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REAL-VALUED CONDITIONAL CONVEX RISK MEASURES IN $L^p(\mathcal{F}, R)$

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Abstract. The numerical representation of convex risk measures beyond essentially bounded financial positions is an important topic which has been the theme of recent literature. In other direction, it has been discussed the assessment of essentially bounded risks taking explicitly new information into account, i.e., conditional convex risk measures. In this paper we combine these two lines of research. We discuss the numerical representation of conditional convex risk measures which are defined in a space $L^p(\mathcal{F}, R)$, for $p \geq 1$, and take values in $L^1(\mathcal{G}, R)$ (in this sense, *real-valued*).
We show how to characterize such a class of real-valued conditional convex risk measures. In the first result of the paper, we see that real-valued conditional convex risk measures always admit a numerical representation in terms of a nice class of "locally equivalent" probability measures $\mathcal{Q}^{q,\infty}_{e,loc}$. To this end, we use the recent extended Namioka-Klee Theorem, due to Biagini and Frittelli. The second result of the paper says that a conditional convex risk measure defined in a space $L^p(\mathcal{F}, R)$ is real-valued if and only if the corresponding minimal penalty function satisfies a coerciveness property, as introduced by Cheridito and Li in the non-conditional case. This characterization, together with an invariance property will allow us to characterize conditional convex risk measures defined in a space $L^\infty(\mathcal{F}, R)$ which can be extended to a space $L^p(\mathcal{F}, R)$, and at the same time continue to be real-valued. In particular we see that the measures of risk, AVaR and Shortfall, assign real values even if we extend their natural domain $L^\infty(\mathcal{F}, R)$ to a space $L^p(\mathcal{F}, R)$.

1. Introduction

The modern point of view of risk-management uses a dynamical approach. Key points in the analysis are the appropriate quantifications of risks and the optimal use of information. Thus, a question of interest is, how to construct measures of risk whose quantifications incorporate the benefit of new information. This question has motivated the development of conditional convex risk measures and more in general of dynamical risk measures. The discussion of this new class and the crucial robust representations have been considered by several authors; see e.g., Bion-Nadal [3], Theorem 3, and Detlefsen and Scandolo [8], Theorem 1. A systematic study of dynamical convex risk measures in L^∞ is presented by Föllmer and Penner [10]. A numerical representation for conditional convex risk measures of bounded stochastic processes in discrete time is presented by Cheridito et al [6], Theorem 3.16.
The study of real-valued convex risk measures beyond essentially bounded financial positions is the topic of recent research; see e.g., Biagini and Frittelli [2], Cheridito and Li [5], Filipović and Svindland [9], Kaina and Rüschendorf [15], Krätschmer [16], Ruszczyński and Shapiro [17].
In this paper we combine these two lines of research. We discuss the numerical representation of conditional convex risk measures which are defined in a space $L^p(\mathcal{F}, R)$, for $p \geq 1$, and take values in $L^1(\mathcal{G}, R)$ (in this

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sense, *real-valued*).

Non trivial quantifications of non bounded positions in L^0 assign infinite risk to some elements; see Delbaen [7]. However, if we consider the spaces L^p, which are in the middle of the spaces L^0 and L^∞, we may insist on real-valued quantifications of risk and still obtain interesting classes of measures of risk. In the static case this phenomenon has been studied in the afore mentioned literature. Here we show how to extend this theory to the conditional case.

The main results of the paper are Theorems 2.7 and 2.9. Theorem 2.7 says that real-valued conditional convex risk measures always admit a numerical representation in terms of a nice class of "locally equivalent" probability measures $\mathcal{Q}^{q,\infty}_{e,loc}$ with bounded penalizations, as defined in (8). Theorem 2.9 says that a conditional convex risk measure defined in a space $L^p(\mathcal{F}, R)$ is real-valued if and only if the corresponding minimal penalty function satisfies the coerciveness property of Definition 2.8. In particular, this property of coerciveness characterizes real-valued conditional convex risk measures in the spaces $L^p(\mathcal{F}, R)$ in terms of the corresponding penalty functions. This fact, together with the invariance property of Proposition 5.1 will allow us to characterize conditional convex risk measures defined in a space $L^\infty(\mathcal{F}, R)$ which can be extended to a space $L^p(\mathcal{F}, R)$ and assign values in the space $L^1(\mathcal{G}, R)$; see Remark 5.2.

The paper is organized as follows. In Section 2, we fix some notation and introduce a class of conditional convex risk measures defined in a space $L^p(\mathcal{F}, R)$, for some exponent $p \geq 1$. Our main interest is in the subclass of conditional convex risk measures which are real-valued in the sense of the Definition 2.2. A crucial concept in the paper is the property of coerciveness of Definition 2.8, a concept introduced in the theory of risk measures by Cheridito and Li [5]. The main results of this section and of the paper are Theorems 2.7 and 2.9. In Section 3, we prove Theorem 2.7. In Section 4, we prove Theorem 2.9. In Section 5, we prove an invariance property with respect to a restriction from L^p to L^∞ of the minimal representation of a real-valued conditional convex risk measures. This invariance property will allow us to characterize those conditional convex risk measures defined in a space L^∞ which can be extended to a space L^p. We give some examples of conditional convex risk measures which can be extended from L^∞ to L^p through this invariance property.

2. Real-valued conditional convex risk measures

Measures of risk were introduced in the seminal paper Artzner et al [1]. Robust numerical representations of risk measures in a general probability space were obtained by Delbaen [7] in the coherent case and by Föllmer and Schied [12,13] and Frittelli and Rosazza Gianin [14] in the convex case. Quantifying risks beyond L^∞ is the subject of recent research; see e.g., Biagini and Frittelli [2], Cheridito and Li [5], Filipović and Svindland [9], Kaina and Rüschendorf [15], Krätschmer [16], Ruszczyński and Shapiro [17].

