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BOUNDS OF RUIN PROBABILITIES FOR INSURANCE COMPANIES IN THE PRESENCE OF STOCHASTIC VOLATILITY ON INVESTMENTS *, **

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Abstract. In this work we consider a model of an insurance company where the insurer has to face a claims process which follows a Compound Poisson process with finite exponential moments. The insurer is allowed to invest in a bank account and in a risky asset described by Geometric Brownian motion with stochastic volatility that depends on an external factor modelled as a diffusion process. By using exponential martingale techniques we obtain upper and lower bounds for the ruin probabilities, that recover the known bounds for constant volatility models. Finally we apply the results to a truncated Scott model.

Résumé. Dans ce travail, nous considérons un modèle d'une compagnie d'assurance où l'assureur doit faire face a un processus de sinistres qui suit un processus de Poisson Composé avec des moments exponentiels finis. L'assureur est autorisé à investir dans un compte bancaire et un actif à risque décrit par le mouvement Brownien Géométrique à volatilité stochastique qui dépend d'un facteur externe modélisé comme un processus de diffusion. En utilisant des techniques de martingale exponentielle, nous obtenons une borne supérieur et une borne inférieur pour les probabilités de ruine, qui récupèrent les bornes connues pour les modèles de volatilité constante. Enfin, nous appliquons les résultats au modèle de Scott tronqué.

Introduction

Since the classical papers of Cramér and Lundberg which first considered the ruin problem of an insurance company in conditions of uncertainty, risk theory has attracted a lot of attention and many works has been done in this area. When the claims admit a Laplace transform, and satisfy the safety loading condition, the ruin probability $\psi(x)$ for the classical Cramér-Lundberg Model satisfies:

$$Ce^{-\nu x} \leq \psi(x) \leq e^{-\nu x},$$

where ν is the Lundberg coefficient and x is the initial wealth, (see for example [1,12]). An additional feature is considered in [3,6,13], the insurer is allowed to invest in a risky asset modelled by the Geometric Brownian motion. Via the Hamilton-Jacobi-Bellman equation Hipp and Plum in [6] prove the existence of an optimal strategy that minimizes the ruin probabilities and Gaier et al. in [3] obtain the exponential bounds:

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$$Ce^{-\theta x} \le \psi(x) \le e^{-\theta x},$$

where θ is the root of the equation

$$\lambda(M_Y(\theta) - 1) = c\theta + \frac{\mu^2}{2\sigma^2},$$

and $M_Y(\theta) = \mathbb{E}[e^{\theta Y}]$. The upper bound is obtained by considering a constant strategy given by

$$K = \frac{\mu}{\theta\sigma^2}, \tag{1}$$

and using maximal exponential martingales inequalities. By using control techniques Hipp and Schmidli in [7] showed that this strategy is asymptotically optimal as the initial wealth goes to infinity. Observe that θ *ties* in some way the return of the risky asset and the wealth of the Cramér-Lundberg process.

Empirical observations of financial market show that some indicators of market volatility behave in a highly erratic manner, which make unrealistic to assume μ, σ and r constants over long time periods. This fact has motivated several authors to study the so called stochastic volatility models (see between others [5, 14, 15]).

In this paper we consider an insurance company whose wealth is given by the classical Cramér-Lundberg process

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

a bank account and a risky asset described by the following stochastic volatility model (see [2, 4])

$$dS_t^0 = S_t^0 r(Z_t)dt, \tag{2}$$

$$dS_t = S_t(\mu(Z_t)dt + \sigma(Z_t)dW_{1t}) \quad \text{with } S_0 = 1, \tag{3}$$

where Z is an external factor modelled as a diffusion process

$$dZ_t = g(Z_t)dt + \beta(\rho dW_{1t} + \varepsilon dW_{2t}) \quad \text{with } Z_0 = z \in \mathbb{R}, \tag{4}$$

where $W_{1,t}$ $W_{2,t}$ are independent Brownian motions. $\mid \rho \mid \le 1, \varepsilon = \sqrt{1-\rho^2}$ and $\beta \neq 0$.

The process Z_t can be interpreted as the behavior of some economic factor, which has an impact on the dynamics of the risky asset and the bank account (see for example [2, 4, 10]).

Our aim is to obtain upper and lower bounds for the ruin probability of exponential type of the form $e^{-\theta(z)x}$ for some function θ, when the insurer is allowed to invest in this market.

One could think that a good function to obtain a bound of the form $e^{-\theta(z)x}$ should be the extension of the constant one given by (1) i.e. for each $z \in \mathbb{R}$, $\theta(z)$ is the root of the equation

$$\lambda(M_Y(\theta(z)) - 1) = c\theta(z) + \frac{\mu^2(z)}{2\sigma^2(z)}.$$

However, as we will see in Example 3.3 this type of models are complex, and it is more difficult to handle with.

The paper is organized as follows: Section 1 is devoted to formulate the problem. In Section 2 we will give general expressions for the exponential martingales that will be used all along the paper. We study the upper bound in Section 3, and we give a theorem with general and abstract conditions for the existence of a function θ that guarantee that the upper bound is of the form $e^{-\theta(z)x}$. We illustrate this theorem with truncated Scott models, and we consider two cases of functions θ, for which a strategy K and the bound are obtained explicitly. Finally in Section 4 we obtain the lower bound, under the usual hypothesis of uniform exponential moment of the tail distribution of the claims (see [3]).

