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FLUCTUATION LIMIT THEOREMS FOR AGE-DEPENDENT CRITICAL BINARY BRANCHING SYSTEMS

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Abstract. We consider an age-dependent branching particle system in $\mathbb{R}^d$, where the particles are subject to α-stable migration ($0 < \alpha \le 2$), critical binary branching, and general (non-arithmetic) lifetimes distribution. The population starts off from a Poisson random field in $\mathbb{R}^d$ with Lebesgue intensity. We prove functional central limit theorems and strong laws of large numbers under two rescalings: high particle density, and a space-time rescaling that preserves the migration distribution. Properties of the limit processes such as Markov property, almost sure continuity of paths and generalized Langevin equation, are also investigated.

1. Introduction

The classical branching random field is a population model which begins at time $t = 0$ with a Poisson-distributed random population, and in which each individual independently develops a simple branching diffusion process. This model enjoys many nice features, such as exponentially distributed individual lifetimes, time homogeneity of its transition probabilities, and strong Markov property, and has been investigated by many authors, specially in the case when the branching is binary and the diffusion process is Brownian motion. In particular, results have been obtained on fluctuation limits under various rescalings and parameterizations, see [4, 7, 10, 13, 15]. In this paper, we consider a random branching population in which the particle lifetimes are not necessarily exponentially distributed. More precisely, we investigate high density and space-time scaling fluctuation limits of a population living in d-dimensional Euclidean space $\mathbb{R}^d$ and evolving as follows. Any given individual independently develops a spherically symmetric α-stable motion during its lifetime τ, where $0 < \alpha \le 2$ and τ is a random variable having a non-arithmetic distribution function, and at the end of its life it either disappears or is replaced at the site where it died by two newborns, each event occurring with probability 1/2. The population starts off from a Poisson random field having Lebesgue measure Λ as its intensity. We postulate the usual independence assumptions in branching systems. Two regimes for the distribution of τ are considered: either τ has finite mean $\mu > 0$, or τ possesses a distribution function F such that $F(0) = 0$, $F(x) < 1$ for all $x \in [0, \infty)$, and

$$\bar{F}(u) := 1 - F(u) \sim u^{-\gamma}\Gamma(1-\gamma)^{-1} \quad \text{as} \quad u \longrightarrow \infty, \tag{1}$$

where $\gamma \in (0, 1)$ and Γ denotes the Gamma function, i.e., F belongs to the normal domain of attraction of a γ-stable law. In particular, this allows to consider lifetimes with infinite mean. Let $X \equiv \{X_t,\ t \ge 0\}$, where X_t denotes the simple counting measure on $\mathbb{R}^d$ whose atoms are the positions of particles alive at time

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t. When τ has an exponential distribution it is well known that the measure-valued process X is Markov. In the literature there is a lot of work about the Markovian model. Our objective here, as we mentioned above, is to investigate the case when τ is not necessarily an exponential random variable, in which case $\{X_t,\, t \geq 0\}$ is no longer a Markov process. Another striking difference with respect to the case of exponential lifetimes arises when the particle lifetime distribution satisfies (1): heavy-tailed lifetimes enhance the mobility of individuals, facilitating in this way the spreading out of particles, and thus counteracting the clumping of the population. Since clumping goes along with local extinction (due to critical branching), a smaller exponent γ favors stability of the population. As a matter of fact, X admits a nontrivial equilibrium distribution if and only if $d \geq \gamma\alpha$, see [9, 19]. This contrasts with the case of exponentially distributed (or general non arithmetic finite-mean) lifetimes, where the necessary and sufficient condition for stability is $d > \alpha$. As we will see, such qualitative departure from the Markovian model propagates also to other properties of the branching particle system, such as the scaling limit theorems mentioned at the beginning of this introduction, which we describe below.

The *high density limit* consists in increasing the initial intensity by a factor K which will tend to infinity, see [16] for a physical motivation of this rescaling. Let $X^{1,K} \equiv \{X_t^{1,K},\, t \geq 0\}$ denote the process X with initial intensity $\mathbb{E}X_0^{1,K} = K\Lambda$. We are interested in the limit behavior, as $K \to \infty$, of the normalized fluctuations process $M^{1,K} \equiv \{M_t^{1,K},\ t \geq 0\}$, where

$$M_t^{1,K} = \frac{X_t^{1,K} - \mathbb{E}X_t^{1,K}}{K^{1/2}}, \quad t \geq 0.$$

For all $t \geq 0$ and $K \geq 1$, $M_t^{1,K}$ takes values in the space $S'(\mathbb{R}^d)$ of Schwartz distributions. We will prove that, as $K \to \infty$, $M^{1,K}$ converges weakly (in the sense of weak convergence of finite-dimensional distributions) to an $S'(\mathbb{R}^d)$-valued, centered Gaussian process M^1 whose covariance functional is explicitly calculated. Also, we prove that the limit process M^1 is Markov, and that its sample paths are almost surely continuous, even in a stronger topology than that of $S'(\mathbb{R}^d)$. These results are shown to hold for any particle lifetime distribution function. When, in addition, the lifetime distribution of particles possesses a continuous density, we also prove that the limit process satisfies a generalized stochastic equation of the form

$$dM_t^1 = AM_t^1 + d\mathcal{W}_t, \quad M^1(0) = W, \tag{2}$$

where A denotes the generator of the particle motion process, W is a centered spatial white noise and $\mathcal{W}$ is a certain generalized Wiener process; see Section 2 for background on generalized random processes and equations of the type (2).

In the *space-time scaling limit*, the coordinates in space and time are respectively Kx and $K^\alpha t$, again K being a parameter which will tend to infinity. This scaling renders the so-called *large scale fluctuation process* [5], and is meant to consider large space-time sets in a way which preserves the migration distribution. For this scaling we need to assume that $d > \alpha\gamma$, i.e. we require dimensions ensuring stability of the branching population. The normalizing constant for the fluctuation process is $K^{-(d+\alpha\gamma)/2}$ (recall that, for exponentially distributed lifetimes, the normalizing factor is $K^{-(d+\alpha)/2}$; see [5, 10]). The limit process is again an $S'(\mathbb{R}^d)$-valued centered Gauss-Markov process that possesses a version which has continuous paths, and satisfies a generalized Langevin equation similar to (2). Heavy-tailed lifetimes play a key role in the space-time scaling because the power γ of the tail decay figures explicitly in the covariance functional of the limit process (see equation (13) below).

The remaining part of the paper is organized as follows. In the next section we briefly recall some basic information on generalized processes that will be used latter on. In Section 3 we compute the mean and covariance functionals of the branching particle system X. These results play a crucial role in our proof of convergence of $\{M^{l,K},\ K \geq 1\}$, $l = 1, 2$. Our main theorems are stated in Section 4, and proved in Section 5.

2. Background on generalized processes

In this section we introduce the background on generalized processes and Langevin equations that we need to develop our arguments. We refer the reader to [2,6,12,18] for further information and more references. The limit processes we consider take values in the space $S'(\mathbb{R}^d)$, where $S'(\mathbb{R}^d)$ denotes the (strong) dual of the Schwartz space $S(\mathbb{R}^d)$ of C^∞ rapidly decreasing test functions. The topology we will use in the space $S(\mathbb{R}^d)$ is the usual one induced by a system of Hilbertian norms $\{\|\cdot\|_p,\ p\geq 0\}$ such that $S(\mathbb{R}^d)=\bigcap_{p=0}^\infty S_p(\mathbb{R}^d)$, where $S_p(\mathbb{R}^d)$ denotes the completion of $S(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_p$. The topology we consider on the space $S'(\mathbb{R}^d)$ is the usual one [18]. We deal with $S'(\mathbb{R}^d)$-valued processes $\mathbf{X}\equiv\{\mathbf{X}_t,\ t\geq 0\}\equiv\{\langle\phi,\mathbf{X}_t\rangle,\ t\geq 0,\ \phi\in S(\mathbb{R}^d)\}$; here $\langle\cdot,\cdot\rangle$ stands for the duality on $S(\mathbb{R}^d)\times S'(\mathbb{R}^d)$, see [18]. The process $\mathbf{X}$ is called *Gaussian* if the family of real random variables $\{\langle\phi,\mathbf{X}_t\rangle,\ t\geq 0,\ \phi\in S(\mathbb{R}^d)\}$ is a Gaussian system. In studying continuity of $S'(\mathbb{R}^d)$-valued Gaussian processes, we will need the stronger topology in the subspace $S'_p(\mathbb{R}^d)$ (dual of $S_p(\mathbb{R}^d)$) of $S'(\mathbb{R}^d)$ given by the customary Hilbertian norm $\|\cdot\|_{-p}$ such that $S'(\mathbb{R}^d)=\bigcup_{p=0}^\infty S'_p(\mathbb{R}^d)$, see [17,18].