In this part we discuss a robust numerical representation of a *real-valued* conditional convex risk measure *defined in* $L^p(\mathcal{F}, R)$; see Theorem 2.7 below. This result provides a bridge which connects two main streams in the literature: Real-valued convex risk measures in L^p and conditional convex risk measures in $L^\infty(\mathcal{F}, R)$.

Let us introduce some notation. We fix a *complete* probability space $(\Omega, \mathcal{F}, R)$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$. We assume that $\mathcal{G}$ contains the null events of R. We fix an exponent p with $1 \leq p < \infty$ and denote by q the conjugate exponent. Typically, we write Z^Q to denote the density of an absolutely continuous probability measure Q. We denote by $\overline{L}^0(\mathcal{G}, R)$ the family of $\mathcal{G}$-measurable functions with values in $\mathbb{R} \cup \{-\infty, +\infty\}$.

In the next definition, relationships of random variables hold R-a.s. true.

Definition 2.1. A conditional convex risk measure ρ in $L^p(\mathcal{F}, R)$ is a mapping

$$\rho : L^p(\mathcal{F}, R) \to \overline{L}^0(\mathcal{G}, R)$$

with the following properties. For all $X, Y \in L^p(\mathcal{F}, R)$:

(1) Conditional cash invariance: For all $Z \in L^p(\mathcal{G}, R)$ $\rho(X + Z) = \rho(X) - Z$.
(2) Monotonicity: If $X \leq Y$ R-as. then $\rho(X) \geq \rho(Y)$.
(3) Conditional convexity: For all $\lambda \in L^p(\mathcal{G}, R)$ with $0 \leq \lambda \leq 1$ R-a.s.:

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

We are going to work with conditional convex risk measures ρ which are normalized:

$$\rho(0) = 0. \tag{1}$$

The axiomatic framework of Definition 2.1 is considered by Detlefsen and Scandolo [8] and Föllmer and Penner [10]. Variants of this formulation are considered, by e.g., Cheridito et al [6] and Weber [18].

We are going to focus in conditional convex risk measures which are real-valued in the sense of the next

Definition 2.2. A conditional convex risk measure $\rho : L^p(\mathcal{F}, R) \to \overline{L}^0(\mathcal{G}, R)$ is real-valued if it takes values in $L^1(\mathcal{G}, R)$. More precisely, for each $X \in L^p(\mathcal{F}, R)$ we have $\rho(X) \in L^1(\mathcal{G}, R)$.

We need to establish a convention for the conditional expectation of a probability measure which is only absolutely continuous with respect to R. Let Q be an absolutely continuous probability measure with density Z^Q. For $X \in L^p(\mathcal{F}, R)$, we are going to chose a specific version of the conditional expectation as follows:

$$E_Q[X \mid \mathcal{G}] := \begin{cases} \frac{1}{E_R[Z^Q|\mathcal{G}]} E_R[Z^Q X \mid \mathcal{G}], & \text{in } \{E_R[Z^Q \mid \mathcal{G}] > 0\}, \\ 0, & \text{in } \{E_R[Z^Q \mid \mathcal{G}] = 0\}. \end{cases} \tag{2}$$

With this convention, the essential supremum in Equation (4) of the next definition, is unambiguous. We are going to distinguish a special class of absolutely continuous probability measures:

$$\mathcal{Q}^q := \left\{ Q \ll R \mid \frac{dQ}{dR} \in L^q(R) \right\}. \tag{3}$$

Definition 2.3. Let $\mathcal{Q} \subset \mathcal{Q}^q$ be a class of absolutely continuous probability measures. A penalty function is a correspondence of the form

$$\alpha : \mathcal{Q} \to \overline{L}^0_+(\mathcal{G}, R).$$

The pair $(\mathcal{Q}, \alpha)$ represents the convex risk measure ρ if

$$\rho(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[-X \mid \mathcal{G}] - \alpha(Q) \right\}, R - a.s, \text{ for each } X \in L^p(\mathcal{F}, R). \tag{4}$$

In this case, we say that the conditional convex risk measure ρ is representable and (4) defines a robust representation.

The numerical representation of Theorem 2.7 below involves the acceptance set and the minimal penalty function associated to ρ in the next definition.

Definition 2.4. The acceptance set of the risk measure ρ is defined by

$$\mathcal{A} := \{a \in L^p(\mathcal{F}, R) \mid \rho(a) \leq 0,\ R - a.s.\}.$$

The minimal penalty function

$$\alpha^{\min} : \mathcal{Q}^q \to \overline{L}^0(\mathcal{G}, R)$$

is given by

$$\alpha^{\min}(Q) := \begin{cases} \operatorname{ess\,sup}_{a\in\mathcal{A}} \{E_Q[-a \mid \mathcal{G}]\}, & \text{in } \{E_R[Z^Q \mid \mathcal{G}] > 0\}, \\ +\infty, & \text{in } \{E_R[Z^Q \mid \mathcal{G}] = 0\}. \end{cases} \tag{5}$$

Remark 2.5. The acceptance set $\mathcal{A}$ is a convex set with the following properties:

1. It is solid. If $X \in L^p(\mathcal{F}, R)$, $Y \in \mathcal{A}$ and $X \geq Y$, then $X \in \mathcal{A}$, due to the monotonicity property of the convex risk measure ρ.
2. If ρ has the Fatou property of Definition 3.2, then $\mathcal{A}$ is sequentially closed with respect to R-a.s. convergence.
3. If $X \in \mathcal{A}$ and $B \in \mathcal{G}$, then $1_B X \in \mathcal{A}$, due to the localization property of Lemma 3.1 below.