1. Formulation of the problem

We start by formulating the model of an insurance company allowed to invest in a risky asset and in a money market in the presence of stochastic volatility. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which carries the following independent stochastic processes.

- A Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda > 0$ and jump times $\{T_i\}_{i\geq 1}$.
- A sequence $\{Y_i\}_{i\geq 1}$ of iid positive random variables with common distribution G.
- W_{1t} and W_{2t} are independent standard Brownian motions.
- The filtration $\mathcal{F}_t$ is defined by

$$\mathcal{F}_t = \sigma\left\{W_{1s}, W_{2s}, Y_i \mathbf{1}_{[i\leq N_s]}, 0 \leq s \leq t, i \geq 1\right\}$$

with the usual conditions.

We will consider that wealth $R_t, t \geq 0$ of the insurance company is given by the Cramér-Lundberg process, i.e.

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

where x is the initial capital and $c > 0$ represents the primes.

Let $S_t^0,\ S_t,\ t \geq 0$ be the bank account and the risky asset respectively described as follows

$$dS_t^0 = S_t^0 r(Z_t)dt, \tag{5}$$

where $r(\cdot)$ is the interest rate function.

$$dS_t = S_t(\mu(Z_t)dt + \sigma(Z_t)dW_{1t}) \quad \text{with } S_0 = 1, \tag{6}$$

where Z is an external factor modelled as a diffusion process

$$dZ_t = g(Z_t)dt + \beta(\rho dW_{1t} + \varepsilon dW_{2t}) \quad \text{with } Z_0 = z \in \mathbb{R}, \tag{7}$$

where $\mid \rho \mid\leq 1, \varepsilon = \sqrt{1-\rho^2}$ and $\beta \neq 0.$ The parameter ρ is the correlation coefficient between W_{1t} and $\tilde{W} = \rho W_{1t} + \varepsilon W_{2t}$, then the external factor can be written as

$$dZ_t = g(Z_t)\, dt + \beta\, d\tilde{W}_t, \quad Z_0 = z.$$

For instance the external factor can be modelled by the mean reverting Ornstein-Uhlenbeck (O-U) process:

$$dZ_t = \gamma(\delta - Z_t)\, dt + \beta\, d\tilde{W}_t, \quad Z_0 = z$$

where γ and δ are constants and the risky asset price can be given by the Scott model, [10]:

$$dS_t = S_t(\mu_0 dt + e^{Z_t} dW_{1t}) \quad \text{with } S_0 = 1, \tag{8}$$

and μ_0 is constant.

We consider an insurer with wealth X_t that invests an amount K_t in the risky asset and the remaining reserve $X_t - K_t$ in the bank account. Let $\mathcal{K}$ the set of admissible strategies given by:

$$\mathcal{K} = \{K = (K_t)_{t\geq 0} : K \text{ is predictable and adapted to } \mathcal{F}_t \\ \text{and } \mathbb{P}[\int_0^t K_s^2\, ds < \infty] = 1 \text{ for all } t \in [0, \infty[\,\}.$$

Then, for each $K \in \mathcal{K}$ the wealth process $X_t^K := X(t,x,z,K)$ can be written as:

$$X_t^K = x + \int_0^t c + K_s\mu(Z_s) + (X_s - K_s)r(Z_s)\,ds + \int_0^t K_s\sigma(Z_s)\,dW_{1s} - \sum_{i=1}^{N_t} Y_i, \tag{9}$$

where $x \geq 0$ is the initial reserve of the insurance company, c is the constant premium rate. For each strategy $K \in \mathcal{K}$, the ruin time in infinite horizon is

$$\tau^K := \tau(x,z,K) = \inf\left\{t : X_t^K < 0\right\}.$$

Then the ruin probability is given by

$$\psi(x,z,K) = \mathbb{P}[\tau^K < \infty].$$

Our aim is to obtain bounds for

$$\psi(x,z) = \inf_{K\in\mathcal{K}} \psi(x,z,K),$$

under the following hypothesis:

Hypothesis A. (1) *The functions $\mu(\cdot), \sigma(\cdot)$ and $g(\cdot)$ are such that there exists a strong solution for equations (6) and (7). The function $r(\cdot)$ is continuous, positive and*

$$r(z) < \mu(z), \text{ for all } z \in \mathbb{R}.$$

(2) *Let Y be a random variable with the common distribution G of the claims. There exists $\theta_\infty \in (0,\infty]$ such that $M_Y(\theta) = \mathbb{E}[e^{\theta Y}] < \infty$ for all $\theta \in [0,\theta_\infty)$ and $h(\theta) = M_Y(\theta) - 1$, satisfies*

$$\lim_{\theta\to\theta_\infty} h(\theta) = \infty.$$

All along the paper we will assume **Hypothesis A.** without no further mention.

2. Decomposition of $e^{-\theta(Z_t)X_t^K}$

In this section we will introduce some notation that will be used all along the paper.

