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# TWO LOCAL TIME STEPPING SCHEMES FOR PARABOLIC PROBLEMS

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**Résumé.** Nous présentons une stratégie avec raffinement local en temps pour la résolution de problèmes instationnaires où des pas de temps différents sont utilisés dans différentes régions de l'espace. Nous proposons deux approximations conservatives basées sur une méthode de Volumes Finis associée à une approche de type décomposition de domaine avec projection sur les fonctions contantes par morceaux. Nous présentons ensuite une méthode itérative pour résoudre le problème algébrique discret qui se ramène à des résolutions standards dans les différentes régions à petits ou grands pas de temps. A chaque étape de la méthode itérative, la conservation est assurée. Nous montrons des résultats numériques pour une équation parabolique et un système d'équations de type mixte parabolique-hyperbolique modélisant un écoulement diphasique en milieu poreux.

**Abstract.** We present a strategy for solving time-dependent problems on grids with local refinements in time using different time steps in different regions of space. We propose two conservative approximations based on finite volume with piecewise constant projections and domain decomposition techniques. Next we present an iterative method for solving the composite-grid system that reduces to solution of standard problems with standard time stepping on the coarse and fine grids. At every step of the algorithm, conservativity is ensured. Finally, numerical results on parabolic and a two-phase flow problems illustrate the accuracy of the proposed methods.

## 1. Introduction

In many physical applications, there are special features which greatly affect the solution globally as well as locally. One important example is the local spatial and temporal behavior of multiphase fluid flow around a production well in the petroleum recovery applications. To capture this local behavior, spatial local refinement is necessary. However, it requires a reduction of the time step, compared to the one used with a coarse mesh, in order to get a solution accurate enough in the refined zone and to avoid convergence problems when solving the non linear discretized equations. When applied uniformly on all the simulation domain, this reduced time step leads to unacceptable cpu-time making the use of local time steps highly desirable. To be efficient, a local time-stepping strategy (numerical scheme and solution method) must

- ensure accuracy of the solution *i.e.* the solution has to be more accurate than the one obtained with a global coarse mesh,
- ensure stability without any too restrictive condition on the time step,
- and has to lead to reduced cpu-time compared to the one obtained when using a small time step on the whole domain.

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In the framework of reservoir simulation where local grid refinement is necessary to represent correctly important local phenomena in the wells vicinity, the corresponding numerical scheme must also be locally conservative in order to be applicable to multi-phase flow simulations where a coupled system of parabolic and hyperbolic equations has to be solved.

For parabolic equations, different approaches have been proposed in the past which extend the classical implicit finite difference scheme to local refinement in time. In [5], the scheme is written as a cell centered Finite Volume scheme. At the interface between the coarse time step zone and the refined time step one, the flux over the coarse step is taken equal to the integral over the corresponding refined steps of the flux computed from the refined zone. This refined flux approximation requires values of the unknowns in the coarse zone at small time steps which are computed using piecewise constant or linear interpolation from the coarse unknowns. The scheme is conservative. Stability and error estimations are obtained for the piecewise constant interpolation. On the contrary, for the linear interpolation, stability is obtained under a sufficient condition which is as restrictive as the time step limitation obtained for an explicit scheme. In [2], Dawson *et al.* proposed to couple classical implicit finite difference schemes in the refined and coarse zones using an explicit approximation at the interface on a larger mesh size in order to attenuate the time step limitation due to the explicit approximation. Although interesting for its simplicity, this approach can not be retained due to its time step limitation. In [4], Ewing and Lazarov proposed an implicit non conservative approach. The scheme for the coarse nodes is straightforward while the fine grid nodes located at the interface between the coarse and fine regions require "slave" points at small time steps on the coarse grid side, which are not grid points. As in [5], the values of the unknown at these slave points are obtained by linear interpolation in time between the corresponding nodes of the coarse grid, and the set of discretized equations involves all the unknowns between two coarse time levels. Stability and error analysis are performed. The solution method uses an iterative method associated to a coarse grid preconditionner of the Schur complement of the system where the refined region unknowns have been eliminated. In the more applied framework of compositional multiphase flow, [3] introduced an implicit time stepping method. For each global time step, the problem is solved implicitly in the whole domain but using a linear approximation of the model in the refined regions which avoids any convergence problem of the non linear solver due to refined mesh. Then, the refined zones are solved using a local time step and taking as boundary conditions the fluxes computed during the first stage at the interface between the refined and coarse zones. This approach ensures that the method is conservative. It is moreover rather efficient as, compared to the cpu-time necessary to solve the problem with a large timestep on the whole domain, it only requires additional cpu-time to solve the equations once in the refined zone. However, the accuracy of the solution is not controlled. Looking for an efficient solution method, [11] used the same finite difference scheme as [4] for linear parabolic equations but proposed a predictor-corrector method. In the predictor stage, the solution is computed at the coarse time step on all the domain and in the correction step, the solution is computed in the refined grid at small time steps using values at slave nodes interpolated from the coarse nodes solution obtained in the first stage. They show that the predictor corrector approach preserves the maximum principle satisfied by the solution of the scheme.

Our paper proposes a local time step strategy based on the domain decomposition framework. It extends the approach introduced in [5] by generalizing the interface conditions used to couple the coarse and refined time-step domains. The method is conservative. Stability and error estimates, which are different from that obtained in [5], are presented. A solution method which improves the predictor-corrector methods of [3] and [11] is proposed. In order to simplify the presentation and to concentrate on the difficulties arising from the local refinement in the time direction, we will first explain the approach in the case of a one-dimensional spatial problem in section 2. Stability and error estimates are stated in the more general case of nD spatial grids in section 3. Finally, some numerical results are presented in section 4.

## 2. Description of the local time stepping strategy

We consider the following problem: Let $T > 0$ and $\Omega$ be an open bounded domain of $\mathbb{R}^d$, $d \geqslant 1$, $p_0 : \Omega \mapsto \mathbb{R}$ and $f : \Omega \times (0,T) \mapsto \mathbb{R}$ be given functions. Find $p : \Omega \times [0,T] \mapsto \mathbb{R}$ such that

$$\frac{\partial p}{\partial t}(x,t) - \Delta p(x,t) = f(x,t), \qquad \forall x \in \Omega, \quad \forall t \in [0,T], \tag{1a}$$

$$p(x,0) = p_0(x), \qquad \forall x \in \Omega, \tag{1b}$$

$$p(x,t) = 0, \qquad \forall x \in \partial\Omega, \quad \forall t \in [0,T]. \tag{1c}$$

In order to explain the scheme, we consider the $d = 1$ case with the cell centered grid shown in Figure 1 and a time step which is variable in space. Namely, the domain $\Omega$ is decomposed into two non overlapping subdomains $\Omega_1$ and $\Omega_2$ where two different time-step sizes are used : the coarser time step is denoted $\delta t_2$ (in $\Omega_2$) and the finer time step is denoted $\delta t_1$ (in $\Omega_1$) such that $\mathcal{K}\delta t_1 = \delta t_2$ with $\mathcal{K} \in \mathbb{N}^*$ (Figure 2).