Remark 2.6. The minimal penalty function can equivalently be defined by

$$\alpha^{\min}(Q) = \operatorname{ess\,sup}_{X\in L^p(\mathcal{F},R)} \{E_Q[-X \mid \mathcal{G}] - \rho(X)\}.$$

At this point we cannot discard that the penalization $\alpha^{\min}(Q)$ may be infinite with Q-positive probability, but strictly less than one, for some $Q \in \mathcal{Q}$. Indeed:

$$\{E_R[Z^Q \mid \mathcal{G}] = 0\} \subset \{\alpha^{\min}(Q) = \infty\}. \tag{6}$$

With this in mind, we introduce a special subclass of $\mathcal{Q}^q$:

$$\mathcal{Q}^{q,\infty} := \{Q \in \mathcal{Q}^q \mid \alpha^{\min}(Q) \in L^\infty(\mathcal{G}, Q)\}. \tag{7}$$

The next theorem is the first main result of this section and of the paper. We need to consider the following class of "locally equivalent" probability measures with bounded penaltizations:

$$\mathcal{Q}^{q,\infty}_{e,loc} := \{Q \in \mathcal{Q}^{q,\infty} \mid E_R[Z^Q \mid \mathcal{G}] > 0,\ R - a.s.\}. \tag{8}$$

Theorem 2.7. *Let ρ be a real-valued conditional convex risk measure defined in the space $L^p(\mathcal{F}, R)$. Then the pair $(\mathcal{Q}^{q,\infty}_{e,loc}, \alpha^{\min})$ represents the risk measure ρ:*

$$\rho(X) = \operatorname{ess\,sup}_{Q\in\mathcal{Q}^{q,\infty}_{e,loc}} \left\{E_Q[-X \mid \mathcal{G}] - \alpha^{\min}(Q)\right\}. \tag{9}$$

Proof. See Section 3. □

Theorem 2.7 says that real-valued conditional risk measures are representable. In Theorem 2.9 below, we are going to see that a conditional convex risk measure is real-valued if and only if the minimal penalty function satisfies the coerciveness property (10) below.

An important concept of this section is that of coerciveness. This concept is introduced by Cheridito and Li [5], Definition 4.6, in the non-conditional case. Let us introduce the class

$$\mathcal{Q}^q_{e,loc} := \{Q \in \mathcal{Q}^q \mid E_R[Z^Q \mid \mathcal{G}] > 0, R - a.s.\}.$$

Definition 2.8. Let $\alpha : \mathcal{Q}^q_{e,loc} \to \overline{L}^0_+(\mathcal{G}, R)$ be a penalty function. We say that α is a coercive penalty function if there exist real constants a, b with $b > 0$ such that

$$E_R[\alpha(Q)] \geq a + b\, E_R \left[\frac{1}{E_R\left[\frac{dQ}{dR} \mid \mathcal{G}\right]} E_R^{\frac{1}{q}} \left[\left(\frac{dQ}{dR}\right)^q \mid \mathcal{G} \right] \right],\ Q \in \mathcal{Q}^q_{e,loc}. \tag{10}$$

The first part of the next theorem is the conditional version of the first part of Proposition 4.7 of Cheridito and Li [5] and the first part of Proposition 2.10 of Kaina and Rüschendorf [15].

Theorem 2.9. (1) *Let $\alpha : \mathcal{Q}^q_{e,loc} \to \overline{L}^0_+(\mathcal{G}, R)$ be a penalty function. Assume there exists $Q^0 \in \mathcal{Q}^q_{e,loc}$ such that $\alpha(Q^0) \in L^\infty(\mathcal{G}, R)$. Let us define a mapping ρ by*

$$\rho(X) := \operatorname{ess\,sup}_{Q \in \mathcal{Q}^q_{e,loc}} \{E_Q[-X \mid \mathcal{G}] - \alpha(Q)\},\ X \in L^p(\mathcal{F}, R).$$

Then ρ is a real-valued conditional risk measure, if α is coercive.

(2) *Conversely, let ρ be a real-valued conditional convex risk measure. If the pair $(\mathcal{Q}^q_{e,loc}, \alpha)$ represents the convex risk measure ρ, then the penalty function α must be coercive.*

Proof. See Section 4. □

3. Proof of Theorem 2.7

We start with a local property.

Lemma 3.1. *A real-valued conditional convex risk measure ρ has the following localization property. For each $A \in \mathcal{G}$:*

$$\rho(1_A X + 1_{A^c} Y) = 1_A \rho(X) + 1_{A^c} \rho(Y).$$

Proof. This property follows from the property of conditional convexity; see Detlefsen and Scandolo [8], Proposition 1. □

We prepare the proof of Theorem 2.7 with the next proposition. It provides a "coarse" representation in terms of the class $\mathcal{Q}^{q,\infty}$, introduced in (7), under the following well-known regularity condition:

Definition 3.2. The convex risk measure ρ has the Fatou property if

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n),\ R-a.s.,$$

for each sequence $\{X_n\}_{n=1}^\infty \subset L^p(\mathcal{F}, R)$ dominated by some $Y \in L^p(\mathcal{F}, R)$ and converging to $X \in L^p(\mathcal{F}, R)$.

In the proof of the next proposition we follow an approach similar to the original papers [7, 12–14].