We will denote by

$$\alpha_t := \int_0^t r(Z_s)\,ds.$$

Let $\theta : \mathbb{R} \to]0,\infty[$ in $C_b^2(\mathbb{R})$ (twice differentiable functions with bounded derivatives), and K an admissible strategy, then Itô's lemma implies that:

$$\begin{aligned} e^{-\alpha_t}\theta(Z_t)X_t^K \quad = \quad & \theta(z)x + \int_0^t e^{-\alpha_s}\theta(Z_s)(c + (\mu(Z_s) - r(Z_s))K_s)\,ds + \\ & \int_0^t e^{-\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)\,dW_{1s} + \\ & \int_0^t \varepsilon\beta e^{-\alpha_s}\theta'(Z_s)X_s\,dW_{2s} + \int_0^t e^{-\alpha_s}\theta'(Z_s)X_s g(Z_s)\,ds + \\ & \frac{1}{2}\int_0^t \beta^2 e^{-\alpha_s}\theta''(Z_s)X_s\,ds + \int_0^t \rho\beta\sigma(Z_s)K_s e^{-\alpha_s}\theta'(Z_s)\,ds - \\ & \int_0^t\int_0^\infty y\theta(Z_s)e^{-\alpha_s}\,\bar{N}(dy,ds). \end{aligned}$$

where $\bar{N}$ is the Poisson random measure on $\mathbb{R}_+ \times [0,\infty[$ defined by

$$\bar{N} = \sum_{n\geq 1} \delta_{(T_n,Y_n)}.$$

Proposition 2.1. *For each function $\theta \in C_b^2((\mathbb{R})$ and $K \in \mathcal{K}$, let*

$$H_\theta^{K,r}(t,x,z) = \exp\left\{-e^{-\alpha_t}\theta(Z_t)X_t^K\right\}.$$

Then

$$H_\theta^{K,r}(t,x,z) = e^{-\theta(z)x} e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)} e^{\int_0^t f_{\theta,r}(e^{-\alpha_s}, X_s, Z_s, K_s)\, ds},$$

where

$$f_{\theta,r} : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},$$

is given by

$$\begin{aligned} f_{\theta,r}(u,x,z,K) \quad &= \quad \lambda h(\theta(z)u) - (c + (\mu(z)-r(z))K)\theta(z)u - xu\theta'(z)g(z) \\ &\quad -\tfrac{1}{2}\beta^2 xu\theta''(z) + \tfrac{1}{2}(Ku\sigma(z)\theta(z) + \rho\beta u\theta'(z)x)^2 \\ &\quad +\tfrac{1}{2}\varepsilon^2\beta^2u^2x^2\theta'^2(z) - \rho\beta uK\sigma(z)\theta'(z). \end{aligned} \tag{10}$$

$\mathcal{D}_\theta^K(t)$ and $\mathcal{E}_\theta(t)$ are the local martingales given by

$$\begin{aligned} \mathcal{D}_\theta^K(t) = &-\int_0^t e^{-\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)dW_{1s} - \int_0^t \varepsilon\beta e^{-\alpha_s}\theta'(Z_s)X_s dW_{2s} \\ &-\frac{1}{2}\int_0^t e^{-2\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)^2 ds - \frac{1}{2}\int_0^t \varepsilon^2\beta^2 e^{-2\alpha_s}\theta'^2(Z_s)X_s^2\, ds, \end{aligned} \tag{11}$$

$$\mathcal{E}_\theta(t) = \int_0^t\int_0^\infty y\theta(Z_s)e^{-\alpha_s}\,\bar{N}(dy,ds) - \lambda\int_0^t\int_0^\infty (e^{\theta(Z_s)ye^{-\alpha_s}} - 1)\, dG(y)\, ds. \tag{12}$$

The proof of the Proposition is straightforward and we leave it.

When the rate $r = 0$ we have the following corollary, that will be used in Example 3.3 and in the estimation for the lower bound.

Corollary 2.2. *If $r = 0$, then:*

$$H_\theta^{K,0}(t,x,z) = e^{-\theta(z)x} e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)} e^{\int_0^t f_\theta^*(X_s, Z_s, K_s)\, ds},$$

where

$$f_\theta^*(x,z,K) := f_{\theta,0}(1,x,z,K).$$

3. The upper Bound

Our aim in this section is to get an upper bound for the ruin probabilities of the form $e^{-\theta(z)x}$, for some function θ.

Theorem 3.1. *If there exists an admissible strategy $K \in \mathcal{K}$ and $\theta : \mathbb{R} \to]0,\infty[$ in $C_b^2(\mathbb{R})$, such that $H_\theta^{K,r}(t,x,z)$ is a supermartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ then:*

$$\psi(x,z) \leq C_r e^{-\theta(z)x},$$

where

$$0 \leq C_r = \inf_{K\in\mathcal{K}} \frac{1}{\mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]} \leq 1.$$

Proof. Since for each strategy $K \in \mathcal{K}$

$$X_t^K < 0 \text{ if and only if } e^{-\alpha_t}\theta(Z_t)X_t^K < 0,$$

it is equivalent to study the ruin probability for this process. By hypothesis $H_\theta^{K,r}(t,x,z)$ is a supermartingale then by the optional sampling theorem we get:

$$\begin{aligned} e^{-\theta(z)x} &\geq \mathbb{E}[H_\theta^{K,r}(t\wedge\tau^K,x,z)] \\ &\geq \mathbb{E}[H_\theta^{K,r}(\tau^K,x,z)\mathbf{1}_{\{\tau^K<t\}}]. \end{aligned}$$

Now letting $t\to\infty$,

$$e^{-\theta(z)x} \geq \mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]\mathbb{P}[\tau^K < \infty],$$

therefore

$$\psi(x,z,K) = \mathbb{P}[\tau^K < \infty] \leq \frac{e^{-\theta(z)x}}{\mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]},$$

and

$$\psi(x,z) = \inf_{K\in\mathcal{K}} \psi(x,z,K) \leq \inf_{K\in\mathcal{K}} \frac{e^{-\theta(z)x}}{\mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]},$$

then

$$\psi(x,z) \leq C_r e^{-\theta(z)x},$$

where

$$C_r = \inf_{K\in\mathcal{K}} \frac{1}{\mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]}.$$

□

Remark 3.2.