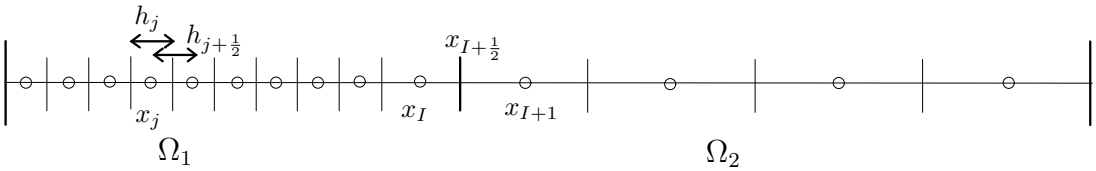


Figure 1. 1D cell-centered grid

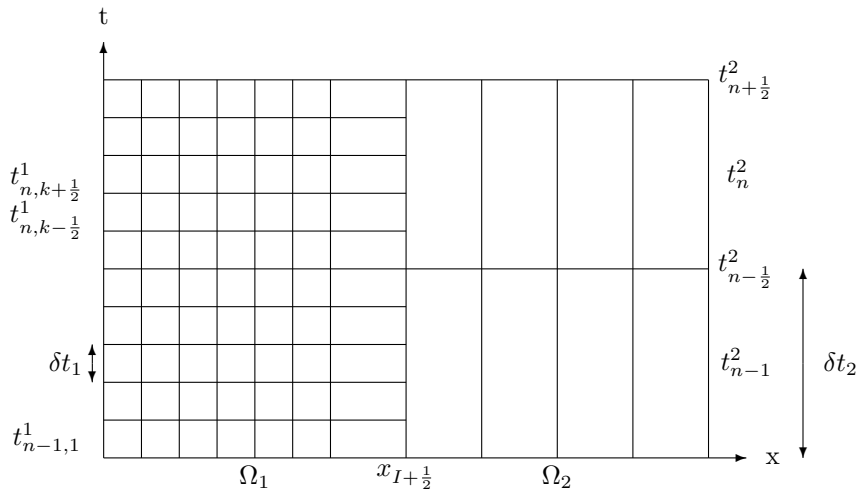


Figure 2. Time-space discretization.

The principle of the method is to consider the space-time domain $\Omega \times [t^2_{n-\frac{1}{2}}, t^2_{n+\frac{1}{2}}]$, decomposed into two non overlapping subdomains $\Omega_1 \times [t^2_{n-\frac{1}{2}}, t^2_{n+\frac{1}{2}}] \cup \Omega_2 \times [t^2_{n-\frac{1}{2}}, t^2_{n+\frac{1}{2}}]$. In a weak sense, one domain imposes its flux to the other one, which in return, imposes its solution. We now detail this idea in the framework for Finite Volumes.

## 2.1. **Discretization**

In each subdomain, the equation is discretised using a classical cell centered finite volume implicit scheme

$$\frac{h_j}{\delta t_1}(p_j^{n,k+1} - p_j^{n,k}) - (u_{j+\frac{1}{2}}^{n,k+1} - u_{j-\frac{1}{2}}^{n,k+1}) = h_j f_j^{n,k+1} \qquad \text{for } \ j \leq I, \tag{2a}$$

$$\frac{h_j}{\delta t_2}(p_j^{n+1} - p_j^{n}) - (u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1}) = h_j f_j^{n+1} \qquad \text{for } \ j > I, \tag{2b}$$

where $p_j^{n,k}$ is an approximation of the unknown in the cell $]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[ \times ]t_{n,k-\frac{1}{2}}, t_{n,k+\frac{1}{2}}[$ and $p_j^n$ in the space-time cell $]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[ \times ]t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}[$ (see Figures 1 and 2). We define

$$f_j^{n,k} = \frac{1}{h_j \delta t_1} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t_{n,k-\frac{1}{2}}}^{t_{n,k+\frac{1}{2}}} f(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad f_j^{n+1} = \frac{1}{h_j \delta t_2} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} f(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3}$$

Except for the boundary nodes (which are handled classically and not precised here) and interface node $x_{I+\frac{1}{2}}$, the flux approximations $u_{j+\frac{1}{2}}$ are given by

$$u_{j+\frac{1}{2}}^{n,k} = \frac{p_{j+1}^{n,k} - p_j^{n,k}}{h_{j+\frac{1}{2}}} \quad \text{if } \ j < I \quad \text{and} \quad u_{j+\frac{1}{2}}^{n+1} = \frac{p_{j+1}^{n+1} - p_j^{n+1}}{h_{j+\frac{1}{2}}} \quad \text{if } \ j > I. \tag{4}$$

For the approximation of the fluxes on the interface $u_{I+\frac{1}{2}}^{n+1}$ and $u_{I+\frac{1}{2}}^{n,k}$, we consider the space-time domain decomposition framework and introduce $p_{I+\frac{1}{2}}^{n+1}$ and $p_{I+\frac{1}{2}}^{n,k}$ the unknown approximations on the interface $x_{I+\frac{1}{2}}$. The flux approximations are classically obtained as

$$u_{I+\frac{1}{2}}^{n,k} = \frac{p_{I+\frac{1}{2}}^{n,k} - p_I^{n,k}}{0.5 h_I} \tag{5a}$$

$$u_{I+\frac{1}{2}}^{n+1} = \frac{p_{I+1}^{n+1} - p_{I+\frac{1}{2}}^{n+1}}{0.5 h_{I+1}} \tag{5b}$$

Discretizations in the two domains are linked by interface conditions which enforce flux and unknown continuity on the interface $x_{I+\frac{1}{2}} \times ]t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}[$. We consider the two possibilities

$$\delta t_2 u_{I+\frac{1}{2}}^{n+1} = \sum_{k=1}^{\mathcal{K}} \delta t_1 u_{I+\frac{1}{2}}^{n,k} \tag{6a}$$

$$p_{I+\frac{1}{2}}^{n,k} = p_{I+\frac{1}{2}}^{n+1}, \qquad k = 1, \ldots, \mathcal{K} \tag{6b}$$

and

$$u_{I+\frac{1}{2}}^{n,k} = u_{I+\frac{1}{2}}^{n+1} \qquad k = 1, \ldots, \mathcal{K} \tag{7a}$$

$$\delta t_2 p_{I+\frac{1}{2}}^{n+1} = \sum_{k=1}^{\mathcal{K}} \delta t_1 p_{I+\frac{1}{2}}^{n,k} \tag{7b}$$

These interface conditions can be rewritten in terms of the L2 orthogonal projections on sets of piecewise constant functions in time (see section 3.3). They correspond to interface scheme ($\mathrm{IS}_1$) defined in section 3.3. Both sets of conditions ensure local conservation for the coarse time step.