Proposition 3.3. *Let ρ be a conditional convex risk measure in $L^p(\mathcal{F}, R)$. If ρ has the Fatou property, then for each $X \in L^p(\mathcal{F}, R)$:*

$$\rho(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{q,\infty}} \left\{E_Q[-X \mid \mathcal{G}] - \alpha^{\min}(Q)\right\},\ R-a.s. \tag{11}$$

Proof. (1) Let $X \in L^p(\mathcal{F}, R)$. We set

$$b := \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{q,\infty}} \left\{E_Q[-X \mid \mathcal{G}] - \alpha^{\min}(Q)\right\}. \tag{12}$$

We must show that $R[\rho(X) = b] = 1$. It is clear that $R[\rho(X) \geq b] = 1$, due to the definition of the minimal penalty function. Now we show the converse inequality. Assume by way of contradiction that

$$R[\rho(X) > b] > 0.$$

Let us call

$$J := \{\rho(X) > b\}.$$

Note that $J \in \mathcal{G}$ and

$$\rho(X) - b = 1_J(\rho(X) - b).$$

Moreover,

$$1_J(\rho(X) - b) = \rho(1_J(X + b)),$$

due to Lemma 3.1. Thus, $1_J(X + b)$ does not belong to the acceptance set $\mathcal{A}$.

(2) Now we separate the sets $\mathcal{A}$ and $\{1_J(X + b)\}$. There exists a linear functional $l : L^p(\mathcal{F}, R) \to \mathbb{R}$ such that

$$\inf_{a \in \mathcal{A}} l(a) \geq x, \tag{13}$$

$$l(1_J(X + b)) < x, \tag{14}$$

due to the Hahn-Banach hyperplane separating Theorem; see e.g., Föllmer and Schied [11], Theorem A.56. Note that $x \leq 0$, since $0 \in \mathcal{A}$ and $l(0) = 0$.

(3) The linear functional l can be selected to be of the form

$$l(X) = E_{Q^0}[X], \text{ for each } X \in L^p(\mathcal{F}, R),$$

where the probability measure Q^0 is absolutely continuous with respect to R and the density $\frac{dQ^0}{dR}$ belongs to $L^q(\mathcal{F}, R)$. Indeed, this follows from the fact that $\mathcal{A}$ is a solid convex set; see Remark 2.5, first part, and the Riesz representation Theorem of linear functionals of $L^p(\mathcal{F}, R)$.

(4) The inequality (13) implies

$$E_{Q^0}[a \mid \mathcal{G}] \geq x, \ Q^0 - a.s., \text{ for each } a \in \mathcal{A}. \tag{15}$$

Indeed, for $a \in \mathcal{A}$, the random variable

$$\widehat{a} := a 1_{\{E_{Q^0}[a|\mathcal{G}] < x\}}$$

belongs to the acceptance set $\mathcal{A}$, since $\{E_{Q^0}[a \mid \mathcal{G}] < x\} \in \mathcal{G}$; see Remark 2.5, third part. Thus, $E_{Q^0}[\widehat{a}] \geq x$. On the other hand,

$$E_{Q^0}[\widehat{a}] = E_{Q^0}[E_{Q^0}[\widehat{a} \mid \mathcal{G}]] = E_{Q^0}[1_{\{E_{Q^0}[a|\mathcal{G}] < x\}} E_{Q^0}[a \mid \mathcal{G}]] \leq x.$$

Thus, $Q^0[\{E_{Q^0}[a \mid \mathcal{G}] < x\}] = 0$ and (15) holds true.

Note that $\alpha^{\min}(Q^0) \leq -x$, $Q^0 - a.s.$, due to (15). Hence, $Q^0 \in \mathcal{Q}^{q,\infty}$.

(5) Now let us define

$$J' := \{E_{Q^0}[1_J(X + b) \mid \mathcal{G}] < x\}.$$

Then, $J' \in \mathcal{G}$ and $J' \subset J$. Moreover

$$Q^0[J'] > 0,$$

due to the inequality (14).

(6) Now we generate a contradiction. In the event J' we have

$$\alpha^{\min}(Q^0) < E_{Q^0}[-1_J(X+b) \mid \mathcal{G}],\ Q^0 - a.s.,$$

due to the definitions of the minimal penalty function and of the event J'. We may rewrite this last inequality to obtain

$$b < E_{Q^0}[-X \mid \mathcal{G}] - \alpha^{\min}(Q^0),\ Q^0 - a.s. \text{ in the event } J'.$$

This contradicts the definition of b given in (12).

□

To some extend, it is unpleasant to select a specific version of the conditional expectation, as fixed in (2). The convention is unnecessary for probability measures $Q \ll R$ with

$$E_R[Z^Q \mid \mathcal{G}] > 0,\ R - a.s.$$

In the next lemma we show that the class of "locally equivalent" probability measures (8)

$$\mathcal{Q}^{q,\infty}_{e,loc} := \{Q \in \mathcal{Q}^{q,\infty} \mid E_R[Z^Q \mid \mathcal{G}] > 0,\ R - a.s.\},$$

is non empty. We use an exhaustion argument started by Halmos and Savage. The exhaustion argument in the theory of risk measures is well known; see e.g., Cheridito et al [6], Lemma 3.22 and Föllmer and Penner [10], Lemma 3.4. This result will allow us to precise the coarse representation of Proposition 3.3.