(1) Observe that given a function θ, for each $x,z\in\mathbb{R}$ fixed the function $f_{\theta,r}(u,x,z,K)$ given by expression (10) is a quadratic form in K. This suggests to take K as a root of this equation. The point here is the existence of it. Example 3.3 is a particular case when this root exists.

(2) If we can take K as a root, then from Proposition 2.1:

$$H_\theta^{K,r}(t,x,z) = e^{-\theta(z)x}e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)}.$$

Then we have that $H_\theta^{K,r}(t,x,z)$ is a local martingale , since it is the product of a continuous and a pure jump local martingales, and since it is positive is a supermartingale.

In the following example we consider a function θ for a truncated Scott model. We could consider a truncation via the stochastic volatility in such a way that it belongs to $C_b^2([-m,m])$ (see for example [2]), however for avoid technicalities we stop the process Z.

Example 3.3. We assume that the claims are exponentially distributed with parameter $\eta > 0$. Also we assume that $\rho = 1$ and $r = 0$. We consider a truncated Scott model in the following sense:

Let $Z_t,\ t \geq 0$ given by

$$dZ_t = \gamma(\delta - Z_t)dt + \beta dW_{1t}, \quad Z_0 = z$$

and for each $m > 0$ define the stopping time τ_m as follows

$$\tau_m = \inf\{t > 0, \mid Z_t \mid > m\}.$$

Let $Z_t^m = Z_{t\wedge\tau_m}$, and

$$dS_t^m = S_t^m(\mu_0 dt + e^{Z_t^m} dW_{1t}), \quad S_0^m = 1.$$

From the convex property of h and **Hypothesis A.2** ($\lim\limits_{\theta\to\theta_\infty} h(\theta) = \infty$), we have that for each $-m \leq z \leq m$, $\theta(z)$ is defined as the positive solution of:

$$\lambda h(\theta(z)) = c\theta(z) + \frac{\mu_0^2}{2e^{2z}}, \quad \text{with } \theta(z) < \eta. \tag{13}$$

Straightforward calculations show that $\theta(z)$ satisfies:

$$\theta^2(z) + \frac{1}{c}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right)\theta(z) - \frac{\eta\mu_0^2}{2c}e^{-2z} = 0, \tag{14}$$

and is given by:

$$\theta(z) = \frac{1}{2}\left[-\frac{1}{c}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right) + \sqrt{\frac{1}{c^2}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right)^2 + \frac{2\eta\mu_0^2}{c}e^{-2z}}\right]. \tag{15}$$

It is clear that $\theta(z) \in C_b^2$, for $z \in [-m, m]$ and $\theta(z) \leq \eta$. Then from the exponential decomposition given in Corollary 2.2 we define the following equation:

$$f_\theta^*(x, z, K) = \frac{1}{2}\theta^2(z)\sigma^2(z)K^2 + b(x, z)K + a(x, z) = 0, \tag{16}$$

where

$$a(x, z) = \frac{1}{2}\beta^2\theta'^2(z)x^2 - \Big(g(z)\theta'(z) + \frac{1}{2}\beta^2\theta''(z)\Big)x + \frac{\mu_0^2}{2}e^{-2z},$$

and

$$b(x, z) = \Big(-\mu_0\theta(z) - \beta\sigma(z)\theta'(z) + \beta x\sigma(z)\theta(z)\theta'(z)\Big).$$

Then from Remark 3.2 we can define $K_*(x, z)$ as the solution of (16) given by

$$K_*(x, z) = \frac{-b(x, z) + \sqrt{b^2(x, z) - 2a(x, z)\theta^2(z)\sigma^2(z)}}{\theta^2(z)\sigma^2(z)},$$

if this root exists. We do not have general conditions on the coefficients for it's existence, however, in some particular cases holds. For example if we take

$$\mu_0 = 0.01,\ c = 1,\ \lambda = 1,\ \beta = 10,\ g(z) = 0.01(1 - z), \eta = 1.1,$$

with the help of Mathematica package v.7 it can be shown that for $m = -\frac{1}{2}\log(2\mu_0 c\eta + 2\mu_0\lambda)$, equation (16) admits a positive solution. The following figures show the behavior of the function $\theta(z)$ and the strategy $K_*(x, z)$ for different values of z.

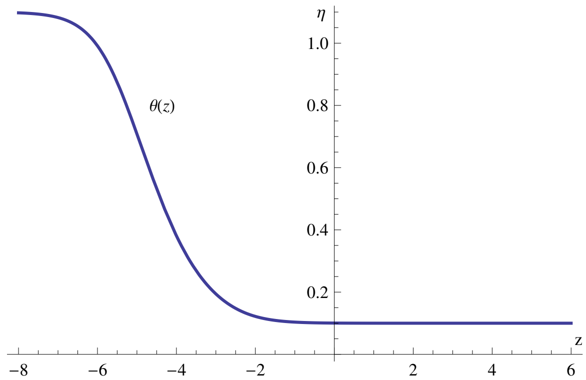

FIGURE 1. The function $\theta(z)$

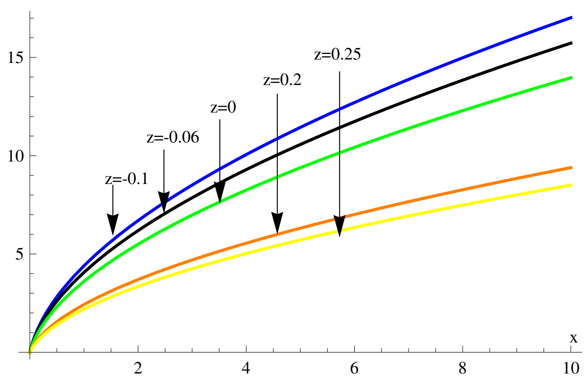

FIGURE 2. The admissible strategy $K_*(x, z)$

The following corollary gives an upper bound for the ruin probability when the return from the bound and the risky asset is bounded from below.