Theorem 2.1. [17] *Let $\{\mathbf{X}_t,\ t\geq 0\}$ be an $S'(\mathbb{R}^d)$-valued Gaussian process. Assume that for every $\varphi\in S(\mathbb{R}^d)$ there exists a function $F_\varphi:\mathbb{R}\to\mathbb{R}_+$ and positive numbers A_φ, M_φ such that $\int_{M_\varphi}^\infty F_\varphi(\exp\{-x^2\})\,dx<\infty$, the function $u\mapsto F_\varphi(u)$ is monotone increasing on $0<u<A_\varphi$ and $\mathbb{E}[\langle\varphi,\mathbf{X}_t-\mathbf{X}_s\rangle^2]\leq F_\varphi^2(|t-s|)$ for any $t,s\geq 0$. Then $\{\mathbf{X}_t,\ t\geq 0\}$ possesses an $S'(\mathbb{R}^d)$-valued continuous version.*

Let $B\subset\mathbb{R}^d$ be a Borel set. An $S'(\mathbb{R}^d)$-valued random variable W is called a *standard white noise concentrated on B*, if its characteristic functional is given by

$$\mathbb{E}\exp\{i\langle\phi,W\rangle\}=\exp\left\{-\frac{1}{2}\int_B\phi^2(x)dx\right\},\quad \phi\in S(\mathbb{R}^d).$$

A generalized Langevin equation is a stochastic evolution equation of the form

$$d\mathbf{X}_t=A^*\mathbf{X}_tdt+d\mathcal{W}_t,\quad t\geq 0, \tag{3}$$

where A^* is the adjoint operator of a continuous linear operator A on $S(\mathbb{R}^d)$ into itself, and $\{\mathcal{W}_t,\ t\geq 0\}$ is an $S'(\mathbb{R}^d)$-Wiener process, i.e., $\{\mathcal{W}_t, t\geq 0\}$ is a continuous $S'(\mathbb{R}^d)$-valued centered Gaussian process whose covariance functional has the form

$$\mathrm{Cov}(\langle\phi,\mathcal{W}_s\rangle,\langle\psi,\mathcal{W}_t\rangle)=\int_0^{s\wedge t}\langle Q_u\phi,\psi\rangle\,du,\quad s,t\geq 0,\quad \phi,\psi\in S(\mathbb{R}^d),$$

where, for each $u\geq 0$, $Q_u:S(\mathbb{R}^d)\to S(\mathbb{R}^d)$ is a symmetric and positive continuous linear operator, and the function $u\mapsto\langle Q_u\phi,\psi\rangle$ is right-continuous with left limits for each $\phi,\psi\in S(\mathbb{R}^d)$. We say in this case that $\mathcal{W}$ is associated to $Q\equiv\{Q_u, u\geq 0\}$. Solutions $\{\mathbf{X}_t,\ t\geq 0\}$ to Equation (3) are going to be interpreted in the sense that

$$\langle\phi,\mathbf{X}_t\rangle=\langle\phi,\mathbf{X}_0\rangle+\int_0^t\langle A\phi,\mathbf{X}_s\rangle\,ds+\langle\phi,\mathcal{W}_t\rangle,\quad t\geq 0, \tag{4}$$

for each $\phi\in S(\mathbb{R}^d)$, where the initial condition $\mathbf{X}_0$ is a random element in $S'(\mathbb{R}^d)$.

Let $C(\mathbb{R}^d)$ denote the space of continuous functions on $\mathbb{R}^d$, and let $C_0(\mathbb{R}^d)\subset C(\mathbb{R}^d)$ be the subset of elements vanishing at infinity. For $p>0$, let $\varphi_p(x)=(1+|x|^2)^{-p}$, $x\in\mathbb{R}^d$. We define

$$C_p(\mathbb{R}^d)=\left\{\varphi\in C(\mathbb{R}^d): [\![\varphi]\!]_p<\infty\right\},$$

where

$$[\![\varphi]\!]_p=\sup_{x\in\mathbb{R}^d}|\varphi(x)/\varphi_p(x)|,$$

and

$$C_{p,0}(\mathbb{R}^d) = \left\{\varphi \in C(\mathbb{R}^d) : \varphi/\varphi_p \in C_0(\mathbb{R}^d)\right\}.$$

Clearly, $S(\mathbb{R}^d) \subset C_{p,0}(\mathbb{R}^d) \subset C_p(\mathbb{R}^d)$ for all $p > 0$. Moreover, $C_{p,0}(\mathbb{R}^d)$ and $C_p(\mathbb{R}^d)$ are Banach spaces for the norm $[\![\cdot]\!]_p$; see [6]. Let $\{\mathcal{S}_t,\, t \geq 0\}$ be the semigroup in $L^2(\mathbb{R}^d)$ with generator $\Delta_\alpha := -(\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$.

Lemma 2.2. [6] *Let $p > d/2$, and additionally $p < (d+\alpha)/2$ in case $\alpha < 2$. For each $t \geq 0$, $\mathcal{S}_t$ is a bounded linear operator from $(C_p(\mathbb{R}^d), [\![\cdot]\!]_p)$ into itself. The operators Δ_α and $\mathcal{S}_t$, $t \geq 0$, are continuous linear mappings from $S(\mathbb{R}^d)$ to $C_{p,0}(\mathbb{R}^d)$, and $t \mapsto \mathcal{S}_t\varphi$ is a continuous curve in $(C_{p,0}(\mathbb{R}^d), [\![\cdot]\!]_p)$ for any $\varphi \in S(\mathbb{R}^d)$.*

3. Some moment calculations

Let Z_t denote the offspring population at time $t \geq 0$, stemming from a single individual at time 0. Following [14] we define

$$Q_t\varphi(x) := \mathbb{E}_x\left[1 - e^{-\langle\varphi, Z_t\rangle}\right], \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{5}$$

where φ belongs to the space $C_c^+(\mathbb{R}^d)$ of non-negative compactly supported continuous functions on $\mathbb{R}^d$, and $\mathbb{E}_x$ denotes expectation when the initial particle is located at $x \in \mathbb{R}^d$. Since the initial population X_0 is Poisson distributed with intensity Lebesgue measure, we have

$$\begin{aligned}\mathbb{E}e^{-\langle\varphi, X_t\rangle} &= \exp\left(-\int \mathbb{E}_x\left[1 - e^{-\langle\varphi, Z_t\rangle}\right] dx\right) \\ &= \exp\left(-\int Q_t\varphi(x)\,dx\right), \quad \varphi \in C_c^+(\mathbb{R}^d). \end{aligned} \tag{6}$$

Let $\{\tau_k, k \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F, and let

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \quad t \geq 0,$$

where the random sequence $\{S_k, k \geq 0\}$ is recursively defined by

$$S_0 = 0, \quad S_{k+1} = S_k + \tau_k, \quad k \geq 0.$$

For any $p = 1, 2, \ldots,$ $0 < t_p \leq t_{p-1}, \ldots, t_1 < \infty$, $\varphi_1, \varphi_2, \ldots, \varphi_p \in C_c(\mathbb{R}^d)$ and $\theta_1, \ldots, \theta_p \in \mathbb{R}$, we define $\bar{t} = (t_1, t_2, \ldots, t_p)$, $\bar{t} - s = (t_1 - s, t_2 - s, \ldots, t_p - s)$, $\theta_{(p)} = (\theta_1, \ldots, \theta_p)'$ and

$$Q_{\bar{t}}^p\theta_{(p)}(x) = \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\langle\varphi_j, Z_{t_j}\rangle}\right].$$

Let $\{B_t,\ t \geq 0\}$ denote the spherically symmetric α-stable process in $\mathbb{R}^d$, with transition density functions $\{p_t(x,y) := p_t(x-y),\, t > 0,\, x, y \in \mathbb{R}^d\}$, and semigroup $\{\mathcal{S}_t,\, t \geq 0\}$.