Lemma 3.4. *If ρ has the Fatou property, then the class $\mathcal{Q}^{q,\infty}_{e,loc}$ is non empty.*

Proof. (1) We define

$$c := \sup\left\{R(E_R[Z^Q \mid \mathcal{G}] > 0) \mid Q \in \mathcal{Q}^{q,\infty}\right\}.$$

There exists $Q^* \in \mathcal{Q}^{q,\infty}$ such that

$$c = R(E_R[Z^{Q^*} \mid \mathcal{G}] > 0).$$

Indeed, let Q^n be a maximizing sequence, so that

$$c = \lim_{n\to\infty} R(E_R[Z^{Q^n} \mid \mathcal{G}] > 0).$$

We define

$$\lambda^n := \frac{1}{2^n}\frac{1}{1 + \|Z^n\|^q_{L^q(R)} + \|\alpha^{\min}(Q^n)\|_{L^\infty(Q^n)}}.$$

Then, the probability measure $Q^* \ll R$ defined by the density

$$\frac{dQ^*}{dR} := \frac{1}{E_R\left[\sum_{n=1}^\infty \lambda^n \frac{dQ^n}{dR}\right]}\sum_{n=1}^\infty \lambda^n \frac{dQ^n}{dR}$$

is an element of $\mathcal{Q}^{q,\infty}$. It attains the value c:

$$c = R(E_R[Z^{Q^*} \mid \mathcal{G}] > 0).$$

(2) Let $A \in \mathcal{G}$ with $R(A) > 0$ be an arbitrary event. Then we have

$$1_A = \rho(-1_A) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{q,\infty}} \left\{ E_Q[1_A \mid \mathcal{G}] - \alpha^{\min}(Q) \right\},$$

due to Proposition 3.3. Hence, we conclude the existence of $\widehat{Q} \in \mathcal{Q}^{q,\infty}$ with

$$\left\{ E_R[Z^{\widehat{Q}} \mid \mathcal{G}] > 0 \right\} \cap A \neq \emptyset, R - a.s.,$$

since ρ is normalized and our convention of the conditional expectation (2).

(3) Now we conclude the proof by showing that $c = 1$. Assume by way of contradiction that $c < 1$. Let the event $A^0 \in \mathcal{G}$ be defined by

$$A^0 := \left\{ E_R[Z^{Q^*} \mid \mathcal{G}] = 0 \right\}.$$

There exists $\widehat{Q} \in \mathcal{Q}^{q,\infty}$ with

$$\left\{ E_R[Z^{\widehat{Q}} \mid \mathcal{G}] > 0 \right\} \cap A^0 \neq \emptyset, R - a.s.,$$

due to the previous step. The probability measure defined by

$$Q^0 := \frac{1}{2}(Q^* + \widehat{Q}),$$

belongs to the class $\mathcal{Q}^{q,\infty}$. It contradicts the optimality of Q^* since

$$\left\{ E_R[Z^{Q^0} \mid \mathcal{G}] > 0 \right\} = \left\{ E_R[Z^{Q^*} \mid \mathcal{G}] > 0 \right\} \cup \left\{ E_R[Z^{\widehat{Q}} \mid \mathcal{G}] > 0 \right\}.$$

□

The Proposition 3.3 and Lemma 3.4 allow us to prove the next proposition.

Proposition 3.5. *If the conditional convex risk measure ρ has the Fatou property, then the pair $(\mathcal{Q}^{q,\infty}_{e,loc}, \alpha^{\min})$ represents the risk measure ρ:*

$$\rho(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{q,\infty}_{e,loc}} \left\{ E_Q[-X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\}. \tag{16}$$

Proof. Let $X \in L^p(\mathcal{F}, R)$ and $Q^1 \in \mathcal{Q}^{q,\infty}$. The identity (16) will be established after we construct a probability measure $\widetilde{Q} \in \mathcal{Q}^{q,\infty}_{e,loc}$ such that

$$E_{Q^1}[-X \mid \mathcal{G}] - \alpha^{\min}(Q^1) \leq E_{\widetilde{Q}}[-X \mid \mathcal{G}] - \alpha^{\min}(\widetilde{Q}),\ R - a.s, \tag{17}$$

due to Proposition 3.3. We set $A := \{E_R[Z^{Q^0} \mid \mathcal{G}] = 0\}$. Assume $0 < R(A) < 1$, otherwise there is nothing to prove.

The class $\mathcal{Q}^{q,\infty}_{e,loc}$ is non empty, due to Lemma 3.4. Without loss of generality, we assume that $R \in \mathcal{Q}^{q,\infty}_{e,loc}$. We define a probability measure $\widetilde{Q} \ll R$ by

$$\frac{d\widetilde{Q}}{dR} := Y 1_A + Z^{Q^1} 1_{A^c}.$$

In this expression, Y is a positive constant selected to satisfy $E_R\left[\frac{d\widetilde{Q}}{dR}\right] = 1$. The probability measure $\widetilde{Q}$ belongs to $\mathcal{Q}_{e,loc}^{q,\infty}$ by construction.

The penalization and conditional expectation of $\widetilde{Q}$ can be computed as follows:

$$\begin{aligned}\alpha^{\min}(\widetilde{Q}) &= 1_A\alpha^{\min}(\widetilde{Q}) + 1_{A^c}\alpha^{\min}(Q^1),\\ E_{\widetilde{Q}}[-X \mid \mathcal{G}] &= \frac{1_A}{Y}E_R[-YX \mid \mathcal{G}] + \frac{1_{A^c}}{E_R[Z^{Q^0} \mid \mathcal{G}]}E_R[-Z^{Q^1}X \mid \mathcal{G}]\\ &= 1_A E_R[-X \mid \mathcal{G}] + 1_{A^c}E_{Q^1}[-X \mid \mathcal{G}].\end{aligned}$$

Thus, (17) holds true, due to the set relationship (6). □

In the next lemma we see that real-valued conditional convex risk measures are continuous, in particular they satisfy the Fatou property.