Corollary 3.4. *We assume that there exists a constant $R_1 > 0$ such that*

$$0 < R_1 \leq \frac{\mu(z) - r(z)}{\sigma(z)} \quad \forall z \in \mathbb{R}. \tag{17}$$

Then the ruin probability $\psi(x, z)$ of an insurer, investing in a risky asset, can be bounded from above by

$$\psi(x, z) \leq e^{-\hat{\theta} x},$$

where $0 < \hat{\theta} < \theta_\infty$ is the unique positive solution of

$$\lambda h(\theta) = c\theta + \frac{1}{2} R_1^2. \tag{18}$$

Proof. The existence of $\hat{\theta}$ is a consequence of the convex property of $h(\theta)$ and **Hypothesis A.2**. From Theorem 3.1 we only need to prove that the process $H_{\hat{\theta}}^{\hat{K},r}(t,x,z)$ is a supermartingale where,

$$\hat{K}_t = \frac{\mu(Z_t) - r(Z_t)}{\hat{\theta}\sigma^2(Z_t)}.$$

Since $\hat{\theta}$ is constant we have that the function $f_{\hat{\theta},r}$ given in Proposition 2.1 is reduced to

$$\begin{aligned} f_{\hat{\theta},r}(e^{-\alpha_s}, x, z, \hat{K}) &= \lambda h(\hat{\theta}e^{-\alpha_s}) - (c + (\mu(z) - r(z))K)\hat{\theta}e^{-\alpha_s} + \frac{1}{2}(\hat{K}e^{-\alpha_s}\hat{\theta}\sigma(z))^2 \\ &= \lambda h(\hat{\theta}e^{-\alpha_s}) - \left(c\hat{\theta} + \frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right)e^{-\alpha_s} + \frac{1}{2}e^{-2\alpha_s}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)} \\ &\leq \lambda h(\hat{\theta}e^{-\alpha_s}) - \left(c\hat{\theta} + \frac{1}{2}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right)e^{-\alpha_s} \\ &\leq e^{-\alpha_s}\left(\lambda h(\hat{\theta}) - c\hat{\theta} - \frac{1}{2}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right). \end{aligned}$$

The last inequality follows from the fact that the function $q(x) = e^{px} - pe^x$ with $x \geq 0$ and $0 \leq p \leq 1$ fixed is decreasing and by taking $x = \hat{\theta}y$, $p = e^{-\alpha_s}$ the inequality follows. Finally, from the definition of $\hat{\theta}$ we have

$$f_{\hat{\theta},r}(e^{-\alpha_s}, X_s, Z_s, \hat{K}_s) \leq 0,$$

and then we get that $H_{\hat{\theta}}^{\hat{K},r}(t,x,z)$ is a supermartingale. □

Remark 3.5. (1) If $c > \lambda\mu$ then the Lundberg coefficient $\nu > 0$ exists and we have $\hat{\theta} > \nu$.

(2) If $\inf_z \frac{\mu(z) - r(z)}{\sigma(z)} = 0$ following a similar procedure as in Corollary 3.4, we can obtain that: $\psi(x,z) \leq e^{-\nu x}$, under the assumption that $c > \lambda\mu$.

(3) Observe that $\mu(z) - r(z)$ represents the premium return from investing in the risky asset, then too small values of $\frac{\mu(z)-r(z)}{\sigma(z)}$ correspond to very large volatilities. So R_1 can be viewed in some sense as a measure of the risk aversion of the investor.

(4) When $\mu(z)$ and $\sigma(z)$ are constants and $r = 0$, we obtain the same bound as in [3].

$$\psi(x,z) \leq e^{-\hat{\theta}x},$$

where $\hat{\theta}$ is the unique positive solution of:

$$\lambda h(\theta) = c\theta + \frac{\mu^2}{2\sigma^2}.$$

(5) The case when μ, σ and r are constants, was studied in [3] under the restriction that the interest force is equal to the inflation force, and only the prime and the claims are affected by the inflation. In this case the wealth process becomes

$$\begin{aligned} X_t^K = x &+ \int_0^t (ce^{rs} + K_s\mu + (X_s - K_s)r)\,ds \\ &+ \int_0^t K_s\sigma\,dW_{1s} - \sum_{i=1}^{N_t} e^{rT_i}Y_i. \end{aligned}$$

It is not so clear for us this assumption, it seems that it is used only for technical reasons. In our case we can deal without it .