Proposition 3.1. [14] *The function $Q_{\bar{t}}^p\theta_{(p)}$ satisfies*

$$\begin{aligned} Q_{\bar{t}}^p\theta_{(p)}(x) = \mathbb{E}_x\Bigg[& 1 - e^{-\sum_{j=1}^p \theta_j\varphi_j(B_{t_j})} - \int_0^{t_p} \frac{1}{2}\left(Q_{\bar{t}-s}^p\theta_{(p)}(B_s)\right)^2 dN_s \\ & - \sum_{i=1}^{p-1}\left(1 - e^{-\sum_{j=i+1}^p \theta_j\varphi_j(B_{t_j})}\right)\int_{t_{i+1}}^{t_i} \frac{1}{2}\left(Q_{\bar{t}-s}^i\theta_{(i)}(B_s)\right)^2 dN_s\Bigg]. \end{aligned}$$

As in (6), since the initial population is Poissonian we have

$$\begin{aligned}\mathbb{E}\left[e^{-\sum_{j=1}^{p}\theta_j\langle\varphi_j,X_{t_j}\rangle}\right] &= \exp\left(-\int\mathbb{E}_x\left[1-e^{-\sum_{j=1}^{p}\theta_j\langle\varphi,Z_{t_j}\rangle}\right]dx\right)\\ &= \exp\left(-\int Q_{\bar{t}}^{p}\theta_{(p)}(x)dx\right). \qquad (7)\end{aligned}$$

Using criticality of the branching and that Lebesgue measure is invariant for the semigroup of the symmetric α-stable process, it is easy to see that

$$m(t,\varphi):=\mathbb{E}[\langle\varphi,X_t\rangle]=\langle\varphi,\Lambda\rangle,\quad t\geq 0,\quad \varphi\in C_c(\mathbb{R}^d). \qquad (8)$$

Throughout the paper we will denote

$$m_x(t,\varphi):=\mathbb{E}_x\left[\langle\varphi,Z_t\rangle\right],\quad x\in\mathbb{R}^d,\quad t\geq 0,\quad \varphi\in C_c(\mathbb{R}^d).$$

Lemma 3.2. *Let $0<s\leq t<\infty$ and $\psi,\varphi\in C_c(\mathbb{R}^d)$. Then,*

$$\begin{aligned}C_x(s,\varphi;t,\psi) &:= \mathbb{E}_x\left[\langle\varphi,Z_s\rangle\langle\psi,Z_t\rangle\right]\\ &= \mathbb{E}_x\left[\varphi(B_s)\psi(B_t)+\int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)\,dN_r\right]. \qquad (9)\end{aligned}$$

Proof: In order to use the same notations as in Proposition 3.1, we put $p=2$, $t_1=t$, $t_2=s$, $\varphi_1=\psi$ and $\varphi_2=\varphi$. Then we have

$$C_x(t_1,\varphi_1;t_2,\varphi_2)=-\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q_{\bar{t}}^2\theta_{(2)}(x)\Big|_{\theta_1=\theta_2=0^+},$$

where

$$\begin{aligned}\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q_{\bar{t}}^2\theta_{(2)}(x) = \mathbb{E}_x\Bigg[&-\varphi_1(B_{t_1})\varphi_2(B_{t_2})e^{-\theta_1\varphi(B_{t_1})-\theta_2\varphi_2(B_{t_2})}\\ &-\int_0^{t_2}\frac{\partial}{\partial\theta_2}Q_{\bar{t}-r}^2\theta_{(2)}(B_r)\frac{\partial}{\partial\theta_1}Q_{\bar{t}-r}^2\theta_{(2)}(B_r)dN_r\\ &-\int_0^{t_2}\left(Q_{\bar{t}-r}^2\theta_{(2)}(B_r)\right)\frac{\partial^2}{\partial\theta_2\partial\theta_1}Q_{\bar{t}-r}^2\theta_{(2)}(B_r)dN_r\\ &-\varphi_2(B_{t_2})e^{-\theta\varphi_2(B_{t_2})}\int_{t_1}^{t_2}\left(Q_{t_2-r}^1\theta_1(B_r)\right)\frac{\partial}{\partial\theta_1}Q_{t_2-r}^1\theta_1(B_r)dN_r\Bigg].\end{aligned}$$

Evaluating at $\theta_1=\theta_2=0$ we finish the proof. □

Proposition 3.3. *Let $0<s\leq t<\infty$ and $\psi,\varphi\in C_c(\mathbb{R}^d)$. Then,*

$$C(s,\varphi;t,\psi):=\mathrm{Cov}\left(\langle\varphi,X_s\rangle,\langle\psi,X_t\rangle\right)=\langle\varphi\mathcal{S}_{t-s}\psi,\Lambda\rangle+\int_0^s\langle\left(\mathcal{S}_{s-r}\varphi\right)\left(\mathcal{S}_{t-r}\psi\right),\Lambda\rangle\,dU(r), \qquad (10)$$

where $U(r)=\sum_{k=1}^{\infty}F^{*k}(r)$.

Proof: We put $p = 2$ in (7) and use the same notations as in the proof of Lemma 3.2. Then,

$$\begin{aligned}\mathbb{E}\left[\langle\varphi_1, X_{t_1}\rangle\langle\varphi_2, X_{t_2}\rangle\right] &= \frac{\partial^2}{\partial\theta_1\partial\theta_2}\exp\left(-\int Q_{\bar t}^2\theta_{(2)}(x)\,dx\right)\Bigg|_{\theta_1=\theta_2=0^+} \\ &= \Bigg[-\frac{\partial^2}{\partial\theta_1\partial\theta_2}\int Q_{\bar t}^2\theta_{(2)}(x)\,dx \\ &\quad + \int\frac{\partial}{\partial\theta_1}Q_{\bar t}^2\theta_{(2)}(x)\,dx\int\frac{\partial}{\partial\theta_1}Q_{\bar t}^2\theta_{(2)}(x)\,dx\Bigg]\Bigg|_{\theta_1=\theta_2=0^+} \\ &= \int C_x(t_1,\varphi_1;t_2,\varphi_2)\,dx + \int m_x(t_1,\varphi_1)\,dx\int m_x(t_2,\varphi_2)\,dx.\end{aligned}$$

Now, from Lemma 3.2 we obtain

$$C(s,\varphi;t,\psi) = \int_{\mathbb{R}^d}\mathbb{E}_x\left[\varphi(B_s)\psi(B_t) + \int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]dx, \tag{11}$$

which completes the proof. □

4. Laws of large numbers and functional central limit theorems

We consider the following two rescalings, parameterized by $K \geq 1$ with $K \to \infty$.

1. High particle density. The initial population intensity is $K\Lambda$. The resulting branching particle system is denoted by $X^{1,K} \equiv \{X_t^{1,K}, t \geq 0\}$.

2. Space-time rescaling. Let us suppose that $d > \alpha\gamma$. The coordinates in space-time are Kx and $K^\alpha t$, respectively. The branching particle system is denoted by $X^{2,K} \equiv \{X_t^{2,K},\ t \geq 0\}$, i.e., for all $\varphi \in S(\mathbb{R}^d)$,

$$\left\langle\varphi, X_t^{2,K}\right\rangle = \left\langle\varphi^K, X_{K^\alpha t}\right\rangle,$$

where $\varphi^K(x) := \varphi(x/K)$, $x \in \mathbb{R}^d$. The fluctuation processes corresponding to these two rescalings are, respectively,

$$M^{l,K} = K^l\left(X^{l,K} - \mathbb{E}X^{l,K}\right), \quad l = 1, 2,$$

where $K^1 = K^{-1/2}$ and $K^2 = K^{-(d+\alpha\gamma)/2}$. We write $\Rightarrow$ for weak convergence of finite-dimensional distributions.

Theorem 4.1. *(Functional central limit theorems)*
(a) Let F be any lifetime distribution function. Then, $M^{1,K} \Rightarrow M^1$ as $K \to \infty$, where M^1, is a continuous centered Gaussian process with covariance functional given by

$$\mathcal{K}^1(s,\varphi;t,\psi) = \langle\varphi\mathcal{S}_{t-s}\psi, \Lambda\rangle + \int_0^s \langle(\mathcal{S}_{s-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda\rangle\,dU(r), \quad 0 \leq s \leq t < \infty, \quad \varphi,\psi \in S(\mathbb{R}^d), \tag{12}$$

where $U(r) = \sum_{k=1}^\infty F^{*k}(r)$.
(b) Let F be a non-arithmetic lifetime distribution satisfying (1). Then, $M^{2,K} \Rightarrow M^2$ as $K \to \infty$, where M^2 is a continuous centered Gaussian process with covariance functional given by

$$\mathcal{K}^2(s,\varphi;t,\psi) = \frac{\gamma}{\Gamma(1+\gamma)}\int_0^s \langle(\mathcal{S}_{t-u}\psi)(\mathcal{S}_{s-u}\varphi), \Lambda\rangle u^{\gamma-1}du, \quad 0 \leq s \leq t < \infty, \quad \varphi,\psi \in S(\mathbb{R}^d). \tag{13}$$

Theorem 4.2. *(Laws of large numbers) Let $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$. The following convergences hold in $L^2(\mathbb{R}^d)$.*
(a) For any lifetime distribution F,

$$\frac{\langle \varphi, X_t^{1,K} \rangle}{K} \to \langle \varphi, \Lambda \rangle \text{ as } K \to \infty.$$

(b) For any non-arithmetic lifetime distribution function F,

$$\frac{\langle \varphi^K, X_t^{2,K} \rangle}{K^d} \to \langle \varphi, \Lambda \rangle \text{ as } K \to \infty.$$

