a third order time integration a CFL of 0.15 is necessary. However, only one flux evaluation per time step is needed compared to four flux evaluations for the RK4 algorithm. Thus, for uniformly sized meshes, CPU costs are equivalent. Anyway, a more precise comparison between different time integrators is still needed.

Other approaches exist for local time-stepping. We can mention for example the ADER method described for example in [8] (see also included references). The ADER method is fully conservative and can be used at any order with reasonable CFL conditions. However, it is more complex than our approach because it requires the symbolic computation of high order space derivatives in order to evaluate the high order time derivatives. We can also mention that the Runge-Kutta strategy has been adapted to local-time stepping. Complexity arises in order to achieve conservation: we can mention the additive Runge-Kutta strategy [10], or the symplectic integrator strategy [17]. We do not know yet if these recent approaches can handle the fully non-linear MHD system.

Our feeling is that the Adams-Bashforth approach is probably the simplest way to achieve high order and conservation in a local time-stepping algorithm. For more robust and precise methods, maybe it is preferable to switch to ADER or symplectic integrators.

4. SLOPE LIMITER

Despite its clean construction, and like other numerical methods for discontinuous solutions, the Discontinuous Galerkin approximation suffers from oscillations in the shock and contact waves. It is thus necessary to add to the method a step of slope limiting before the computation of the fluxes. Several approaches exist. In this work, we use a very simple method that is entropy dissipative. It is very satisfying for one-dimensional problems. However, it is sometimes too robust and dissipative for higher space dimensions.

The principle is classical: we have to avoid, or at least limit, the apparition of a local extremum. For this, we consider in each cell L a vector of limiters which is set initially to

$$\beta_L^i = 1, \quad i = 1 \cdots m \quad (35)$$

FIGURE 1. 2D mesh and its 7 levels of CFL

FIGURE 2. 2D mesh: zoom on the small cells region

(no limitation). The maximum limitation is achieved when all the components are equal to zero. After the slope limiting step, each component i of the conservative variables vector will be changed to the following convex combination between the non-limited value and the mean value of the approximation on the cell

$$\tilde{w}_{L,j}^i(t) = \beta_L^i w_{L,j}^i(t) + (1 - \beta_L^i) \bar{w}_L^i, \quad (36)$$

where $\bar{w}_L^i$ denotes the mean value of the i th component of w_L . We observe that when all the limiter components $\beta_L^i = 0$, we recover the mean value of the approximation and the scheme degenerates to first order. It is also easy to see that the slope limiting step is necessarily conservative. Finally, thanks to the Jensen's inequality, it is also entropy dissipative. This kind of approach is described in [6] in a more sophisticated way. See also included references.

Now, in order to compute β , we use the following algorithm. First, we loop on all the cell edges L/R . We then loop on all the Gauss points G of the edge and compute the value of $\mathbf{w}_L(G)$ and $\mathbf{w}_R(G)$. If for a component i , $\tilde{w}_L^i(G)$ (respectively $\tilde{w}_R^i(G)$) is not between $\bar{w}_L^i$ and $\bar{w}_R^i$ then decrease the limiter β_R^i (respectively β_L^i) accordingly. More precisely, we take as new limiter value

$$\beta_L^i \leftarrow \max(\min(\beta_L^i, \frac{\bar{w}_R^i - \bar{w}_L^i}{w^i(G) - \bar{w}_L^i}), 0) \quad (37)$$

(and the symmetric formula for β_R^i). At the end of this algorithm, the values of w_L in the cell L are replaced by the values given in (36).

5. NUMERICAL RESULTS

We present in this section several numerical results. We have implemented the described algorithms (DG and Adams-Bashforth) in a fully three-dimensional fortran parallel code named CM2 (for Code Multi-physique Multi-échelle). This code can also compute multifluid flows. It is possible to perform one-dimensional and two-dimensional test cases by using fictitious edges in the supplementary (y, z) or z directions.

5.1. One-dimensional test cases

The MHD equations possess seven eigenvalues, some of which may coincide depending on the direction and the strength of the magnetic field. As pointed out in [5], [21], [20], [14] the MHD system is thus non-strictly hyperbolic and possesses non-convexity. As a consequence a solution of the Riemann problem may be composed not only of ordinary shock and rarefaction waves but also of other waves as compound waves.