Lemma 3.6. *Let ρ be a real-valued conditional convex risk measure in $L^p(\mathcal{F}, R)$. Let $\{X^n\}_{n=1}^{\infty} \subset L^p(\mathcal{F}, R)$ be a sequence strongly converging to $X^0 \in L^p(\mathcal{F}, R)$. Then*

$$\lim_{n\to\infty} \left\|\rho(X^n) - \rho(X^0)\right\|_{L^1} = 0.$$

Proof. We start with $X^0 = 0$. We assume that

$$\lim_{n\to\infty} 2^n \left\|X^n\right\|_{L^p} = 0,$$

by taking a subsequence if necessary. The sequence defined by

$$Y^n := \sum_{i=1}^{n} \frac{1}{2^j \left\|X^j\right\|_{L^p}} \left|X^j\right|$$

is increasing. It is easy to see that the sequence has the Cauchy property. Thus, it converges to some $Y \in L^p(\mathcal{F}, R)$.

Now we get

$$|\rho(X^n)| \le \rho(-|X^n|) \le 2^n \left\|X^n\right\|_{L^p} \rho(-\frac{1}{2^n \left\|X^n\right\|_{L^p}} |X^n|),$$

due to monotonicity and convexity of ρ. Hence

$$|\rho(X^n)| \le 2^n \left\|X^n\right\|_{L^p} \rho(-Y).$$

In order to obtain the result for arbitrary X^0, we define a new convex risk measure by $\rho^0(X) := \rho(X + X^0) - \rho(X^0)$. It is easy to see that ρ^0 satisfies the conditions of the proposition. Thus, we may apply the previous step. □

Remark 3.7. The arguments in the proof of Lemma 3.6 are due to Biagini and Frittelli [2], Theorem 2 (the "extended Namioka-Klee Theorem"). Note that Lemma 3.6 does not follow directly from Theorem 2 [2] by considering the functional $E_R[\rho]$.

Corollary 3.8. *A real-valued convex risk measure has the Fatou property.*

Proof. The statement of the corollary follows from Lemma 3.6. □

Now we are ready to conclude the proof of Theorem 2.7.

Proof. The real-valued conditional convex risk measure ρ has the Fatou property, due to Corollary 3.8. Thus, the risk measure ρ is represented by the pair $(\mathcal{Q}^{q,\infty}_{e,loc}, \alpha^{\min})$, due to Proposition 3.5. □

4. Proof of Theorem 2.9

In this section we prove Theorem 2.9 which characterizes real-valued conditional convex risk measures defined in $L^p(\mathcal{F}, R)$.

We are going to denote by $L^p_+(\mathcal{F}, R)$ the non negative elements of $L^p(\mathcal{F}, R)$. We need to introduce the following family of random variables:

$$S^+ := \{X \in L^p_+(\mathcal{F}, R) \mid \|X\|_{L^p} = 1\}.$$

The second part of the next lemma uses a well known argument about the linear functionals of the space $L^p(\mathcal{F}, R)$; see e.g., Werner [19], Beispiel (j), p.50.

Lemma 4.1. *Let $Z \in L^q_+(\mathcal{F}, R)$. Then the random variable $\lceil Z \rceil$ defined by*

$$\lceil Z \rceil := \operatorname{ess\,sup}_{X \in S^+} E_R[ZX \mid \mathcal{G}], \tag{18}$$

belongs to $L^1(\mathcal{G}, R)$. Moreover

$$\lceil Z \rceil = E_R^{\frac{1}{q}}[Z^q \mid \mathcal{G}]. \tag{19}$$

Proof. (1) Let $\{Y_n\}_{n=1}^{\infty} \subset S^+$ be a maximizing sequence:

$$\lim_{n\to\infty} E_R[E_R[ZY_n \mid \mathcal{G}]] = E_R[\lceil Z \rceil].$$

As a consequence

$$0 \le E_R[\lceil Z \rceil] \le \lim_{n\to\infty} E_R[ZY_n].$$

Moreover,

$$E_R[ZY_n] \le \|Z\|_{L^q} \|Y_n\|_{L^p} = \|Z\|_{L^q},$$

due to Hölder's inequality. Thus:

$$0 \le E_R[\lceil Z \rceil] \le \|Z\|_{L^q}.$$

(2) Now we prove (19). We define

$$X^0 := \frac{Z^{\frac{q}{p}}}{E_R^{\frac{1}{p}}[Z^q \mid \mathcal{G}]} 1_{\{E_R[Z|\mathcal{G}]>0\}}.$$

The random variable X^0 is well defined and belongs to S^+, since

$$\{E_R[Z^q \mid \mathcal{G}] = 0\} \subset \{Z^q = 0\}.$$

Moreover

$$E_R[X^0 Z \mid \mathcal{G}] = E_R^{\frac{1}{q}}[Z^q \mid \mathcal{G}].$$

Hence

$$\lceil Z \rceil \ge E_R^{\frac{1}{q}}[Z^q \mid \mathcal{G}].$$

□

Lemma 4.2. *Let $Z \in L^q_+(\mathcal{F}, R)$ and $X \in L^p_+(\mathcal{F}, R)$. Then*

$$E_R[ZX \mid \mathcal{G}] \le \lceil Z \rceil \, \|X\|_{L^p} . \tag{20}$$

Proof. Without loss of generality we may assume that $\|X\|_{L^p} > 0$. The following relationship is clear

$$E_R[ZX \mid \mathcal{G}] = E_R\left[Z \frac{X}{\|X\|_{L^p}} \mid \mathcal{G} \right] \|X\|_{L^p} .$$

Hence

$$E_R[ZX \mid \mathcal{G}] \le \lceil Z \rceil \, \|X\|_{L^p} .$$

□

Now we are ready to prove the first part of Theorem 2.9.