Theorem 4.3. *(Properties of the fluctuation limits)*
(a) For $l = 1, 2$, M^l is a Markov process and, for every $\psi \in S(\mathbb{R}^d)$,

$$\langle \psi, M_t^l \rangle - \int_0^t \langle \Delta_\alpha \psi, M_s^l \rangle \, ds, \ t \geq 0, \tag{14}$$

is a martingale with respect to the filtration $\mathcal{F}_t^l = \sigma\{\langle \phi, M_r^l \rangle, \, r \leq t, \, \phi \in S(\mathbb{R}^d)\}$, $t \geq 0$.
(b) There exists $p \geq 1$ such that M^l, $l = 1, 2$, has a continuous version in the norm $\| \cdot \|_{-p}$.
(c) Let $\alpha = 2$, and assume that F has a continuous density f. The process M^1 satisfies the generalized Langevin equation

$$dM_t^1 = \Delta_\alpha M_t^1 + d\mathcal{W}_t^1, \quad M_0^1 = W, \tag{15}$$

where W is a centered spatial white noise, and the Wiener process $\mathcal{W}^1$ is associated to the family of operators $\{Q_t^1, t \geq 0\}$ such that, for each $\varphi, \psi \in S(\mathbb{R}^d)$,

$$\langle Q_t^1 \varphi, \psi \rangle = \langle \varphi\psi, \Lambda \rangle u(t) - 2\langle \varphi \Delta_\alpha \psi, \Lambda \rangle, \tag{16}$$

where $u(t) = dU(t)/dt$. The process M^2 satisfies the generalized Langevin equation

$$dM_t^2 = \Delta_\alpha M_t^2 + d\mathcal{W}_t^2, \quad M_0^2 = 0, \tag{17}$$

where the generalized Wiener process $\mathcal{W}^2$ has covariance functional

$$\mathbb{E}\left[\langle \varphi, \mathcal{W}_s^2 \rangle \langle \psi, \mathcal{W}_t^2 \rangle\right] = \frac{(s \wedge t)^\gamma}{\Gamma(1+\gamma)} \langle \varphi, \psi \rangle, \quad 0 \leq s, t, \quad \varphi, \psi \in S(\mathbb{R}^d).$$

Remark 4.4. (a) By the renewal theorem, Theorem 4.1(b) is still true in case of particle lifetimes with finite mean $\mu > 0$. In this case, the covariance functional of the limit process is given by

$$\mathcal{K}^2(s, \varphi; t, \psi) = \frac{1}{\mu} \int_0^s \langle (\mathcal{S}_{t-u}\psi)(\mathcal{S}_{s-u}\varphi), \Lambda \rangle \, du, \quad 0 \leq s \leq t < \infty, \quad \varphi, \psi \in S(\mathbb{R}^d).$$

(b) The meaning of equations (15) and (17), with $\alpha = 2$, is that

$$\langle \varphi, M_t^l \rangle = \langle \varphi, M_0^l \rangle + \int_0^t \langle \Delta_\alpha \varphi, M_s^l \rangle \, ds + \langle \varphi, \mathcal{W}_t^l \rangle, \quad l = 1, 2, \quad t \geq 0, \quad \varphi \in S(\mathbb{R}^d). \tag{18}$$

We remark that, when $\alpha = 2$, the operator Δ_α is the generator of the d-dimensional Brownian motion with variance parameter 2, and only for this value of α the inclusion $\Delta_\alpha(S(\mathbb{R}^d)) \subset S(\mathbb{R}^d)$ and the interpretation (18) are valid. It can be shown that $\Delta_\alpha(S(\mathbb{R}^d)) \not\subset S(\mathbb{R}^d)$ when $\alpha \neq 2$, and therefore the integral term in (18) is not defined when $0 < \alpha < 2$. This situation motivated Dawson and Gorostiza [6] to formulate the following generalized notion of solution to equations of the form (3) (of which (15) and (17) are special cases): a generalized solution to (3) on $[0,T]$ is a process Y in $D([0,T],S'(\mathbb{R}^d))$, defined on the same probability space $(\Omega,\mathcal{F},P)$ on which M^l is defined, such that

C1. There exist a Banach space $V(\mathbb{R}^d)$ of real functions on $\mathbb{R}^d$ satisfying $S(\mathbb{R}^d)\subset V(\mathbb{R}^d)\subset L^2(\mathbb{R}^d)$, where $S(\mathbb{R}^d)$ is densely and continuously embedded in $V(\mathbb{R}^d)$.

C2. For each $\phi\in(\mathrm{Dom}(A)\cap V(\mathbb{R}^d))\hat{\otimes}\mathcal{D}([-\delta,T])$, the expression $\int_0^T\langle A\phi_t,Y_t^l\rangle\,dt$ is a random variable on $(\Omega,\mathcal{F},P)$.

C3. The equality

$$\int_0^T\left\langle A\phi_t+\frac{d}{dt}\phi_t,M_t^l\right\rangle dt=-\langle\phi_0,M_0^l\rangle+\int_0^T\left\langle\frac{d}{dt}\phi_t,\mathcal{W}_t^l\right\rangle dt$$

holds in $\mathcal{L}^0(\Omega,\mathcal{F},P)$, $l=1,2$.

Here $\delta>0$ is a given constant, $\mathcal{D}([-\delta,T])$ is the usual space of C^∞-functions with supports contained in $[-\delta,T]$, and $\mathcal{L}^0(\Omega,\mathcal{F},P)$ is the space of equivalence classes of real random variables on a complete probability space.

Thus, in order to ensure that a given generalized process $\{Z_t\}$ satisfies a generalized equation in the sense of [6], it is necessary to corroborate that $\{Z_t\}$ fulfills the three conditions C1-C3. When the operator A figuring in (3) is the fractional power $-(-\Delta)^{\alpha/2}=:\Delta_\alpha$ of the Laplacian, $0<\alpha<2$, one can prove that the space $C_{p,0}(\mathbb{R}^d)$ defined in Section 2 satisfies Condition C1 above, provided $d/2<p<(d+\alpha)/2$. Nonetheless, conditions C2 and C3 need to be validated for each particular instance of (3) (in [6], such a validation is carried out for a generalized process which arises as the fluctuation limit of a branching particle system in a random medium). For our generalized equations (15) and (17) with $0<\alpha<2$, verification of the above mentioned conditions C2 and C3 is beyond the scope of the present paper, and will be developed latter on.
(c) By Remark (a) of Theorem 3.6 in [2], without any regularity condition on F we still have, again for $\alpha=2$, that

$$\langle\varphi,M_t^1\rangle=\langle\varphi,W\rangle+\int_0^t\langle\Delta_\alpha\varphi,M_s^1\rangle\,ds+\langle\varphi,\mathcal{W}_t\rangle,\quad t\geq 0,$$

where $\{\mathcal{W}_t,t\geq 0\}$ is a continuous $S'(\mathbb{R}^d)$-valued Gaussian process whose covariance functional is given by

$$\mathbb{E}\left[\langle\varphi,\mathcal{W}_s\rangle\langle\varphi,\mathcal{W}_t\rangle\right]=\mathcal{K}^1(s\wedge t,\varphi;s\wedge t,\psi)-\int_0^{s\wedge t}(\mathcal{K}^1(u,\Delta_\alpha\varphi;u,\psi)+\mathcal{K}^1(u,\varphi;u,\Delta_\alpha\psi))\,du,$$

for all $s,t\geq 0$ and $\varphi,\psi\in S(\mathbb{R}^d)$.
(d) The assumption that F has a continuous density cannot be dropped in Theorem 4.3(c); without such assumption we cannot guarantee differentiability of the function $t\mapsto\mathcal{K}^1(t,\varphi;t,\varphi)$.
(e) Assuming that $\bar{F}(t)=e^{-Vt}$, $t\geq 0$, and $\alpha=2$ we get that $U(dt)\equiv V\,dt$. In this case (16) is equivalent to

$$\langle Q_t^1\varphi,\psi\rangle=V\langle\varphi\psi,\Lambda\rangle+\langle\nabla\varphi\cdot\nabla\psi,\Lambda\rangle,$$

which recovers a result from [10] for critical binary branching.

5. Proofs

We are going to consider only the case $\alpha=2$. The proofs for the case $0<\alpha<2$ are formally the same as for $\alpha=2$, but the generalized Langevin equations in Theorem 4.3 need to be interpreted in an extended sense; see Remark 4.4(b).