Thanks to Torrilhon's Riemann solver, available at <http://www.math.ethz.ch/~matorril/mhdsolver>, exact solutions can be computed in most cases. We have also presented in [1] a rather general method for solving the Riemann problem for the MHD equations.

For all one-dimensional test cases $\gamma = \frac{5}{3}$, $b = B_x$ is constant and $x_0 = 0$ denotes the position where the initial discontinuity is applied.

5.1.1. Compound shocks

We propose to study first a compound shocks test case. As shown in Figure 3, Figure 4 and Figure 5, this problem shows the formation of a left-going slow compound wave with a weak right-going slow shock and a contact discontinuity.

FIGURE 3. Compound shocks : ρ at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

Left state	Right state
$\rho_L = 1.$	$\rho_R = 0.125$
$u_L = 0.$	$u_R = 0.$
$v_L = 0.$	$v_R = 0.$
$w_L = 0.$	$w_R = 0.$
$p_L = 1.$	$p = 0.1$
$B_{x,L} = 0.75$	$B_x = 0.75$
$B_{y,L} = 1.$	$B_y = -1.$
$B_{z,L} = 0.$	$B_z = 0.$

5.1.2. All seven waves

This problem shows the formation of all seven possible MHD waves. See Figure 6, Figure 7 and Figure 8.

Left state	Right state
$\rho_L = 1.08$	$\rho_R = 1.$
$u_L = 1.2$	$u_R = 0.$
$v_L = 0.$	$v_R = 0.$
$w_L = 0.$	$w_R = 0.$
$p_L = 0.95$	$p = 1.$
$B_{x,L} = 2./\sqrt{4\pi}$	$B_x = 2.\sqrt{4\pi}$
$B_{y,L} = 3.6\sqrt{4\pi}$	$B_y = 4.\sqrt{4\pi}$
$B_{z,L} = 2.\sqrt{4\pi}$	$B_z = 2.\sqrt{4\pi}$

FIGURE 4. Compound shocks : B_y at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

On Figure 9, Figure 10, Figure 11 and Figure 12 we present a comparison between the first and the third order schemes with or without the slope limiter.

FIGURE 5. Compound shocks : p at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

Figure 13 gives a measure of the L^1 norm of the error, in the case of a sinusoidal density ρ convected on a domain with periodic boundaries. The straight line corresponds to first order, the dashdot line to second order and finally the dashed line refers to third order. The convergence curves do not all present a perfect linear behavior and we observe some slight slope changes: this is probably due to the fact that the Adams-Bashforth integration is initialized by several time steps of a lower order method and by the unstructured nature of the mesh.

5.2. 2D academic test cases

5.2.1. Convergence test case

As for 1D test cases, we give a measure of the L^1 norm of the error. The test case is again a sinusoidal density, of period 2, convected on the whole domain at a constant velocity (the magnetic field vanishes and the pressure is constant). Figure 14 presents the error curves for first and second order schemes. As before, the slope changes in some convergence curves are probably due to the fact that the Adams-Bashforth is initialized by several time steps of a lower order method, but also because the mesh is unstructured and contains several small cells where the local time-stepping algorithm is activated (in practice, the automatic mesh generator produces two CFL levels). However, the error curves are coherent and the mean slope decreases with the order of the scheme.

5.2.2. The MHD vortex

This test case consists in introducing a variation of the magnetic field, transported through the computational domain at an angle of 45° . The domain $[-5, 5] \times [-5, 5]$ is periodic in both directions. The problem is calculated up to $t = 10$. At that time, the vortex should have crossed the computational domain exactly once and should have reached the starting position. It enables thus an accuracy analysis by comparing the final results to the

FIGURE 6. All seven waves : ρ at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

initial conditions. Initial conditions are given by:

$$(\rho, u, v, w, p, B_x, B_y, B_z) = (1, 1, 1, 0, 1, 0, 0, 0) \quad (38)$$

The ratio of specific heats is $\gamma = 5/3$. The vortex is initialized at the center of the computational domain by way of fluctuations in the velocity and magnetic fields given by:

$$\begin{pmatrix} \partial \rho \\ \partial u \\ \partial v \\ \partial w \\ \partial p \\ \partial B_x \\ \partial B_y \\ \partial B_w \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\kappa}{2\pi} e^{\frac{1}{2}(1-r^2)} y \\ \frac{\kappa}{2\pi} e^{\frac{1}{2}(1-r^2)} x \\ 0 \\ \frac{1}{2} \left(\frac{\mu}{2\pi} \right)^2 (1-r^2) e^{(1-r^2)} - \frac{1}{2} \left(\frac{\kappa}{2\pi} \right)^2 e^{(1-r^2)} \\ -\frac{\mu}{2\pi} e^{\frac{1}{2}(1-r^2)} y \\ \frac{\mu}{2\pi} e^{\frac{1}{2}(1-r^2)} x \\ 0 \end{pmatrix} \quad (39)$$

For this test case, $\kappa = 1.0$ and $\mu = 1.0$.