Proof. We only prove that ρ is real-valued. Note that it is only necessary to consider the minimal penalty function $\alpha^{\min}$.

(1) We first consider the case where X is non positive. There exists $\widetilde{X} \in L^\infty$ such that $X \le \widetilde{X} \le 0$ and

$$\left\| X - \widetilde{X} \right\|_{L^p} \le \frac{b}{2},$$

due to Lebesgue Dominated Convergence Theorem. Hence

$$\begin{aligned} \rho(X) &= \operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[-\widetilde{X} \mid \mathcal{G}] + E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\} \\ &\le \left\| \widetilde{X} \right\|_{L^\infty} + \operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\}. \end{aligned} \tag{21}$$

(2) If we take expectation with respect to R, then we get

$$E_R\left[\operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\} \right] = \sup_{Q \in \mathcal{Q}} E_R\left[\left\{ E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\} \right]. \tag{22}$$

Indeed, let $\{Q^n\}_{n=1}^\infty \subset \mathcal{Q}$ be a maximizing sequence:

$$\operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\} = \lim_{n \to \infty} \left\{ E_{Q^n}[\widetilde{X} - X \mid \mathcal{G}] - \alpha(Q^n) \right\}.$$

We claim that we may assume the sequence of functions $\left\{ E_{Q^n}[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q^n) \right\}$ to be bounded from below. Hence, we can apply Fatou's Lemma to obtain:

$$E_R\left[\operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{ E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q) \right\} \right] \le \liminf_{n \to \infty} E_R\left[E_{Q^n}[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q^n) \right].$$

This proves the inequality $\le$. The converse direction is clear.

Now we prove the claim. For Q^n we define the event

$$A^n := \left\{ E_{Q^n}[\widetilde{X} - X \mid \mathcal{G}] - \alpha(Q^n) \ge - \left\| \alpha^{\min}(Q^0) \right\|_{L^\infty} \right\}.$$

We construct a probability measure $\widetilde{Q}^n$ by

$$\frac{d\widetilde{Q}^n}{dR} := 1_{A^n}\frac{dQ^n}{dR} + 1_{(A^n)^c}\frac{dQ^0}{dR}Y.$$

The penalization and conditional expectation of $\widetilde{Q}$ can be computed as follows:

$$\alpha^{\min}(\widetilde{Q}^n) = 1_{A^n}\alpha^{\min}(Q^n) + 1_{(A^n)^c}\alpha^{\min}(Q^0),$$
$$E_{\widetilde{Q}^n}[-X \mid \mathcal{G}] = 1_{A^n}E_{Q^n}[\widetilde{X} - X \mid \mathcal{G}] + 1_{(A^n)^c}E_{Q^0}[\widetilde{X} - X \mid \mathcal{G}].$$

(3) Now we put together (21) and (22) to obtain

$$E_R[\rho(X)] \leq \left\|\widetilde{X}\right\|_{L^\infty} + \sup_{Q\in\mathcal{Q}} E_R\left[E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha^{\min}(Q)\right].$$

Moreover

$$E_R\left[E_Q[\widetilde{X} - X \mid \mathcal{G}]\right] = E_R\left[\frac{1}{E_R[Z^Q \mid \mathcal{G}]}E_R[Z^Q(\widetilde{X} - X) \mid \mathcal{G}]\right]$$
$$\leq E_R\left[\frac{1}{E_R[Z^Q \mid \mathcal{G}]}\lceil Z^Q\rceil\right]\left\|X - \widetilde{X}\right\|_{L^p},$$

due to Lemma 4.2. Thus

$$\sup_{Q\in\mathcal{Q}} E_R\left[E_Q[\widetilde{X} - X \mid \mathcal{G}] - \alpha(Q)\right] \leq -a - \frac{b}{2}\sup_{Q\in\mathcal{Q}} E_R\left[\frac{1}{E_R[Z^Q \mid \mathcal{G}]}\lceil Z^Q\rceil\right].$$

Hence, we conclude:

$$E_R[\rho(X)] \leq \left\|\widetilde{X}\right\|_{L^\infty} - a.$$

(4) Now we consider the case $X \geq 0$. We have

$$0 \geq \rho(X) \geq E_{Q^0}[-X \mid \mathcal{G}] - \alpha^{\min}(Q^0).$$

The conditional expectation $E_{Q^0}[-X \mid \mathcal{G}]$ is integrable since $X \in L^p(\mathcal{F}, R)$ and $\frac{dQ^0}{dR} \in L^q(\mathcal{F}, R)$. The penalization $\alpha^{\min}(Q^0)$ is integrable by hypothesis. Hence $\rho(X)$ belongs to $L^1(\mathcal{G}, R)$.

(5) For general X we write $X = X^+ + X^-$ with $X^+ \geq 0$ and $X^- \leq 0$. We have

$$\rho(X^-) \geq \rho(X) \geq \rho(X^+),$$

due to the monotonicity of conditional expectation. We conclude the desired integrability of $\rho(X)$ from the previous steps.

□

The first part of Theorem 2.9 says that coercive penalty functions induce real-valued conditional risk measures. Now we prove the converse. That is, we are ready to prove the second part of Theorem 2.9.