Proof of Theorem 4.1 (a). The proof of this theorem uses Minlos-Sasonov's Theorem [12]. First we note that

$$\begin{aligned}
\mathbb{E}\left[e^{i\sum_{j=1}^{p}\theta_j\langle\varphi_j,M^{1,K}_{t_j}\rangle}\right] &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\frac{\langle\varphi_j,X^{1,K}_{t_j}\rangle-K\langle\varphi_j,\Lambda\rangle}{K^{-1/2}}\right)\right]\\
&= \exp\left(-K\int_{\mathbb{R}^d}\mathbb{E}_x\left[1-e^{i\sum_{j=1}^{p}\theta_jK^{-1/2}\langle\varphi_j,Z_{t_j}\rangle}\right]dx\right)\\
&\quad\times\exp\left(-iK^{1/2}\sum_{j=1}^{p}\theta_j\langle\varphi_j,\Lambda\rangle\right)\\
&= \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2dx\right)\\
&\quad\times\exp\left(\int_{\mathbb{R}^d}K\left[\mathbb{E}_x\left(e^{i\sum_{j=1}^{p}K^{-1/2}\theta_j\langle\varphi_j,Z_{t_j}\rangle}-1\right)\right.\right.\\
&\quad\left.\left.-iK^{-1/2}\sum_{j=1}^{p}\theta_j\mathbb{E}_x\langle\varphi_j,Z_{t_j}\rangle+\frac{1}{2}K^{-1}E_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2\right]dx\right),
\end{aligned}$$

where, in the right-hand side of the last equality, the integrand in the rightmost exponential converges to 0 as $K\to\infty$, and is bounded by $c\sum_{j=1}^{p}\theta_j^2\mathbb{E}_x\langle\varphi_j,Z_{t_j}\rangle^2$ for some constant $c>0$ (see [3], Proposition 8.44). Hence,

$$\lim_{K\to\infty}\mathbb{E}\left[e^{i\sum_{j=1}^{p}\theta_j\langle\varphi_j,M^{1,K}_{t_j}\rangle}\right]=\exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}\mathbb{E}_x\left[\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right]^2dx\right),$$

and

$$\int_{\mathbb{R}^d}\mathbb{E}_x\left[\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right]^2dx=\sum_{j=1}^{p}\sum_{k=1}^{p}\theta_j\theta_k\mathcal{K}^1(t_j,\varphi_j;t_k,\varphi_k).$$

This shows that $M^{1,K}\Rightarrow M^1$ as $K\to\infty$. There remains to prove that the Gaussian process M^1 has a version whose paths are a.s. continuous in the strong topology of $S'(\mathbb{R}^d)$. According to Theorem 2.1, it suffices to show that, for every $\varphi\in S(\mathbb{R}^d)$ and any number $T>0$, there exists a positive constant $c_T(\varphi)$ such that

$$\mathbb{E}\left[\langle\varphi,M^1_t\rangle-\langle\varphi,M^1_s\rangle\right]^2\le c_T(\varphi)|t-s|,\quad 0\le s,t\le T. \tag{19}$$

Let $T>0$ and $0\le s<t\le T$. For any $\varphi\in S(\mathbb{R}^d)$ we have that

$$\begin{aligned}
\mathbb{E}\left[\langle\varphi,M^1_t\rangle-\langle\varphi,M^1_s\rangle\right]^2 &\le \left|\frac{\mathcal{K}^1(s,\phi;t,\phi)-\mathcal{K}^1(s,\phi;s,\phi)}{t-s}\right||t-s|\\
&+\left|\frac{\mathcal{K}^1(s,\phi;t,\phi)-\mathcal{K}^1(t,\phi;t,\phi)}{t-s}\right||t-s|.
\end{aligned}\tag{20}$$

Hence, from (12) we get

$$\begin{aligned}&\mathcal{K}^1(s,\varphi;t,\varphi)-\mathcal{K}^1(s,\varphi;s,\varphi)\\ &\quad= \langle\varphi(\mathcal{S}_{t-s}\varphi-\varphi),\Lambda\rangle+\int_0^s\langle(\mathcal{S}_{s-r}\varphi)\,\mathcal{S}_{s-r}\left(\mathcal{S}_{t-s}\varphi-\varphi\right),\Lambda\rangle\,dU(r).\end{aligned}$$

Let $p\in(d/2,(d+\alpha)/2)$. It follows from the definition of $[\![\cdot]\!]_p$ and Lemma 2.2 that

$$\frac{|\mathcal{S}_{t-s}\varphi-\varphi|}{t-s}\le\frac{1}{t-s}\int_0^{t-s}|\mathcal{S}_r\Delta_\alpha\varphi|\,dr\le\frac{\text{Const.}}{t-s}\int_0^{t-s}[\![\Delta_\alpha\varphi]\!]_p\,dr=\text{Const.}[\![\Delta_\alpha\varphi]\!]_p.$$

Moreover, for any $\varphi\in S(\mathbb{R}^d)$, $\|\varphi\|_{L^1(\mathbb{R}^d)}\le[\![\varphi]\!]_p\|\varphi_p\|_{L^1(\mathbb{R}^d)}<\infty$. Therefore,

$$\begin{aligned}\left|\frac{\mathcal{K}^1(s,\varphi;t,\varphi)-\mathcal{K}^1(s,\varphi;s,\varphi)}{t-s}\right| &\le \text{Const.}\left([\![\Delta_\alpha\varphi]\!]_p\|\varphi\|_{L^1(\mathbb{R}^d)}+U(s)[\![\Delta_\alpha\varphi]\!]_p\|\varphi\|_{L^1(\mathbb{R}^d)}\right)\\ &\le \text{Const.}\left([\![\Delta_\alpha\varphi]\!]_p\|\varphi\|_{L^1(\mathbb{R}^d)}+U(T)[\![\Delta_\alpha\varphi]\!]_p\|\varphi\|_{L^1(\mathbb{R}^d)}\right)\end{aligned}$$

because the renewal function U is monotonically increasing. We conclude that

$$\left|\frac{\mathcal{K}^1(s,\varphi;t,\varphi)-\mathcal{K}^1(s,\varphi;s,\varphi)}{t-s}\right|\le c_T^1(\varphi),$$

and in a similar way one can show that

$$\left|\frac{\mathcal{K}^1(s,\varphi;t,\varphi)-\mathcal{K}^1(t,\varphi;t,\varphi)}{t-s}\right|\le c_T^2(\varphi),$$

where $c_T^1(\varphi)$ and $c_T^2(\varphi)$ are positive constants. Setting $c_T(\varphi)=c_T^1(\varphi)+c_T^2(\varphi)$ yields (19). □

Proof of Theorem 4.2 (a). From Proposition 3.3 we get that, for all $\varphi\in S(\mathbb{R}^d)$,

$$\mathbb{E}\left(\frac{\langle\varphi,X_t^{1,K}\rangle}{K}-\langle\varphi,\Lambda\rangle\right)^2=\frac{1}{K^2}\text{Var}\left(\langle\varphi,X_t^{1,K}\rangle\right)=\frac{1}{K}\mathcal{K}^1(t,\varphi;t,\varphi).$$

Letting $K\longrightarrow\infty$ yields the result. □

The following lemmas 5.1 and 5.2 are going to be useful in proving Theorem 4.1(b).

Lemma 5.1. *Let* $\mathcal{K}^{2,K}(t_1,\varphi_1;t_2,\varphi_2):=\text{Cov}(\langle\varphi_1,M_{t_1}^{2,K}\rangle,\langle\varphi_2,M_{t_2}^{2,K}\rangle)$*, where* $0\le t_2\le t_1<\infty$ *and* $\varphi_1,\varphi_2\in S(\mathbb{R}^d)$*. Then,*

$$\mathcal{K}^{2,K}(t_1,\varphi_1;t_2,\varphi_2)\longrightarrow\mathcal{K}^2(t_1,\varphi_1;t_2,\varphi_2)\ \textit{ as }\ K\longrightarrow\infty,$$

where

$$\mathcal{K}^2(t_1,\varphi_1;t_2,\varphi_2)=\frac{1}{\Gamma(1+\gamma)}\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)\,dx\,d(u^\gamma).$$

Proof: Notice that,

$$\begin{aligned}\text{Cov}(\langle\varphi_1^K,X_{K^\alpha t_1}\rangle;\langle\varphi_2^K,X_{K^\alpha t_2}\rangle) &= \langle\varphi_2^K\mathcal{S}_{K^\alpha(t_1-t_2)}\varphi_1^K,\Lambda\rangle\\ &\quad+\int_0^{K^\alpha t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{K^\alpha t_2-r}\varphi_2^K)(x)(\mathcal{S}_{K^\alpha t_1-r}\varphi_1^K)(x)dx\,dU(r).\end{aligned}$$

Performing the change of variables $u = r/K^\alpha$ and using the self-similarity property of the α-stable semigroup, the above equality renders

$$\begin{aligned}\mathrm{Cov}(\langle\varphi_1^K, X_{K^\alpha t_1}\rangle;\langle\varphi_2^K, X_{K^\alpha t_2}\rangle) &= K^d\langle\varphi_2\mathcal{S}_{t_1-t_2}\varphi_1,\Lambda\rangle \\ &+K^d\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)\,dx\,dU(K^\alpha u).\end{aligned} \tag{21}$$