Example 3.6. As an application of Corollary 3.4, we consider another truncated Scott Model. Assume $0 < r < \mu_0$ is constant, and let $Z_t,\ S_t^{1,m}$ $t \geq 0$ be given by

$$dZ_t = \gamma(\delta - Z_t)dt + \beta dW_{1t}, \quad Z_0 = z,$$

and for each $m > 0$,

$$dS_t^{1,m} = S_t^{1,m}(\mu_0 dt + \sigma(Z_t)dW_{1t}), \quad S_0^{1,m} = 1,$$

where

$$\sigma(z) = \begin{cases} e^z & \text{if} \quad z \in [-m, m], \\ e^m & \text{if} \quad z \in]m, \infty[,\ \text{for some } m > 0. \\ e^{-m} & \text{if} \quad z \in]-\infty, -m[, \end{cases}$$

Then $R_1 = (\mu_0 - r)e^{-m}$. In particular if the claims are exponentially distributed with parameter η, we have that (18) becomes:

$$\lambda\left(\frac{\eta}{\eta - \theta} - 1\right) = c\theta + \frac{(\mu_0 - r)^2}{2}e^{-2m}$$

which leads to:

$$\theta^2 + \frac{1}{c}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right) - \frac{\eta(\mu_0 - r)^2}{2c}e^{-2m} = 0.$$

Then

$$\begin{aligned} \hat{\theta} \quad &= \quad \frac{1}{2}\left[-\frac{1}{c}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right)\right. \\ &\quad \left. + \sqrt{\frac{1}{c^2}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right)^2 + \frac{2\eta(\mu_0 - r)^2}{c}e^{-2m}}\right]. \end{aligned}$$

The admissible strategy is given by:

$$\hat{K}_t = \frac{\mu_0 - r}{\hat{\theta}\sigma^2(Z_t)}.$$

Then by Corollary 3.4 we get that for all $(x, z) \in [0, \infty[\times \mathbb{R}$

$$\psi(x, z) \leq e^{-\hat{\theta}x}.$$

4. The Lower Bound

In this section $r = 0$, i.e. the bank account is not taken in consideration. Then in order to get a lower bound for the ruin probabilities, we assume the following:

Hypothesis B.

$$\text{There exists a constant} \quad R_2 > 0 \text{ such that} \quad 0 < \frac{\mu(z)}{\sigma(z)} \leq R_2 \ \ \forall z \in \mathbb{R}. \tag{19}$$

Definition 4.1. Let $0 < \theta < \theta_\infty$ be given. We say that Y has a uniform exponential moment in the tail distribution for θ, if the following condition holds true

$$\sup_{z \geq 0} \mathbb{E}[e^{-\theta(z-Y)} \mid Y > z] < \infty.$$

Theorem 4.2. *Assume that Y has a uniform exponential moment in the tail distribution for θ^*. Then:*

$$\psi(x,z) \geq C^* e^{-\theta^* x} \quad \forall z \in \mathbb{R},$$

where

$$0 < C^* = \inf_{y\geq 0} \frac{\displaystyle\int_y^\infty dG(u)}{\displaystyle\int_y^\infty e^{-\theta^*(y-z)} dG(z)} \leq 1,$$

and θ^* is the unique positive solution of:

$$\lambda h(\theta) = c\theta + \frac{1}{2}R_2^2. \tag{20}$$

For the proof of Theorem 4.2 we need the following lemma. For ease of notation we use the following notation in the rest of this work $\tau := \tau^K$.

Lemma 4.3. *Suppose that Y has a uniform exponential moment in the tail distribution for θ^*. Then for each $K \in \mathcal{K}$, the process $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a uniformly integrable submartingale.*

Proof. The proof will be given in two steps. In Step 1 we will prove that $H_*^K = \sup_{t\geq 0} H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ has a finite first moment, and in Step 2 that $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a local submartingale.
Step 1. The existence of θ^* is a consequence of the convex property of $h(\theta)$ and **Hypothesis A.2**. Our aim is to prove that H_*^K has a first finite moment. We observe that:

$$H_*^K = \begin{cases} H_{\theta^*}^{K,0}(\tau,x,z) > 1 & \text{on} \quad [\tau<\infty]\cap[X(\tau^-,x,z)>0], \\ H_{\theta^*}^{K,0}(\tau,x,z) = 1 & \text{on} \quad [\tau<\infty]\cap[X(\tau^-,x,z)=0], \\ \sup_{t\geq 0} H_{\theta^*}^{K,0}(t,x,z) \leq 1 & \text{on} \quad [\tau=\infty]. \end{cases}$$

Then

$$\begin{aligned} \mathbb{E}[H_*^K] &= \mathbb{E}[H_*^K \mathbb{1}_{[\tau=\infty]} + H_*^K \mathbb{1}_{\{[\tau<\infty]\cap[X(\tau^-,x,z)=0]\}}] \\ &\quad + \mathbb{E}[H_*^K \mathbb{1}_{\{[\tau<\infty]\cap[X(\tau^-,x,z)>0]\}}] \\ &\leq 2 + \mathbb{E}[H_*^K \mathbb{1}_{\{[\tau<\infty]\cap[X(\tau^-,x,z)>0]\}}] \end{aligned} \tag{21}$$

On the other hand, given that ruin occurs at a jump time means that the jump is conditioned to be greater or equal of $X^K(\tau^-,x,z) > 0$. More precisely, let Y be an independent copy of $(Y_i)_{i\geq 1}$ then

$$\begin{aligned} &\mathbb{E}[H_{\theta^*}^{K,0}(\tau,x,z) \mid \tau=t, X(\tau-,x,z)=v] \\ &= \mathbb{E}[e^{-\theta^*(X(\tau^-,x,z)-Y)} \mid \tau=t, X(\tau-,x,z)=v] \\ &= \mathbb{E}[e^{-\theta^*(v-Y)} \mid \tau=t, X(\tau-,x,z)=v] \\ &= \mathbb{E}[e^{-\theta^*(v-Y)} \mid Y>v] = \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_0^\infty dG(s)}. \end{aligned} \tag{22}$$