Another way to couple the fine and coarse grid which is somewhat more natural, is to introduce the unknown approximations $p_{I+\frac{1}{2}}^{n+1}$ and $p_{I+\frac{1}{2}}^{n,k}$ not on the interface but rather in the neighbouring cells $p_I^{n+1}$ and $p_{I+1}^{n,k}$. The flux approximations on the interface are then directly expressed as for an interior edge and instead of (5a)–(5b), we have

$$u_{I+\frac{1}{2}}^{n,k} = \frac{p_{I+1}^{n,k} - p_I^{n,k}}{h_{I+\frac{1}{2}}} \tag{8a}$$

$$u_{I+\frac{1}{2}}^{n+1} = \frac{p_{I+1}^{n+1} - p_I^{n+1}}{h_{I+\frac{1}{2}}} \tag{8b}$$

and the interface conditions become

$$\delta t_2 u_{I+\frac{1}{2}}^{n+1} = \sum_{k=1}^{\mathcal{K}} \delta t_1 u_{I+\frac{1}{2}}^{n,k} \tag{9a}$$

$$p_{I+1}^{n,k} = p_{I+1}^{n+1}, \qquad k = 1, \dots, \mathcal{K} \tag{9b}$$

or

$$u_{I+\frac{1}{2}}^{n,k} = u_{I+\frac{1}{2}}^{n+1}, \qquad k = 1, \dots, \mathcal{K} \tag{10a}$$

$$\delta t_2 p_I^{n+1} = \sum_{k=1}^{\mathcal{K}} \delta t_1 p_I^{n,k}. \tag{10b}$$

We have thus four possible coupling schemes

$$\text{(5a)–(5b)–(6)} \; ; \quad \text{(5a)–(5b)–(7)} \; ; \quad \text{(8a)–(8b)–(9)} \; ; \quad \text{(8a)–(8b)–(10)}$$

which are analyzed in the sequel. Equations (8a), (8b), (9) are the approximations proposed in [5]. These conditions correspond to interface conditions ($\mathrm{IS}_2$) of section 3.3. For technical reasons in the error estimate, we rewrite the unknowns $p_{I+1}^{n+1}$, $p_{I+1}^{n,k}$ in (9b) as a function of interface unknowns, i.e.,

$$p_{I+1}^{n+1} = p_{I+\frac{1}{2}}^{n+1} + \frac{h_{I+1}}{2} u_{I+\frac{1}{2}}^{n+1} \tag{11a}$$

$$p_{I+1}^{n,k} = p_{I+\frac{1}{2}}^{n,k} + \frac{h_{I+1}}{2} u_{I+\frac{1}{2}}^{n,k} \tag{11b}$$

Similarly,

$$p_I^{n+1} = p_{I+\frac{1}{2}}^{n+1} - \frac{h_I}{2} u_{I+\frac{1}{2}}^{n+1} \tag{12a}$$

$$p_I^{n,k} = p_{I+\frac{1}{2}}^{n,k} - \frac{h_I}{2} u_{I+\frac{1}{2}}^{n,k} \tag{12b}$$

in (10b). From a theoretical point of view, interface conditions ($\mathrm{IS}_1$) are easier to analyze than ($\mathrm{IS}_2$).

### 2.2. **Solution method**

To solve the system of algebraic equations for the unknowns values of the approximate solution between two coarse time levels, which includes all intermediate local time levels, we propose a method which combines the attractive feature of predictor-corrector approaches with the accuracy of domain decomposition type iterative algorithms. The method includes a predictor stage which corresponds to the computation of the solution on the coarse grid time and an iterative corrector stage where refined and coarse grids unknowns are solved alternatively, until interface conditions are satisfied, using a Schwarz multiplicative Dirichlet-Neumann algorithm [10]. If we consider the coupling strategy (8a)–(8b)–(9), the algorithm consists of

- Predictor stage: compute an approximate solution at coarse time step on the whole grid by

$$\frac{h_j}{\delta t_2}(\tilde{p}_j^{n+1} - p_j^n) - (\tilde{u}_{j+\frac{1}{2}}^{n+1} - \tilde{u}_{j-\frac{1}{2}}^{n+1}) = h_j f_j^{n+1} \quad \text{for all } j. \tag{13}$$

- Corrector iterative stage: solve alternatively the equations in domain $\Omega_1$ using (9b) interface condition and in domain $\Omega_2$ using (9a), until both interface conditions are satisfied simultaneously.

If the coupling strategy (8a)–(8b)–(10) is used, then the corrector iterative stage is changed to

- Corrector iterative stage: solve alternatively the equations in domain $\Omega_1$ using (10a) interface condition and in domain $\Omega_2$ using (10b), until both interface conditions are satisfied simultaneously.

These algorithms can also be easily written for the coupling strategies (5a)–(5b)–(6) and (5a)–(5b)–(7). We can notice that it is not necessary to iterate the corrector stage until convergence to obtain a conservative solution. It is sufficient to stop the process after the resolution of the domain where Neumann interface conditions are imposed. The algorithm proposed in [3] consists in the predictor stage (13) and in the first iteration of the corrective stage with (10a) interface condition. This solution method is not limited to linear discrete equations and can be extended to the resolution of the non linear equations that arise in petroleum recovery applications. Following the idea of [3], the predictor stage would then use a linear approximation of the problem in the refined domain in order to avoid convergence problems of the Newton algorithm, while the iterative corrector stage would consider the non linear problem.