Comparing the final output with the initial conditions, we obtain an L^1 error plot for the density ρ (see Figure 15).

FIGURE 7. All seven waves : B_y at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

5.3. Two-dimensional test case: Kelvin-Helmholtz instabilities

This test case is proposed in [7]. The initial condition is defined as follows

$$\begin{aligned}
 (\rho, u_1, u_2, u_3, p, B_1, B_2, B_3) &= (1, u_x^0, u_y^0, 0, 1, 0, 0, 50) \text{ with} \\
 u_x^0(x, y) &= 5(\tanh(20(y + 0.5)) - (\tanh(20(y - 0.5)) + 1)) \\
 u_y^0(x, y) &= 0.25 \sin(2\pi x)(e^{-100(y+0.5)^2} - e^{-100(y-0.5)^2})
 \end{aligned}$$

The computational domain is $[0; 1] \times [-1; 1]$ and the boundary conditions are periodic. In this case, we have also improved the damping of the magnetic field divergence by adding a zero order source term described in [7]. The CFL is set to 0.4 for a second order approximation without slope limiter. The final time is $T_{final} = 0.5$ s. The mesh is represented in Figure 16. As can be seen it has three CFL levels. Without the local time-stepping algorithm, the computations last 5,967 s. With the local time-stepping activated, the computations last 4,573 s. The x -component of the magnetic field is plotted in Figure 17. It compares well with the results of [7]. A slight Gibbs phenomenon is observed because the slope limiter is not activated.

CONCLUSION

In this paper, we have constructed a numerical method for the simulation of the ideal MHD equations.

The spatial part of the method is based on a classical Discontinuous Galerkin approach. It can be applied to 3D MHD flows on unstructured grids, it is of high order and entropy dissipative. The CEMRACS 2008 session gave us the opportunity to develop a new 3D parallel software for solving hyperbolic conservation laws. Several

FIGURE 8. All seven waves : p at time $t = 0.1$ with a resolution $\Delta x = 0.001$.

other problematics have also partially been addressed: resolution of the Riemann problem, theoretical CFL stability study, numerical fluxes (see [1]).

For the time approximation, we have proposed a new local time-stepping technique for Discontinuous Galerkin schemes. The local time-stepping is achieved thanks to nice properties of the Adams-Bashforth time integration and is rather simple to implement. It allows interesting time saving when cells of different sizes are mixed in the unstructured mesh.

Numerically speaking, we have performed academic test cases in order to certify the whole approach. The next step would be to test the software on more complicated 2D and 3D situations where the parallel implementation becomes more interesting.

REFERENCES

- [1] C. Altmann, W. Angulo, T. Belat, M. Gutnic, P. Helluy, J.-M. Hérard, H. Mathis, and E. Sonnendrücker. Galerkin discontinuous approximation of the mhd equations. Technical report, INRIA project CALVI, 2008. <http://hal.archives-ouvertes.fr/hal-00337063/fr/>.
- [2] T. Barberon, P. Helluy, and S. Rouy. Practical computation of axisymmetrical multifluid flows. *International Journal of Finite Volumes*, 1(1):1–34, 2003.
- [3] T. Barth. On the role of involutions in the discontinuous Galerkin discretization of Maxwell and magnetohydrodynamic systems. In *Compatible spatial discretizations*, volume 142 of *IMA Vol. Math. Appl.*, pages 69–88. Springer, New York, 2006.
- [4] François Bouchut, Christian Klingenberg, and Knut Waagan. A multiwave approximate Riemann solver for ideal MHD based on relaxation. I. Theoretical framework. *Numer. Math.*, 108(1):7–42, 2007.
- [5] M. Brio and C. C. Wu. An upwind differencing scheme for the equations of ideal magnetohydrodynamics. *J. Comput. Phys.*, 75(2):400–422, 1988.
- [6] F. Coquel, P. Helluy, and J. Schneider. Second-order entropy diminishing scheme for the Euler equations. *Internat. J. Numer. Methods Fluids*, 50(9):1029–1061, 2006.