Now let $M(dt,dv)$ be the joint distribution of $(\tau, X(\tau-,x,z))$ then from (21) and (22) we have

$$\begin{aligned}
\mathbb{E}[H_*^K] &\le 2 + \mathbb{E}[\mathbb{E}[H_*^K \mathbb{1}_{[\tau<\infty]\cap[X^K(\tau-,x,z)>0]} \mid \tau, X^K(\tau-,x,z)]] \\
&= 2 + \int_0^\infty \int_0^\infty M(dt,dv) \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} \\
&\le 2 + \mathbb{P}[\tau<\infty, X^K(\tau-,x,z)>0] \sup_{v\ge 0} \int_y^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} \\
&\le 2 + \sup_{v\ge 0} \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} < \infty,
\end{aligned}$$

where the last inequality follows from the hypothesis that Y has a uniform exponential moment in the tail distribution.
Step 2. We know that

$$H_{\theta^*}^{K,0}(t,x,z) = e^{-\theta^* x} e^{\mathcal{D}_{\theta^*}^K(t) + \mathcal{E}_{\theta^*}(t)} e^{\int_0^t f_{\theta^*}^*(X_s, Z_s, K_s)\, ds}.$$

By using **Hypothesis B.** and (20), we get that for all $T \ge 0$ such that $t \le T$:

$$f_{\theta^*}^*(X_t, Z_t, K_t) \ge \frac{1}{2}\theta^{*2}\sigma^2(Z_t)\left(K_t - \frac{\mu(Z_t)}{\theta^*\sigma^2(Z_t)}\right)^2 \ge 0.$$

Following a similar procedure as in the proof of Corollary 3.4 we obtain that $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a local submartingale. By using Step 1 and the dominated convergence theorem we get that $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a uniformly integrable submartingale (see [9], Theorem I.51). □

Lemma 4.4. *If Y has a uniform exponential distribution moment in the tail distribution for θ^*, then for arbitrary $K \in \mathcal{K}$ and $(x,z) \in \mathbb{R}_+ \times \mathbb{R}$, the stopped wealth process $X^K(t\wedge\tau,x,z)$ converges almost surely on $\tau = \infty$ to ∞ for $t \to \infty$.*

Proof. Lemma 4.3 implies that $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a uniformly integrable submartingale then. By Doob's convergence theorem (see [11],Theorem II.69.1),

$$\lim_{t\to\infty} H_{\theta^*}^{K,0}(t\wedge\tau,x,z) \text{ exists a.s.}$$

Then, $X_{t\wedge\tau}$ converges for $t\to\infty$. Now to prove that

$$\mathbb{P}[\lim_{t\to\infty} X_{t\wedge\tau} = \infty \mid \tau = \infty] = 1,$$

we work toward a contradiction. We assume that

$$\mathbb{P}[\lim_{t\to\infty} X_{t\wedge\tau} < \infty, \tau = \infty] > 0,$$

then there exists $m_0 > 0$ such that $\mathbb{P}[\sup_{t\ge 0} X_t \le m_0, \tau = \infty] > 0$, because the jump is down. We consider the following event $[\sum_{i=1}^{N(1)} Y_i > 2m_0 + c]$. Since the Compound Poisson process has unbounded support then:

$$\mathbb{P}[\sum_{i=1}^{N(1)} Y_i > 2m_0 + c] = p_0 > 0.$$

Let $(t_k)_{k\geq 0}$ a sequence of points given by:

$$t_1 < t_1 + 1 < t_2 < t_2 + 1 < \dots$$

Because the Compound Poisson process has stationary and independent increments, then $(\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i)_{k\geq 0}$ is a sequence of iid random variables and represent the number of claims in an interval of length 1. Now we consider the following sequence of r.v $(D_k)_{k\geq 0}$ defined by:

$$D_k = \mathbf{1}_{\left\{\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right\}}$$

By the strong law of large numbers we get that:

$$\frac{1}{n}\sum_{k=1}^{n} D_k \xrightarrow{a.s} p_0,$$

then

$$\mathbb{P}\left[\sum_{k=1}^{\infty} \mathbf{1}_{\left\{\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right\}} = \infty\right] = 1,$$

which implies that

$$\mathbb{P}\left[\bigcap_n \bigcup_{k\geq n} \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right] = 1.$$

Now let

$$A = [\sup_{t\geq 0} X_t \leq m_0, \tau = \infty],$$

and

$$B = [\bigcap_n \bigcup_{k\geq n} \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0].$$

Observe that on A we have that for $T_i \leq t < T_{i+1}$

$$X_t = X_{T_i} + c + \int_{T_i}^{t} \mu(Z_s)K_s\,ds + \int_{t_k}^{t} \sigma(Z_s)K_s\,dW_{1s} < m_0, \tag{23}$$

and then

$$\int_{T_i}^{t} \mu(Z_s)K_s\,ds + \int_{T_i}^{t} \sigma(Z_s)K_s\,dW_{1s} \leq m_0 - c \quad \text{on} \quad A.$$