## 3. Finite volume discretization

Problem (1) is rewritten as a domain decomposition problem. The domain $\Omega$ is decomposed into two non overlapping subdomains $\Omega_1$ and $\Omega_2$ ($\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and $\Omega_1 \cap \Omega_2 = \emptyset$). The interface is denoted by $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Problem (1) is equivalent to finding $p^1 : \Omega_1 \times [0,T] \mapsto \mathbb{R}$ and $p^2 : \Omega_2 \times [0,T] \mapsto \mathbb{R}$ such that

$$\frac{\partial p^i}{\partial t}(x,t) - \Delta p^i(x,t) = f(x,t), \qquad \forall x \in \Omega_i, \qquad \forall t \in [0,T], \quad \forall i \in \{1,2\}, \tag{14a}$$

$$p^i(x,0) = p_0(x), \qquad \forall x \in \Omega_i, \qquad \forall i \in \{1,2\}, \tag{14b}$$

$$p^i(x,t) = 0, \qquad \forall x \in \partial\Omega_i, \cap \partial\Omega, \quad \forall t \in [0,T], \quad \forall i \in \{1,2\}, \tag{14c}$$

$$\frac{\partial p^1}{\partial n_1}(x,t) + \frac{\partial p^2}{\partial n_2}(x,t) = 0, \qquad \forall x \in \Gamma, \qquad \forall t \in [0,T], \tag{14d}$$

$$p^1(x,t) = p^2(x,t), \qquad \forall x \in \Gamma, \qquad \forall t \in [0,T], \tag{14e}$$

where $n_i$ is the outward normal to domain $\Omega_i$, $i = 1, 2$. Problem (14) is discretized using a cell centered finite volume scheme in each subdomain [6]. We choose this scheme as an example but other schemes would be possible as well.

### 3.1. **Mesh and definitions**

For $i = 1, 2$, let $\mathcal{T}_i$ be a set of closed polygonal subsets associated with $\Omega_i$ such that $\bar{\Omega}_i = \cup_{K\in\mathcal{T}_i} K$. We shall denote $h = \max_{i\in\{1,2\},K\in\mathcal{T}_i} \mathrm{diam}(K)$ its mesh size. We shall use the following notations for all $i = 1, 2$.

- $\mathcal{E}_{\Omega_i}$ is the set of faces of $\mathcal{T}_i$.
- $\mathcal{E}_{iD}$ is the set of faces such that $\partial\Omega_i \cap \partial\Omega = \bigcup_{\epsilon\in\mathcal{E}_{iD}} \epsilon$ (let us recall that a Dirichlet boundary condition will be imposed on $\partial\Omega_i \cap \partial\Omega$).
- $\mathcal{E}_i$ is the set of faces such that $\partial\Omega_i \backslash \partial\Omega = \bigcup_{\epsilon\in\mathcal{E}_i} \epsilon$. Grids are matching on the interface so that

$$\mathcal{E} := \mathcal{E}_1 = \mathcal{E}_2 \,.$$

- $\forall K \in \mathcal{T}_i$,
  $\mathcal{E}(K)$ denotes the set of faces of $K$.
  $\mathcal{E}_{iD}(K) = \mathcal{E}(K) \cap \mathcal{E}_{iD}$ is the set of faces of $K$ which are on $\partial\Omega_i \cap \partial\Omega$.
  $\mathcal{E}_i(K) = \mathcal{E}(K) \cap \mathcal{E}_i$ is the set of faces of $K$ which are on $\partial\Omega_i \backslash \partial\Omega$.
  $\mathcal{N}_i(K) = \{K' \in \mathcal{T}_i | K \cap K' \in \mathcal{E}_{\Omega_i}\}$ is the set of the control cells adjacent to $K$ in $\Omega_i$.
  $K_i(\epsilon)$ denotes the cell of $\mathcal{T}_i$ adjacent to $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$.
- The time step in subdomain $\Omega_i$ is denoted by $\delta t_i$, and $N_i$ denotes the number of time steps of the simulation so that $N_1\delta t_1 = N_2\delta t_2 = T$. Parameters $\delta t_i$, $N_i$ satisfy

$$\frac{N_1}{N_2} = \frac{\delta t_2}{\delta t_1} = \mathcal{K} \in \mathbb{N}^* \tag{15}$$

- Let $[0, T]_{\delta t_i}$ denotes the discretization of the time interval $[0, T]$ in each subdomain $\Omega_i$ : $[0, T]_{\delta t_i} = (t_n^i)_{n=1,\ldots,N_i}$, with $t_n^i = (n - \frac{1}{2})\delta t_i$ ; since the time discretization in $\Omega_1$ is a refinement of that in $\Omega_2$, we shall also write : $t_{n,k}^1 = (n-1)\delta t_2 + (k - \frac{1}{2})\delta t_1$, $n = 1, \ldots, N_2$, $k = 1, \ldots, \mathcal{K}$

We make the following geometrical assumptions on the global mesh : $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$

**Assumption 3.1.** $\mathcal{T}$ is a finite volume admissible mesh, i.e., $\mathcal{T}$ is a set of closed subsets of dimension $d$ such that

- for any $(K, K') \in \mathcal{T}^2$ with $K \neq K'$, one has either $[K\,K'] := K \cap K' \in \mathcal{E}_{\Omega_1} \cup \mathcal{E}_{\Omega_2}$ or $\dim(K \cap K') < d-1$
- for $i = 1, 2$, there exist points $(y_\epsilon)_{\epsilon\in\mathcal{E}_{\Omega_i}}$ on the faces and points $(x_K)_{K\in\mathcal{T}_i}$ inside the cells such that (see figure 3)
  - for any adjacent cells $K$ and $K'$, the straight line $[x_K, x_{K'}]$ is perpendicular to the face $[K\,K']$ and $[x_K, x_{K'}] \cap [K\,K'] = \{y_\epsilon\}$
  - for any face $\epsilon \in \mathcal{E}_{iD}$, let $K(\epsilon) \in \mathcal{T}_i$ be such that $\epsilon \subset K$: then the straight line $[x_{K(\epsilon)}, y_\epsilon]$ is perpendicular to $\epsilon$
- Each mesh $\mathcal{T}_i$, $i = 1, 2$ has at least one interior cell.