FIGURE 9. All seven waves : B_y at time $t = 0.1$ with a resolution $\Delta x = 0.001$, without slope limiter

- [7] A. Dedner, F. Kemm, D. Kröner, C.-D. Munz, T. Schnitzer, and M. Wesenberg. Hyperbolic divergence cleaning for the MHD equations. *J. Comput. Phys.*, 175(2):645–673, 2002.
- [8] Michael Dumbser, Dinshaw S. Balsara, Eleuterio F. Toro, and Claus-Dieter Munz. A unified framework for the construction of one-step finite volume and discontinuous Galerkin schemes on unstructured meshes. *J. Comput. Phys.*, 227(18):8209–8253, 2008.
- [9] J.-M. Hérard, P. Helluy, H. Mathis, and S. Müller. A simple parameter-free entropy correction for approximate riemann solvers. Technical report, EDF and IRMA, 2008.
- [10] Christopher A. Kennedy and Mark H. Carpenter. Additive Runge-Kutta schemes for convection-diffusion-reaction equations. *Appl. Numer. Math.*, 44(1-2):139–181, 2003.
- [11] T. Miyoshi and K. Kusano. A multi-state hll approximate riemann solver for ideal magnetohydrodynamics. *J. Comput. Phys.*, 208:315–344, 2005.
- [12] Siegfried Müller and Youssef Stiriba. Fully adaptive multiscale schemes for conservation laws employing locally varying time stepping. *J. Sci. Comput.*, 30(3):493–531, 2007.
- [13] C.-D. Munz, P. Omnes, R. Schneider, E. Sonnendrücker, and U. Voß. Divergence correction techniques for maxwell solvers based on a hyperbolic model. *J. Comput. Phys.*, 161(2):484–511, 2000.
- [14] R. S. Myong. *Theoretical and computational investigations of nonlinear waves in magnetohydrodynamics*. PhD thesis, University of Michigan, 1996.
- [15] Q.-L. Nguyen. *Adaptation dynamique de maillage pour les écoulements diphasiques en conduite pétrolières. Application à la simulation des phénomènes de terrain-slugging et severe slugging*. PhD thesis, Université Paris VI, june 2009.
- [16] Stanley Osher and Richard Sanders. Numerical approximations to nonlinear conservation laws with locally varying time and space grids. *Math. Comp.*, 41(164):321–336, 1983.
- [17] Serge Piperno. Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems. *M2AN Math. Model. Numer. Anal.*, 40(5):815–841 (2007), 2006.
- [18] K. G. Powell. An approximate Riemann solver for magnetohydrodynamics (that works in more than one space dimension). Technical Report NASA/CR-194902 ICASE Report No. 94-24, ICASE-NASA Langley, ICASE, NASA Langley Research Center, April 1994.

FIGURE 10. All seven waves : ρ , B_y at time $t = 0.1$ with a resolution $\Delta x = 0.001$, without slope limiter

- [19] Hua-zhong Tang and Gerald Warnecke. High resolution schemes for conservation laws and convection-diffusion equations with varying time and space grids. *J. Comput. Math.*, 24(2):121–140, 2006.
- [20] M. Torrilhon. Non-uniform convergence of finite volume schemes for Riemann problems of ideal magnetohydrodynamics. *J. Comput. Phys.*, 192(1):73–94, 2003.
- [21] M. Torrilhon. Uniqueness conditions for Riemann problems of ideal magnetohydrodynamics. *J. Plasma Phys.*, 69(3):253–276, 2003.

FIGURE 11. All seven waves : ρ at time $t = 0.1$ with a resolution $\Delta x = 0.001$, with slope limiter

FIGURE 12. All seven waves : B_y at time $t = 0.1$ with a resolution $\Delta x = 0.001$, with slope limiter

FIGURE 13. L^1 error norm: ρ without slope limiter

FIGURE 14. L^1 error norm: ρ without slope limiter FIGURE 15. L^1 error norm: ρ without slope limiter

FIGURE 16. Mesh for the Kelvin-Helmholtz instability case

FIGURE 17. Magnetic field B_1 (the range is $[-4, 6.5]$)