Now, by definition

$$\mathcal{K}^{2,K}(t_1,\varphi_1;t_2,\varphi_2) = K^{-(d+\alpha\gamma)}\mathrm{Cov}(\langle\varphi_1^K, X_{K^\alpha t_1}\rangle;\langle\varphi_2^K, X_{K^\alpha t_2}\rangle),$$

and from (21),

$$\begin{aligned}\mathcal{K}^{2,K}(t_1,\varphi_1;t_2,\varphi_2) &= K^{-\alpha\gamma}\langle\varphi_2\mathcal{S}_{t_1-t_2}\varphi_1,\Lambda\rangle \\ &+K^{-\alpha\gamma}\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u})\varphi_2(x)(\mathcal{S}_{t_1-u})\varphi_1(x)\,dx\,dU(K^\alpha u).\end{aligned} \tag{22}$$

Using (1) and Karamata's Tauberian theorem we conclude, as in [1] p. 361, that $U(K^\alpha u) \sim (K^\alpha u)^\gamma/\Gamma(1+\gamma)$ for all K large enough, and therefore

$$\mathcal{K}^{2,K}(t_1,\varphi_1;t_2,\varphi_2) \longrightarrow \mathcal{K}^2(t_1,\varphi_1;t_2,\varphi_2),$$

as $K \longrightarrow \infty$. □

Lemma 5.2. *For each $0 \le t_3 \le t_2 \le t_1 < \infty$ and $\varphi_j \in S(\mathbb{R}^d)$, $j = 1,2,3$,*

$$\begin{aligned}\mathbb{E}_x\left[\prod_{j=1}^3\langle\varphi_j, Z_{t_j}\rangle\right] &= \mathbb{E}_x\left[\prod_{j=1}^3\varphi_j(B_{t_j})\right] \\ &+\int_0^{t_3}\mathbb{E}_x\left[C_{B_s}(t_3-s,\varphi_3;t_2-s,\varphi_2)m_{B_s}(t_1-s,\varphi_1)\right. \\ &+C_{B_s}(t_3-s,\varphi_3;t_1-s,\varphi_1)m_{B_s}(t_2-s,\varphi_2) \\ &\left.+C_{B_s}(t_2-s,\varphi_2;t_1-s,\varphi_1)m_{B_s}(t_3-s,\varphi_3)\right]\,dU(s) \\ &-\mathbb{E}_x\left[\varphi_3(B_{t_3})\int_{t_3}^{t_2}\prod_{j=1}^2 m_{B_s}(t_j-s,\varphi_j)dU(s)\right].\end{aligned} \tag{23}$$

Proof: Keeping in mind the notations in Lemma 3.1, we have that, for $p = 3$,

$$\mathbb{E}_x\left[\prod_{j=1}^3\langle\varphi_j, Z_{t_j}\rangle\right] = \frac{\partial^3}{\partial\theta_3\theta_2\theta_1}Q_t^3\theta_{(3)}(x)|_{\theta_j=0},$$

where

$$
\begin{aligned}
\frac{\partial^3}{\partial\theta_3\theta_2\theta_1} Q^3_{\bar t}\theta_{(3)}(x) &= \mathbb{E}_x \Bigg\{ \prod_{j=1}^{3} \varphi_j(B_{t_j}) e^{-\sum_{j=1}^{3}\theta_j\varphi_j(B_{t_j})} \\
&- \int_0^{t_1} \Bigg[\Psi'''(Q^3_{\bar t-s}\theta_{(3)}(B_s)) \prod_{j=1}^{3} \frac{\partial}{\partial\theta_j} Q^3_{\bar t-s}\theta_{(3)}(B_s) \\
&+ \Psi''(Q^3_{\bar t-s}\theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_3\partial\theta_2} Q^3_{\bar t-s}\theta_{(3)}(B_s) \frac{\partial}{\partial\theta_1} Q^3_{\bar t-s}\theta_{(3)}(B_s) \\
&+ \Psi''(Q^3_{\bar t-s}\theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_3\partial\theta_1} Q^3_{\bar t-s}\theta_{(3)}(B_s) \frac{\partial}{\partial\theta_2} Q^3_{\bar t-s}\theta_{(3)}(B_s) \\
&+ \Psi''(Q^3_{\bar t-s}\theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_2\partial\theta_1} Q^3_{\bar t-s}\theta_{(3)}(B_s) \frac{\partial}{\partial\theta_3} Q^3_{\bar t-s}\theta_{(3)}(B_s) \\
&+ \Psi'(Q^3_{\bar t-s}\theta_{(3)}(B_s)) \frac{\partial^3}{\partial\theta_3\partial\theta_2\partial\theta_1} Q^3_{\bar t-s}\theta_{(3)}(B_s) \Bigg] dN_s \\
&+ \prod_{j=2}^{3} \varphi_j(B_{t_j}) e^{-\theta_j\varphi_j(B_{t_j})} \int_{t_2}^{t_1} \Psi'(Q^1_{\bar t-s}\theta_{(1)}(B_s)) \frac{\partial}{\partial\theta_1} Q^1_{\bar t-s}\theta_{(1)}(B_s) dN_s \\
&- \varphi_3(B_{t_3}) e^{-\theta_3\varphi_3(B_{t_3})} \int_{t_3}^{t_2} \Psi'(Q^2_{\bar t-s}\theta_{(2)}(B_s)) \frac{\partial^2}{\partial\theta_2\partial\theta_1} Q^2_{\bar t-s}\theta_{(2)}(B_s) dN_s \\
&- \varphi_3(B_{t_3}) e^{-\theta_3\varphi_3(B_{t_3})} \int_{t_3}^{t_2} \Psi''(Q^2_{\bar t-s}\theta_{(2)}(B_s)) \prod_{j=1}^{2} \frac{\partial}{\partial\theta_j} Q^2_{\bar t-s}\theta_{(2)}(B_s) dN_s \Bigg\}.
\end{aligned}
$$

Since $\Psi(s) = \frac{1}{2}s^2$, we have that

$$
\Psi'(0) = 0, \quad \Psi''(0) = 1, \quad \Psi^{(K)}(0) = 0, \quad \text{for} \quad K = 3, 4, \cdots.
$$

The proof ends by recalling that $Q^3_{\bar t}\theta_{(3)}(x)|_{\theta_1=\theta_2=\theta_3=0} = 0$. □

Lemma 5.3. *Assume that $d > \alpha\gamma$. For each $0 \le t_3 \le t_2 \le t_1 < \infty$ and $\varphi_j \in S(\mathbb{R}^d)$, $j = 1, 2, 3$,*

$$
K^{-(d+\alpha\gamma)3/2} \int_{\mathbb{R}^d} \mathbb{E}_x \left[\prod_{j=1}^{3} \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right] dx \longrightarrow 0 \quad \textit{as} \quad K \longrightarrow \infty.
$$

Proof: We are going to use the formula (23) given in Lemma 5.2. We start with the first term in the right hand side of (23). Using the Markov property of the α-stable process, we obtain

$$
\begin{aligned}
&\mathbb{E}_x\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\right]\\
&= \mathbb{E}_x\left\{\mathbb{E}\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\middle| B_{K^\alpha t_3}\right]\right\}\\
&= \mathbb{E}_x\left\{\varphi_3^K(B_{K^\alpha t_3})\mathbb{E}\left[\varphi_1^K(B_{K^\alpha t_1})\varphi_2^K(B_{K^\alpha t_2})|B_{K^\alpha t_3}\right]\right\}\\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}\left[\varphi_1^K(B_{K^\alpha t_1})\varphi_2^K(B_{K^\alpha t_2})|B_{K^\alpha t_3}=y\right]dy\\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}_y\left[\varphi_1^K(B_{K^\alpha (t_1-t_3)})\varphi_2^K(B_{K^\alpha (t_2-t_3)})\right]dy\\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}_y\left\{\mathbb{E}\left[\varphi_1^K(B_{K^\alpha (t_1-t_3)})\varphi_2^K(B_{K^\alpha (t_2-t_3)})|B_{K^\alpha (t_2-t_3)}\right]\right\}dy\\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\int p_{K^\alpha (t_2-t_3)}(y,z)\varphi_2^K(z)\mathbb{E}_z[\varphi_1^K(B_{K^\alpha (t_1-t_2)})]dzdy\\
&= \left(\mathcal{S}_{K\alpha t_3}\left(\varphi_3^K(\cdot)\left(\mathcal{S}_{K^\alpha (t_2-t_3)}\varphi_2^K\right)(\cdot)\left(\mathcal{S}_{K^\alpha (t_1-t_2)}\varphi_1^K\right)(\cdot)\right)\right)(x)\\
&= \left(\mathcal{S}_{t_3}\left(\varphi_3(\cdot)\left(\mathcal{S}_{t_2-t_3}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-t_2}\varphi_1\right)(\cdot)\right)\right)^K(x).
\end{aligned}
$$

Then, after a change of variables and using that the Lebesgue measure is invariant for the α-stable semigroup, we get

$$
\int_{\mathbb{R}^d}\mathbb{E}_x\left[\prod_{j=1}^{3}\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle\right]dx = K^d\int_{\mathbb{R}^d}\varphi_3(x)\left(\mathcal{S}_{t_2-t_3}\varphi_2\right)(x)\left(\mathcal{S}_{t_1-t_2}\varphi_1\right)(x)dx. \tag{24}
$$