**Notation 3.2.** *For all $\epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}$, $i = 1, 2$, $d_\epsilon^i$ denotes the distance between $x_{K_i(\epsilon)}$ and $y_\epsilon$.*

### 3.2. **Cell centered finite volume scheme in the subdomains**

The unknowns of the scheme and what they aim to approximate are ($i = 1, 2$)

$$p_K^{i,n} \simeq p^i(x_K, t_n), \qquad K \in \mathcal{T}_i, \tag{16a}$$

$$p_\epsilon^{i,n} \simeq p^i(y_\epsilon, t_n), \qquad \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}, \tag{16b}$$

$$u_\epsilon^{i,n} \simeq \frac{\partial p^i}{\partial n_i}(y_\epsilon, t_n), \qquad \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}, \tag{16c}$$

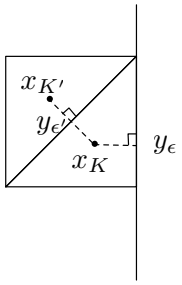


FIGURE 3. Assumption 3.1

The numerical flux is defined by

$$u_\epsilon^{i,n} = \frac{p_\epsilon^{i,n} - p_{K_i(\epsilon)}^{i,n}}{d_\epsilon^i}, \qquad \forall \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD}, \quad \forall n \in \{0, \dots, N_i - 1\}, \quad \forall i \in \{1, 2\}. \tag{17}$$

The scheme is defined by (see [6] for its derivation)

$$\begin{aligned} &\frac{p_K^{i,n+1} - p_K^{i,n}}{\delta t_i}\text{meas}(K) - \sum_{K' \in \mathcal{N}_i(K)} \frac{p_{K'}^{i,n+1} - p_K^{i,n+1}}{d(x_K, x_{K'})}\text{meas}([KK']) \\ &\quad - \sum_{\epsilon \in \mathcal{E}_i(K) \cup \mathcal{E}_{iD}(K)} u_\epsilon^{i,n+1}\text{meas}(\epsilon) = f_K^{i,n+1}\text{meas}(K), \qquad \forall K \in \mathcal{T}_i, \quad \forall n \in \{0, \dots, N_i - 1\}, \end{aligned} \tag{18}$$

where $d(x_K, x_{K'})$ is the distance between $x_K$ and $x_{K'}$ and $f_K^{i,n}$ is an approximation to

$$\frac{1}{\delta t_i \text{meas}(K)} \int_{t_{n-1/2}^i}^{t_{n+1/2}^i} \int_K f(x,t)\, dx\, dt. \tag{19}$$

The initial and boundary conditions are discretized by

$$p_K^{i,0} = p_0(x_K), \qquad \forall K \in \mathcal{T}_i, \quad \forall i \in \{1, 2\} \tag{20a}$$

$$p_\epsilon^{i,n+1} = 0, \qquad \forall \epsilon \in \mathcal{E}_{iD}, \quad \forall i \in \{1, 2\}, \quad \forall n \in \{0, \dots, N_i - 1\}. \tag{20b}$$

When there is no domain decomposition, this scheme has been analyzed in [6] in the more general case of discontinuous coefficients, and it is proven to be of order $O(h)$ for a discrete $H^1$-norm. In order to define the domain decomposition discretization scheme, we shall define in section 3.3 the matching conditions for the diffusive fluxes.

**Discrete spaces**. In all definitions, $C^t$ denotes a possibly different constant at each time on each cell.

- $$\begin{aligned} P_0(\mathcal{T}_i \times [0,T]_{\delta t_i}) = \big\{ p : \bar{\Omega}_i \times [0,T] \mapsto \mathbb{R} \backslash\ \forall n \in \{0, \dots, N_i - 1\} \\ \forall K \in \mathcal{T}_i \ \ p_{|K \times (t_{n-1/2}^i, t_{n+1/2}^i)} \equiv C^t \\ \text{and } \forall \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \ , \ p_{|\epsilon \times (t_{n-1/2}^i, t_{n+1/2}^i)} \equiv C^t \big\} \end{aligned} \tag{21}$$
- Similarly, $P_0(\mathcal{E} \times [0,T]_{\delta t_i})$ is the space of piecewise constant functions on the interface for the time mesh of subdomain $\Omega_i$.

- $P_0([0,T]_{\delta t_i}) = \left\{ p : [0,T] \mapsto \mathbb{R} \backslash \ \ \forall n \in \{0,\dots,N_i-1\}\ p_{|(t^i_{n-1/2},t^i_{n+1/2})} \equiv C^t \right\}$

These are spaces of piecewise constant functions.

Let $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, we denote its restriction

- to $\bar{\Omega}_i \times \{t^i_n\}$ by $p^{i,n}$, for $n \in \{0,\dots,N_i-1\}$
- to $\epsilon \times [0,T]_{\delta t_i}$ by $p^i_\epsilon$ for $\epsilon \in \mathcal{E}$
- to $\mathcal{E} \times [0,T]_{\delta t_i}$ by $p^i_{\mathcal{E}}$

For any $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, we introduce the norms

$$\|p^{i,n}\|^2_{L^2(\Omega_i)} = \sum_{K\in\mathcal{T}_i} (p^{i,n}_K)^2 \mathrm{meas}(K), \tag{22a}$$

$$\|p^i\|^2_{L^2(0,T;L^2(\Omega_i))} = \sum_{n=0}^{N_i} \delta t_i \|p^{i,n}\|^2_{L^2(\Omega_i)}, \tag{22b}$$

and the semi-norms

$$\begin{aligned} |p^{i,n}|^2_{1,\mathcal{T}_i} = \sum_{K\in\mathcal{T}_i} \Bigg( & \sum_{K'\in\mathcal{N}_i(K)} \frac{(p^{i,n}_{K'} - p^{i,n}_K)^2}{d(x_K,x_{K'})} \mathrm{meas}([KK']) \\ & + \sum_{\epsilon\in\mathcal{E}_i(K)} \frac{(p^{i,n}_\epsilon - p^{i,n}_K)^2}{d(x_K,y_\epsilon)} \mathrm{meas}(\epsilon) + \sum_{\epsilon\in\mathcal{E}_{iD}(K)} \frac{(p^{i,n}_K)^2}{d(x_K,y_\epsilon)} \mathrm{meas}(\epsilon) \Bigg) \end{aligned} \tag{23}$$

and

$$|p^i|^2_{1,\mathcal{T}_i,\delta t_i} = \sum_{n=0}^{N_i} \delta t_i |p^{i,n}|^2_{1,\mathcal{T}_i}. \tag{24}$$

**Définition 3.1.** Let $p^i, u^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, $i = 1,2$, we define a discrete scalar product by

$$\sum_{i=1}^{2} \left\langle u^i, p^i \right\rangle_{L^2(0,T;L^2(\Gamma))} := \sum_{i=1}^{2} \sum_{n=0}^{N_i-1} \delta t_i \sum_{\epsilon\in\mathcal{E}} u^{i,n+1}_\epsilon p^{i,n+1}_\epsilon \mathrm{meas}(\epsilon).$$

**Notation 3.3.** *Let $i = 1,2$, $p^i \in P_0(\mathcal{T}_i \times [0,T]_{\delta t_i})$, for $\epsilon \in \mathcal{E}$, $u^i_\epsilon(p)$ denotes the associated numerical flux defined by* (17)*. Very often, we will simply write $u^i_\epsilon$ and $u^i_{\mathcal{E}} = (u_\epsilon(p^i))_{\epsilon\in\mathcal{E}_i}$.*