On the other hand, on B the following occurs infinitely often:

$$\begin{aligned} X_{t_k+1} &= X_{t_k} + c + \int_{t_k}^{t_k+1} \mu(Z_s)K_s\,ds + \int_{t_k}^{t_k+1} \sigma(Z_s)K_s\,dW_{1s} - \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i \\ &\leq X_{t_k} + c + \int_{t_k}^{t_k+1} \mu(Z_s)K_s\,ds + \int_{t_k}^{t_k+1} \sigma(Z_s)K_s\,dW_{1s} - 2m_0. \end{aligned} \tag{24}$$

Then from (23) and (24) $P[A\cap B] = 0$, since $P[B] = 1$ then $P[A] = 0$. □

Proof. **Theorem 4.2.** Since $H_{\theta^*}^{K,0}(t\wedge\tau,x,z)$ is a submartingale then

$$e^{-\theta^* x} \le \mathbb{E}[H_{\theta^*}^{K,0}(t\wedge\tau,x,z)]$$

and

$$\mathbb{E}[H_{\theta^*}^{K,0}(t\wedge\tau,x,z)] = \mathbb{E}[H_{\theta^*}^{K,0}(\tau,x,z),\tau<t] + \mathbb{E}[H_{\theta^*}^{K,0}(t,x,z),\tau>t].$$

By letting $t\to\infty$

$$e^{-\theta^* x} \le \mathbb{E}[H_{\theta^*}^{K,0}(\tau,x,z)\mid\tau<\infty]\mathbb{P}[\tau<\infty] + \mathbb{E}[\lim_{t\to\infty} H_{\theta^*}^{K,0}(t,x,z)\mid\tau=\infty]\mathbb{P}[\tau=\infty].$$

By lemma 4.4 we get that:

$$e^{-\theta^* x} \le \mathbb{E}[H_{\theta^*}^{K,0}(\tau,x,z)\mid\tau<\infty]\mathbb{P}[\tau<\infty],$$

and

$$\begin{aligned}\psi(x,z,K)=\mathbb{P}[\tau<\infty] &\ge \frac{e^{-\theta^* x}}{\mathbb{E}[H_{\theta^*}^{K,0}(\tau,x,z)\mid\tau<\infty]}\\ &\ge \inf_{y\ge 0}\frac{1}{\int_y^\infty e^{-\theta^*(y-z)}\frac{dG(z)}{\int_y^\infty dG(u)}}\\ &\ge C^* e^{-\theta^* x}.\end{aligned}$$

Finally

$$\psi(x,z)\ge C^* e^{-\theta^* x},$$

where

$$C^* = \inf_{y\ge 0}\frac{\int_y^\infty dG(u)}{\int_y^\infty e^{-\theta^*(y-z)}dG(z)}.$$

□

As a consequence of Corollary 3.4 and Theorem 4.2 we get the following estimations for the ruin probability when the initial capital goes to infinity.

Corollary 4.5. *If θ^+ is the positive solution of (18) when $r=0$ then:*

$$C^* e^{-\theta^* x}\le\psi(x,z)\le e^{-\theta^+ x}.$$

Furthermore for all $z\in\mathbb{R}$:

$$-\theta^*\le\liminf_{x\to\infty}\frac{1}{x}\ln(\psi(x,z))\le\limsup_{x\to\infty}\frac{1}{x}\ln(\psi(x,z))\le-\theta^+.$$

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References

[1] Asmussen, S. (2000). *Ruin probabilities.* World Scientific, Singapore.
[2] Castañeda, N., Hernández, D. (2005). Optimal consumption-investment problems in incomplete marckets with stochastic coefficients. Siam J. Control Optim. Vol 44, No.4, pp. 1322-1344.
[3] Gaier, J., Grandits, P. & Schachermeyer, W. (2003). Asymptotic ruin probabilities and optimal investment. *Ann. Appl. Probab.* 13, 1054-1076.
[4] Fleming, W.H., Hernández, D. (2005). The tradeoff between consumption and investment in incomplete financial markets. Appl Math Optim 52:219-235.
[5] Fleming, W.H., Sheu S. J. (1999). Risk sensitive control and an optimal investment model. Math. Finance 10, 197-213.
[6] Hipp, C. & Plum, M. (2000). Optimal investment for investors. Insurance: Math. Econom. 27, 215-228
[7] Hipp, C. & Schmidli, H. (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scand. Actuarial J.,*. 5: 321-355.
[8] Ikeda, N. & Watanabe, S. (1989). *Stochastic differential equations and diffusion processes.* North-Holland/Kodansha.
[9] Protter, P. (2004). *Stochastic integration and differential equations.* Springer-verlag.
[10] Rama, C., Peter, T. (2003). *Financial modelling with jump processes.* Chapman & Hall.
[11] Rogers, L., Williams, D. (1994). *Diffusions, markov processes and martingales.* Volume 1, 2nd Edition. Wiley & and Sons.
[12] Rolski, T., Schmidli, H., Schmidt, V. & Teugels, J.L. (1999). *Stochastic Processes for insurance and Finance.* Wiley, Chichester.
[13] Schmidli, H. (2002). On minimising the ruin probability by investesment and reinsurance. *Ann. Appl. Probab.* 12, 890-907.
[14] Scott, L. O., (1997). Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: applications of Fourier inversion methods, Math. Finance, 7, pp. 413-426.
[15] Zariphopoulou T. (2001). A solution approach to valuation with unhedgeable risks. Finance Stochast. 5, 61-88.