Hence,

$$
K^{-(d+\alpha\gamma)3/2}\int_{\mathbb{R}^d}\mathbb{E}_x\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\right]dx \longrightarrow 0, \text{ as } K\longrightarrow\infty. \tag{25}
$$

Now we deal with the second term in the right-hand side of (23). Namely,

$$
\begin{aligned}
&\int_0^{K^\alpha t_3}\mathbb{E}_x\left[C_{B_s}(K^\alpha t_3-s,\varphi_3^K;K^\alpha t_2-s,\varphi_2^K)m_{B_s}(K^\alpha t_1-s,\varphi_1^K)\right]dU(s)\\
&= \int_0^{t_3}\mathbb{E}_x[C_{B_{K^\alpha s}}(K^\alpha(t_3-s),\varphi_3;K^\alpha(t_2-s),\varphi_2)m_{B_{K^\alpha s}}(K^\alpha(t_1-s),\varphi_1)]dU(K^\alpha s)\\
&= \int_0^{t_3}\int_{\mathbb{R}^d}p_{K^\alpha s}(x,y)\left[\left(\varphi_3^K\mathcal{S}_{K^\alpha(t_2-t_3)}\varphi_2^K\right)(y)\right.\\
&\quad\left.+\int_0^{K^\alpha(t_3-s)}\left(\mathcal{S}_{K^\alpha t_3-r}\varphi_3^K\right)(y)\left(\mathcal{S}_{K^\alpha t_2-r}\varphi_2^K\right)(y)dU(r)\left(\mathcal{S}_{K^\alpha(t_1-s)}\varphi_1^K\right)(y)\right]dy\,dU(K^\alpha s),
\end{aligned}
$$

where

$$\begin{aligned}
&\int_0^{t_3} \int_{\mathbb{R}^d} p_{K^\alpha s}(x,y)\varphi_3^K(y)(\mathcal{S}_{K^\alpha(t_2-t_3)}\varphi_2^K)(y)dydU(K^\alpha s) \\
&\quad = \int_0^{t_3} \left(\mathcal{S}_{K^\alpha s}(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)^K\right)(x)dU(K^\alpha s) \\
&\quad = \int_0^{t_3} \left(\mathcal{S}_s(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)\right)^K(x)dU(K^\alpha s),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t_3} \int_{\mathbb{R}^d} p_{K^\alpha s}(x,y) \int_0^{K^\alpha(t_3-s)} \left(\mathcal{S}_{K^\alpha t_3 - r}\varphi_3^K\right)(y)\left(\mathcal{S}_{K^\alpha t_2 - r}\varphi_2^K\right)(y)dU(r) \\
&\qquad \times \left(\mathcal{S}_{K^\alpha(t_1-s)}\varphi_1^K\right)(y)dydU(K^\alpha s) \\
&\quad = \int_0^{t_3} \mathcal{S}_{K^\alpha t_3} \int_0^{t_3-s} \left[\left(\mathcal{S}_{K^\alpha(t_3-h)}\varphi_3^K\right)(\cdot)\left(\mathcal{S}_{K^\alpha(t_2-h)}\varphi_2^K\right)(\cdot)dU(K^\alpha h)\right. \\
&\qquad \left.\times \left(\mathcal{S}_{K^\alpha(t_1-s)}\varphi_1^K\right)(\cdot)\right](x)dU(K^\alpha s) \\
&\quad = \int_0^{t_3}\int_0^{t_3-s} \mathcal{S}_{K^\alpha s}\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)^K(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)^K(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)^K(\cdot)\right](x)dU(K^\alpha h)dU(K^\alpha s) \\
&\quad = \int_0^{t_3}\int_0^{t_3-s} \left(\mathcal{S}_s\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)(\cdot)\right](x)\right)^K dU(K^\alpha h)dU(K^\alpha s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^{K^\alpha t_3} \mathbb{E}_x[C_{B_s}(K^\alpha t_3 - s, \varphi_3^K; K^\alpha t_2 - s, \varphi_2^K)m_{B_s}(K^\alpha t_1 - s, \varphi_1^K)]dU(s)dx \\
&\quad = \int_{\mathbb{R}^d}\int_0^{t_3} \left(\mathcal{S}_s(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)\right)^K(x)dU(K^\alpha s)dx \\
&\qquad + \int_{\mathbb{R}^d}\int_0^{t_3}\int_0^{t_3-s} \left(\mathcal{S}_s\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)(\cdot)\right](x)\right)^K dU(K^\alpha h)dU(K^\alpha s)dx,
\end{aligned}$$

then as in (24),

$$\int_{\mathbb{R}^d}\int_0^{K^\alpha t_3} \mathbb{E}_x[C_{B_s}(K^\alpha t_3 - s, \varphi_3^K; K^\alpha t_2 - s, \varphi_2^K)m_{B_s}(K^\alpha t_1 - s, \varphi_1^K)]dU(s)dx = O(K^{d+\alpha\gamma}) + O(K^{d+2\alpha\gamma}).$$

Similarly, we have that

$$\int_{\mathbb{R}^d}\int_0^{K^\alpha t_3} \mathbb{E}_x[C_{B_s}(K^\alpha t_3 - s, \varphi_3^K; K^\alpha t_1 - s, \varphi_1^K)m_{B_s}(K^\alpha t_2 - s, \varphi_2^K)]dU(s)dx = O(K^{d+\alpha\gamma}) + O(K^{d+2\alpha\gamma}),$$

and

$$\int_{\mathbb{R}^d}\int_0^{K^\alpha t_3} \mathbb{E}_x[C_{B_s}(K^\alpha t_2 - s, \varphi_2^K; K^\alpha t_1 - s, \varphi_1^K)m_{B_s}(K^\alpha t_3 - s, \varphi_3^K)]dU(s)dx = O(K^{d+\alpha\gamma}) + O(K^{d+2\alpha\gamma}).$$

Also, it can be shown as in the preceding calculations that,

$$\int_{\mathbb{R}^d} \mathbb{E}_x \left[\varphi_3^K(B_{K^\alpha t_3}) \int_{K^\alpha t_3}^{K^\alpha t_2} \prod_{j=1}^{2} m_{B_s}(K^\alpha t_j - s, \varphi_j^K) dU(s) \right] dx = O(K^{d+\alpha\gamma}).$$

In this way, putting together all these calculations, we obtain that

$$\int_{\mathbb{R}^d} \mathbb{E}_x \left[\prod_{j=1}^{3} \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right] dx = O(K^d) + O(K^{d+\alpha\gamma}) + O(K^{d+2\alpha\gamma}) - O(K^{d+\alpha\gamma}).$$

Finally, since $d > \alpha\gamma$,

$$K^{-(d+\alpha\gamma)3/2} \int_{\mathbb{R}^d} \mathbb{E}_x \left[\prod_{j=1}^{3} \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right] dx \to 0 \quad \text{as} \quad K \to \infty.$$

□

Proof of Theorem 4.1 (b). Given $0 \le t_p \le t_{p-1} \le \cdots \le t_1 < \infty$ and $\varphi_1, \cdots, \varphi_p \in S(\mathbb{R}^d)$, we have that, for each $\theta_1, \cdots, \theta_p \in \mathbb{R}$,

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} \theta_j \langle \varphi_j, M_{t_j}^{2,K} \rangle \right) \right] \\
= \ & \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} \theta_j \frac{\langle \varphi_j^K, X_{K^\alpha t_j} \rangle - \mathbb{E}\langle \varphi_j^K, X_{K^\alpha t_j} \rangle}{K^{(d+\alpha\gamma)/2}} \right) \right] \\
= \ & \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2} \theta_j \langle \varphi_j^K, X_{K^\alpha t_j} \rangle \right) \right] \\
& \times \exp\left(-i \sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2} \theta_j \mathbb{E}\langle \varphi_j^K, X_{K^\alpha t_j} \rangle \right),
\end{aligned}$$

where, due to the fact that the initial population is Poisson distributed,

$$\begin{aligned}
= \ & \mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2} \theta_j \langle \varphi_j^K, X_{K^\alpha t_j} \rangle \right) \right] \\
& \times \exp\left(- \int_{\mathbb{R}^d} \mathbb{E}_x \left[1 - e^{i \sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2} \theta_j \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle} \right] dx \right) \\
= \ & \exp\left(\int_{\mathbb{R}^d} \left[i K^{-(d+\alpha\gamma)/2} \mathbb{E}_x \left(\sum_{j=1}^{p} \theta_j \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right) - \frac{K^{-(d+\alpha\gamma)}}{2} \mathbb{E}_x \left(\sum_{j=1}^{p} \theta_j \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right)^2 \right. \right. \\
& \left. \left. - \frac{i}{3!} K^{-(d+\alpha\gamma)3/2} \mathbb{E}_x \left(\sum_{j=1}^{p} \theta_j \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \right)^3 + \cdots \right] dx \right).
\end{aligned}$$