## 3.3. **Finite volume on the interface**

In order to enforce the weak continuity of the primary unknown $p$ and of its normal derivative (denoted by $u$) across the interface $\Gamma \times [0,T]$, we introduce $\mathcal{Q}_i$ the $L^2$ orthogonal projection onto $P_0([0,T]_{\delta t_i})$. We have the compatibility condition

**Lemme 3.2.** For all $u_i \in P_0([0,T]_{\delta t_i})$, $i = 1,2$,

$$\langle \mathcal{Q}_1(u_2), u_1 \rangle_{L^2([0,T])} = \langle u_2, \mathcal{Q}_2(u_1) \rangle_{L^2([0,T])}.$$

As in mortar methods [1], we consider that one subdomain enforces the weak continuity of the primary unknown which is interpreted as the Dirichlet interface condition. This subdomain is called the master. The other subdomain is called the slave. It enforces the weak continuity of the normal derivative which corresponds to a Neumann interface condition. Since here the interfaces are non matching only in the time direction, it is possible to define matching conditions locally on each interface face $\epsilon \in \mathcal{E}$.

**Interface Scheme** ($\mathrm{IS}_1$) **based on interface unknowns**. The subscript $m$ will denote the master subdomain and $e$ the slave ($\{m,e\}=\{1,2\}$), the interface conditions on $\Gamma\times[0,T]$ read

$$\begin{aligned} u_\epsilon^m &= \mathcal{Q}_m(-u_\epsilon^e),\\ p_\epsilon^e &= \mathcal{Q}_e(p_\epsilon^m), \end{aligned} \tag{$\mathrm{IS}_1$}$$

for all $\epsilon\in\mathcal{E}$.

**Overlapping interface scheme** ($\mathrm{IS}_2$). Let $d_\epsilon^i$ denotes the quantity $d(x_{K_i(\epsilon)},y_\epsilon)$. The Dirichlet boundary condition is modified but not the Neumann one

$$\begin{aligned} u_\epsilon^m &= \mathcal{Q}_m(-u_\epsilon^e),\\ p_\epsilon^e + d_\epsilon^m u_\epsilon^e &= \mathcal{Q}_e(p_\epsilon^m - d_\epsilon^m u_\epsilon^m), \end{aligned} \tag{$\mathrm{IS}_2$}$$

for all $\epsilon\in\mathcal{E}$. The modified Dirichlet interface condition comes from the relations

$$u_\epsilon^e = \frac{\mathcal{Q}_e(p^m_{K_m(\epsilon)}) - p^e_{K_e(\epsilon)}}{d_\epsilon^m+d_\epsilon^e} = \frac{\mathcal{Q}_e(p_\epsilon^m - d_\epsilon^m u_\epsilon^m) - p_\epsilon^e + d_\epsilon^e u_\epsilon^e}{d_\epsilon^m+d_\epsilon^e}\,. \tag{25}$$

The first part of the above formula is somewhat natural. When writing the finite volume scheme for a cell $K_e(\epsilon)$ adjacent to the interface in the "slave" subdomain, it is necessary to approximate the flux on the face $\epsilon$. This is done using pressure values from both sides of the interface: the pressure in the "slave" subdomain and pressures values in the neighboring "master" subdomain. These last values are made compatible with the "slave" unknowns by using the transmission operator $\mathcal{Q}_e$. Finally, all quantities are expressed in terms of interface values in order to ease a comparison with ($\mathrm{IS}_1$).

Due to the fact that the large time step $\delta t_2$ is a multiple of the small time step $\delta t_1$, we have a simple form for the $L^2$ projection operators.

**Lemme 3.3.** *We have* $\mathcal{Q}_2 : P_0([0,T]_{\delta t_1}) \mapsto P_0([0,T]_{\delta t_2})$ *and* $\mathcal{Q}_1 : P_0([0,T]_{\delta t_2}) \mapsto P_0([0,T]_{\delta t_1})$

$$\text{For } v_2 \in P_0([0,T]_{\delta t_2}),\quad \mathcal{Q}_1(v_2)_{|(t^1_{n-1/2},t^1_{n+1/2})} = v_{2|(t^1_{n-1/2},t^1_{n+1/2})},\quad \forall\, n\in\{0,\dots,N_1-1\}, \tag{26a}$$

$$\text{For } v_1 \in P_0([0,T]_{\delta t_1}),\ \mathcal{Q}_2(v_1))_{|(t^2_{n-1/2},t^2_{n+1/2})} = \frac{1}{\delta t_2}\int_{t^2_{n-1/2}}^{t^2_{n+1/2}} v_1,\quad \forall\, n\in\{0,\dots,N_2-1\}. \tag{26b}$$

We also need the following technical assumptions. For the family of meshes we consider, the mesh close to the interface is not too stretched. This is expressed by

**Assumption 3.4.** *There exists a constant* $\alpha>0$ *such that* $d_\epsilon^m \leqslant \alpha d_\epsilon^e$ *for all* $\epsilon\in\mathcal{E}$.

We also need a geometric assumption

**Assumption 3.5.** *For all* $\epsilon\in\mathcal{E}$, $y_\epsilon$ *is the barycenter of the face* $\epsilon$ *and for* $i=1,2$,

$$\frac{\mathrm{diam}(\epsilon)^2}{d(x_{K_i(\epsilon)},y_\epsilon)} = O(h)$$

Let $p^1$, $p^2$ be the solution to the continuous problem (14a)–(14d). We define the interpolation on the mesh $\mathcal{T}_1 \cup \mathcal{T}_2$ at time $t_n$ by

$$\tilde{p}_K^{i,n} = p^i(x_K, t_n) \quad \forall K \in \mathcal{T}_i \tag{27a}$$

$$\tilde{p}_\epsilon^{i,n} = \frac{1}{\delta t_i} \int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\epsilon)} \int_\epsilon p^i \quad \forall \epsilon \in \mathcal{E}_i \tag{27b}$$

$$\tilde{p}_\epsilon^{i,n} = 0 \quad \forall \epsilon \in \mathcal{E}_{iD} \tag{27c}$$

$$\tilde{u}_\epsilon^{i,n} = \frac{1}{\delta t_i} \int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \frac{1}{\text{meas}(\epsilon)} \int_\epsilon \frac{\partial p^i}{\partial n_i} \quad \forall \epsilon \in \mathcal{E}_i \cup \mathcal{E}_{iD} \tag{27d}$$

We estimate the error terms $e_K^i = p_K^i - \tilde{p}_K^i$, $e_\epsilon^i = p_\epsilon^i - \tilde{p}_\epsilon^i$ and $q_\epsilon^i = u_\epsilon^i - \tilde{u}_\epsilon^i$.

**Théorème 3.4.** *We suppose that the solution has the regularity*

$$p \in C^1(0, T; C^2(\bar{\Omega})) \tag{28}$$

*and that the numerical right hand side is such that*

$$\frac{1}{\delta t_i \text{meas}(K)} \int_{t^i_{n-1/2}}^{t^i_{n+1/2}} \int_K (f_K^{i,n} - f) = O(\text{diam}(K) + \delta t_i), \quad \forall n \in \{0, \dots, N_i\}, \quad \forall i \in \{1, 2\}. \tag{29}$$

*We assume that Assumptions 3.1, 3.5 hold and that transmission scheme* ($\text{IS}_1$) *is used. Then, we have the estimate*

$$\sum_{i=1}^{2} |e^i|_{1,\mathcal{T}_i,\delta t_i} + \sum_{i=1}^{2} \|e^{i,N_i}\|_{L^2(\Omega_i)} = O(h + \delta t)$$

*where* $\delta t = \max(\delta t_1, \delta t_2)$. *The same estimate holds for transmission scheme* ($\text{IS}_2$) *if in addition Assumption 3.4 holds.*

The proof is given in [7].

## 4. Numerical results

### 4.1. Parabolic equation

We first illustrate the method with a parabolic equation coming from a previous article of Ewing and Lazarov [4]. We consider the ($\text{IS}_2$) interface conditions, i.e. equation (9) in one dimension, which are more natural, see § 2.1. We solve the following model problem :

$$\frac{\partial p}{\partial t}(x,t) - \frac{\partial^2 p}{\partial x^2}(x,t) = f(x,t) \qquad \forall x \in [0,1], \quad \forall t \in [0, 0.1], \tag{30a}$$

$$p(x,t) = 0, \qquad \forall x \in \partial\Omega, \tag{30b}$$

$$p(x,0) = 0, \qquad \forall x \in \partial\Omega. \tag{30c}$$

The function

$$p(x,t) = \exp(20(t - t^2) - 37x^2 + 8x - 1) \tag{31}$$

is used as the exact solution. This function represents a bump with a maximum value near the position $x = 0.15$. In the interval $[0.5, 1]$, the function is close to 0. In this interval, the function changes negligibly in time. In contrast, the function changes rapidly in time in the interval $[0, 0.5]$ and simulates a local behavior.

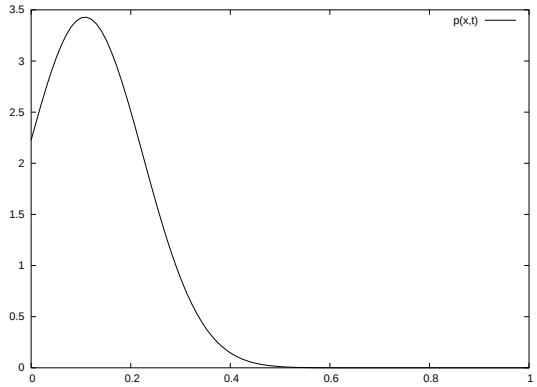


FIGURE 4. Exact solution for t=0.1

We use two different time step sizes, namely,

- a fine time step $\delta t_1 = 0.002$ in the subdomain $\Omega_1 = [0, 0.25]$, discretized with a fine grid $\delta x_1 = 0.01$
- a coarse time step $\delta t_2 = 0.02$ in the subdomain $\Omega_2 = [0.25, 1]$, discretized with a coarse grid $\delta x_2 = 0.05$

The interface is placed at $x = 0.25$. This is a worst case since the domain with local refinement only partially covers the interval $[0, 0.25]$ where the solution changes quickly.

We consider two cases. The first one (coarse master) is when the coarse domain enforces the Dirichlet condition, see equation (9). The second one (fine master) is when the refined domain enforces the Dirichlet condition, see equation (10). We make a comparison with the algorithm given by Mlacnik and Heinemann [8, 9]. In the following pictures, we plot the evolution of the errors in space and the time evolution of the $L^2$ norm of the error between the exact solution, the two local time step methods and the solution with the fine or coarse time step on the whole domain.

At each coarse time step, we solve the set of discretized equations using the iterative algorithm explained in section 2.2 with the stopping criterium $\varepsilon = 10^{-5}$. In the following figures, we plot the error between these two solutions and the exact solution. For completeness, we also plot the error for a computation with either the coarse or the fine time step on the whole domain. The number of iterations needed to reach the convergence is quite small; it is about 6 for the fine master method and about 8 for the coarse master method.

We notice that for both cases, the error is significantly smaller than the one of the coarse time step. Morever, for the fine master method, see equation(10), the error is close to the fine time step error in the refined zone, see figure 5.

As explained in section 2.2, it is not necessary to iterate until convergence the algorithm to obtain a conservative method. In figure 6, we plot the error after only one iteration of the corrector stage. As expected,the errors are larger than with the converged solutions. Let us recall, that the method proposed by Mlacnik corresponds to the fine master curve in figure 6.

## 4.2. **Two-phase flow**

We consider a simple model of an immiscible two-phase flow

$$\frac{\partial}{\partial t}(\rho_p \phi S_p) + \mathrm{div}(\rho_p \nu_p^r) = f_p \tag{32a}$$

$$\nu_p^r = -K \frac{k_{rp}(S_p)}{\mu_p} \mathrm{grad}(P - P_{c_p}) - \rho_p g, \tag{32b}$$

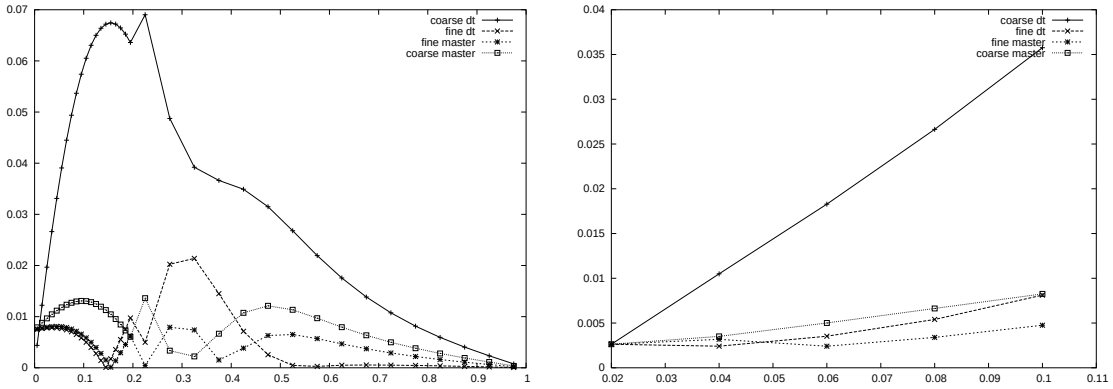


FIGURE 5. After convergence, left panel: error in space with the exact solution at time $t = 0.1$; right panel: time evolution of the $L^2$ norm of the error.