Thus, from the preceding calculations and lemmas 5.1 and 5.3, we get that

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\langle\varphi_j, M^{2,K}_{t_j}\rangle\right)\right] = \exp\left(-\frac{1}{2}\sum_{j=1}^{p}\sum_{l=1}^{p}\theta_j\theta_l\mathcal{K}^K(t_j,\varphi_j;t_l,\varphi_l) + o(K^{(d+\alpha\gamma)3/2})\right),$$

and therefore,

$$\lim_{K\longrightarrow\infty}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\langle\varphi_j, M^{2,K}_{t_j}\rangle\right)\right] = \exp\left(-\frac{1}{2}\sum_{j=1}^{p}\sum_{l=1}^{p}\theta_j\theta_l\mathcal{K}(t_j,\varphi_j;t_l,\varphi_l)\right).$$

It follows from the last equality that $M^{2,K} \Rightarrow M^2$, as $K \longrightarrow \infty$. The a.s. continuity of M^2 is proved following the same steps as in the case of M^1; see the proof of Theorem 4.1(a). □

Proof of Theorem 4.2 (b). We only consider distribution functions F satisfaying (1), the proof for the case of finite-mean particle lifetimes being similar and simpler. Given $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$ we have that,

$$\begin{aligned}\mathbb{E}\left(\frac{\langle\varphi^K, X_{K^\alpha t}\rangle}{K^d} - \langle\varphi,\Lambda\rangle\right)^2 &= K^{-2d}\mathbb{E}\left(\langle\varphi^K, X_{K^\alpha t}\rangle - \mathbb{E}\langle\varphi^K, X_{K^\alpha t}\rangle\right) \\ &= K^{-2d}C^2_K(t,\varphi;t,\varphi).\end{aligned}$$

Using Karamata's Tauberian theorem (as in [1], pag. 361), we get from (21) that

$$\begin{aligned}&\lim_{K\longrightarrow\infty}\mathbb{E}\left(\frac{\langle\varphi^K, X_{K^\alpha t}\rangle}{K^d} - \langle\varphi,\Lambda\rangle\right)^2 \\ &= \lim_{K\longrightarrow\infty}\left(K^{-d}\langle\varphi^2,\Lambda\rangle + K^{-d}\int_0^t\langle(\mathcal{S}_{t-s}\varphi)^2,\Lambda\rangle dU(K^\alpha s)\right) \\ &= \lim_{K\longrightarrow\infty}\frac{K^{-d+\alpha\gamma}}{\Gamma(1+\gamma)}\int_0^t\langle(\mathcal{S}_{t-s}\varphi)^2,\Lambda\rangle d(s^\gamma) = 0,\end{aligned}$$

because of $-d+\alpha\gamma < 0$. This ends the proof. □

5.1. **Proof of Theorem 4.3**

Proof of (a): According to [16], in order to prove that the limit processes are Markovian, it suffices to verify that

$$\mathcal{K}^l(s,\varphi;s,\mathcal{S}_{t-s}\psi) = \mathcal{K}^l(s,\varphi;t,\psi) \text{ for all } s\leq t \text{ and } \varphi,\psi\in S(\mathbb{R}^d),\ l=1,2.$$

Indeed,

$$\begin{aligned}\mathcal{K}^1(s,\varphi;s,\mathcal{S}_{t-s}\psi) &= \langle\varphi\mathcal{S}_{s-s}\mathcal{S}_{t-s}\psi,\Lambda\rangle + \int_0^s\langle(\mathcal{S}_{s-r}(\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi),\Lambda\rangle dU(r) \\ &= \langle\varphi\mathcal{S}_{t-s}\psi,\Lambda\rangle + \int_0^s\langle(\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi),\Lambda\rangle dU(r) \\ &= \mathcal{K}^1(s,\varphi;t,\psi). \qquad (26)\end{aligned}$$

In a similar way one can verify that $\mathcal{K}^2(s,\varphi;s,\mathcal{S}_{t-s}\psi) = \mathcal{K}^2(s,\varphi;t,\psi)$. Hence, the Markov property follows from Theorem 6 in [16]. The martingale property of (14) follows directly from (26) and Theorem 18 in [8]. □

Proof of (b): We already have proved that M^1 is almost surely continuous in the strong topology of $S'(\mathbb{R}^d)$.

In order to prove that there exists $p \geq 1$ such that M^1 is almost surely continuous in the norm $\|\cdot\|_{-p}$, it suffices to show that

$$\sup_{T\in\mathbb{R}_+} \frac{V_T(\phi)}{g(T)} < \infty, \tag{27}$$

where g is a positive locally bounded function on $[0,\infty)$ and

$$V_T(\phi) := \mathbb{E}\left[\sup_{0\leq t\leq T} \langle \phi, M_t^1\rangle^2\right],$$

with $\phi \in S(\mathbb{R}^d)$. Taking for granted (27), the $\|\cdot\|_p$–continuity of M^1 for some $p \geq 1$ follows from Theorem 4 in [17]. The proof of (27) goes along the same lines as in [10], see page 386 there. Namely, by applying Doob's inequality to the martingale (14). In the same fashion it is proved the a.s. continuity of M^2 in the norm $\|\cdot\|_{-p}$ for some $p > 0$. We omit the details. □

Proof of (c): Since, by Theorem 4.1 (a), M^1 is a continuous, centered $S'(\mathbb{R}^d)$-valued Gaussian process, due to Theorem 3.6 in [2] it suffices to verify that:

(1) for each $\varphi \in S(\mathbb{R}^d)$, the function $s \mapsto \mathcal{K}^1(s,\varphi;s,\varphi)$ is continuously differentiable;

(2) for any $0 \leq s \leq t$, and $\varphi,\psi \in S(\mathbb{R}^d)$, $\mathcal{K}^1$ satisfies $\mathcal{K}^1(s,\varphi;t,\varphi) = \mathcal{K}^1(s,\varphi;s,\mathcal{S}_{t-s},\varphi)$.

Notice that (2) above follows from Theorem 4.3(a). Let us show that (1) also holds. We have that, for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{K}^1(t,\varphi;t,\varphi) &= \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle dU(r)\\ &= \langle \varphi^2, \Lambda\rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle u(r)dr,\end{aligned}$$

where the second inequality is a consequence of our assumption that the lifetimes distribution possesses a continuous density. Hence, the function $t \longmapsto \mathcal{K}^1(t,\varphi;t,\varphi)$ is continuously differentiable and M^1 satisfies all the conditions in Theorem 3.6 in [2]. There remains to verify (16). Notice that, for $s = t$, (12) can be written as

$$\mathcal{K}^1(t,\varphi;t,\psi) = \langle \varphi\psi, \Lambda\rangle + \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\psi), \Lambda\rangle u(r)\,dr, \quad t \geq 0, \quad \varphi,\phi \in S(\mathbb{R}^d).$$

Therefore,

$$\langle Q_t^1\varphi,\varphi\rangle \equiv \frac{d}{dt}\mathcal{K}^1(t,\varphi;t,\varphi) - 2\mathcal{K}^1(t,\Delta_\alpha\varphi;t,\varphi) = \langle \varphi^2,\Lambda\rangle - 2\langle (\Delta_\alpha\varphi)\varphi,\Lambda\rangle.$$

Notice that (16) follows also from the expression

$$\langle Q_t^1\varphi,\psi\rangle = \frac{1}{2}\left[\langle Q_t^1(\varphi+\psi),(\varphi+\psi)\rangle - \langle Q_t^1\varphi,\varphi\rangle - \langle Q_t^1\psi,\psi\rangle\right].$$

Equation (17) is obtained in the same way as above. In this case

$$\begin{aligned}\langle Q_t^2\varphi,\psi\rangle &= \frac{d}{dt}\mathcal{K}^2(t,\varphi;t,\psi) - \mathcal{K}^2(t,\Delta_\alpha\varphi;t,\psi) - \mathcal{K}^2(t,\varphi;t,\Delta_\alpha\psi)\\ &= \frac{\gamma t^{\gamma-1}}{\Gamma(1+\gamma)}\langle\varphi,\psi\rangle, \quad t > 0,\end{aligned}$$

which shows that

$$\mathbb{E}\left[\langle\varphi,\mathcal{W}_s^2\rangle\langle\psi,\mathcal{W}_t^2\rangle\right]=\int_0^{s\wedge t}\langle Q_r^2\varphi,\psi\rangle\,dr=\frac{(s\wedge t)^\gamma}{\Gamma(1+\gamma)}\langle\varphi,\psi\rangle,\quad 0\le s,t,\quad \varphi,\psi\in S'(\mathbb{R}^d).$$

□

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