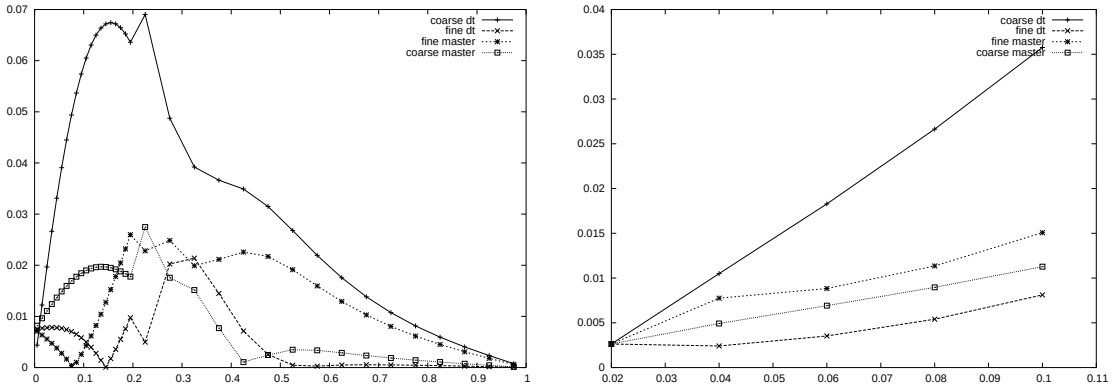


FIGURE 6. After one iteration, left panel: error in space with the exact solution at time $t = 0.1$; right panel: time evolution of the $L^2$ norm of the error.

where $p = w$ (for water) or $p = o$ (for oil). The main unknowns are the pressure $P$ and the saturation $S_p$ of phase $p$. The system is parabolic with respect to $P$ and non linear hyperbolic or degenerate parabolic if the capillary pressure $P_{c_p}$ is not neglected.

We further simplify the model by considering a one-dimensional well/reservoir coupling. The well is discretized with a fine mesh and a small time step $dt$ whereas the reservoir has a coarse mesh and a large time step $DT$. As the problem is non linear, the predictor stage uses a linear approximation of the problem in the well (in order to avoid convergence problems of the Newton algorithm) but keeps the non linear scheme in the reservoir domain. The iterative corrector stage considers the fully non linear problem in both domains. When solving in the well, the interface condition consists in the reservoir transmitting the fluxes $F_w$ and $F_o$ for water and oil. This corresponds to Neumann solves. When solving in the reservoir, the well provides the Dirichlet data for the pressure $P$ and one of the saturation (say $S_o$). This corresponds to Dirichlet solves.

For the numerical test shown here, $DT/dt = 10.$ and we had to use 2 to 5 iterations per coarse time step to solve the coupled system. In figures 7 and 8, the solution obtained with the same fine time step $dt$ in the reservoir and the well is compared at various time steps to the solution obtained with a coarse time step in the reservoir keeping the fine time step in the well.

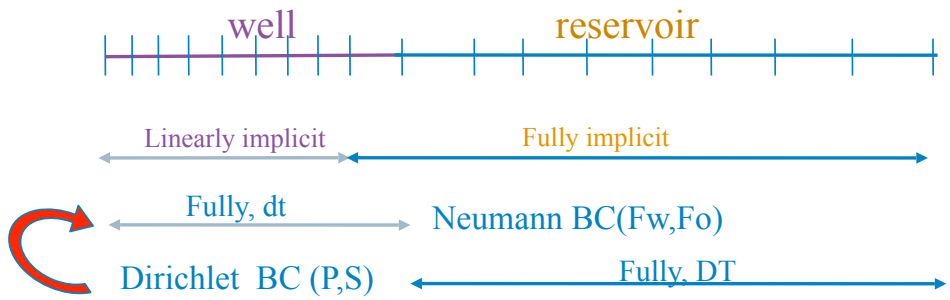


FIGURE 7. Well/Reservoir coupling for a two-phase flow

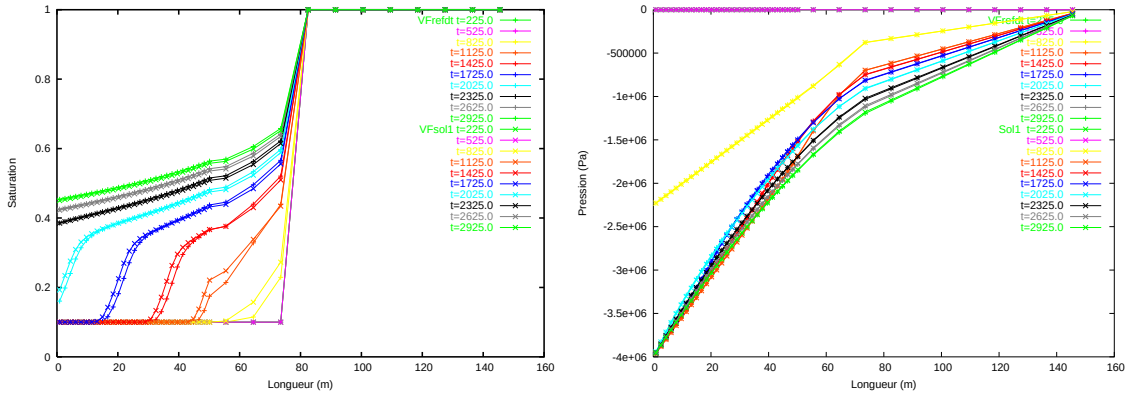


FIGURE 8. Saturation (left) and pressure (right) in space at different time steps. A reference solution is compared to the local time step solution.

## 5. CONCLUSION

We have proposed a local time step strategy for solving problems on grids with different time steps in different regions. We have proposed two schemes: ($IS_1$) and ($IS_2$). In ($IS_1$), the coupling involves additional interface unknowns. Scheme ($IS_2$) is written only in terms of "classical" finite volume unknowns. Both schemes are conservative, of order one in space and time. The assumptions are more restrictive for ($IS_2$) than for ($IS_1$). We have presented an iterative solution method for solving the composite grid system. Its main feature is that at every stage, conservativity is ensured. Numerical tests on toy problems (a parabolic equation and a one dimensional immiscible two phase flow) confirm the capabilities of the method. The scheme is being implemented in a multiphase three dimensional simulation code.

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