Proof. By way of contradiction assume there exists a sequence $\{Q^n\}_{n=1}^\infty \subset \mathcal{Q}^q_{e,loc}$ such that

$$E_R\left[\alpha(Q^n)\right] < -n + 2^{-n-1}E_R\left[\frac{1}{E_R[Z^n \mid \mathcal{G}]}\lceil Z^n\rceil\right], \tag{23}$$

where Z^n denotes the density of Q^n. There exists $X^n \in S^+$ such that

$$E_R\left[\frac{E_R[Z^nX^n \mid \mathcal{G}]}{E_R[Z^n \mid \mathcal{G}]}\right] \geq \frac{1}{2}E_R\left[\frac{\lceil Z^n \rceil}{E_R[Z^n \mid \mathcal{G}]}\right],$$

due to Lemma 4.1 and Fatou's Lemma. Now we define

$$X := \sum_{n=1}^{\infty} 2^{-n}X^n.$$

We get

$$\rho(-X) \geq \rho(-2^{-n}X^n) \geq E_{Q^n}[2^{-n}X^n \mid \mathcal{G}] - \alpha(Q^n).$$

Moreover

$$\begin{aligned} E_R\left[E_{Q^n}[2^{-n}X^n \mid \mathcal{G}] - \alpha(Q^n)\right] &\geq 2^{-n-1}E_R\left[\frac{\lceil Z^n \rceil}{E_R[Z^n \mid \mathcal{G}]}\right] + n - 2^{-n-1}E_R\left[\frac{\lceil Z^n \rceil}{E_R[Z^n \mid \mathcal{G}]}\right] \\ &= n. \end{aligned}$$

Thus:

$$E_R[\rho(-X)] \geq n,$$

a clear contradiction. □

5. Invariance of the minimal representation

Recall that ρ is a real-valued conditional convex risk measure defined in $L^p(\mathcal{F}, R)$. We may define a new risk measure ρ^∞ in $L^\infty(\mathcal{F}, R)$ by restriction:

$$\rho^\infty(X) = \rho(X), \text{ for each } X \in L^\infty(\mathcal{F}, R).$$

The new risk measure ρ^∞ has associated a minimal penalty function:

$$\alpha^{\min,\infty}(Q) := \operatorname{ess\,sup}_{X \in L^\infty(\mathcal{F},R)} \left\{E_Q[-X \mid \mathcal{G}] - \rho^\infty(X)\right\}, Q \ll R. \tag{24}$$

We next show that the minimal representation is invariant to this restriction in the sense that $\alpha^{\min,\infty} = \alpha^{\min}$ in $\mathcal{Q}^q_{e,loc}$.

Proposition 5.1. *The minimal representation of ρ keeps invariant in $L^\infty(\mathcal{F}, R)$. Thus, $\alpha^{\min,\infty} = \alpha^{\min}$ in $\mathcal{Q}^q_{e,loc}$.*

Proof. Let $Q \in \mathcal{Q}^q_{e,loc}$. Let $Y \in L^p(\mathcal{F}, R)$. We define a sequence by $Y^n := (Y \wedge n) \vee (-n)$. It is clear that $Y^n \in L^\infty(\mathcal{F}, R)$ and the sequence converges to Y in $L^p(\mathcal{F}, R)$. Moreover,

$$\lim_{n\to\infty} E_Q[-Y^n \mid \mathcal{G}] = E_Q[-Y \mid \mathcal{G}], R - a.s.$$

due to Lebesgue's Dominated Convergence Theorem and Hölder's inequality. Furthermore,

$$\lim_{n\to\infty} \rho(Y^n) = \rho(Y),$$

due to Lemma 3.6. We conclude that

$$E_Q[-Y \mid \mathcal{G}] - \rho(Y) \leq \operatorname{ess\,sup}_{X \in L^\infty(\mathcal{F},R)} \left\{E_Q[-X \mid \mathcal{G}] - \rho(X)\right\} = \alpha^{\min,\infty}(Q).$$

This proves the claims of the proposition. □

Remark 5.2. Theorem 2.9 and Proposition 5.1 characterize conditional convex risk measures defined in $L^\infty(\mathcal{F}, R)$ which can be extended to real-valued conditional convex risk measures in $L^p(\mathcal{F}, R)$.

Example 5.3. The risk measure Average Value at Risk at level $\lambda \in (0, 1]$, $AVaR_\lambda$ is defined by

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_t\, dt,\ for\, X \in L^\infty(\mathcal{F}, R),$$

where VaR_t denotes Value at Risk at level t; see e.g., Föllmer and Schied [11], Definition 4.43. The Average Value at Risk is a coherent measure of risk with the representation:

$$AVaR_\lambda(X) = \max_{Q\in\mathcal{Q}_\lambda} E_Q[-X], \tag{25}$$

where $\mathcal{Q}_\lambda$ is the set of all probability measures whith R-densities essentially dominated by $\frac{1}{\lambda}$; see Föllmer and Schied [11], Theorem 4.47. The robust representation (25) involves the minimal penalty function:

$$\alpha^{\min}(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{Q}_\lambda, \\ +\infty, & \text{if } Q \notin \mathcal{Q}_\lambda, \end{cases} \tag{26}$$

which is clearly coercive. Thus, the risk of measure AVaR defines a (non-conditional) real-valued risk measure in $L^1(\mathcal{F}, R)$. A particular case of a general result due to Filipović and Svindland [9], Theorem 3.8. Note however that we do not require the σ-algebra $\mathcal{F}$ to be separable.

Example 5.4. Now we consider a conditional version of the risk measure AVaR; see e.g., Cheridido and Kupper [4], Example 3.3.1. Let λ be a $\mathcal{G}$-measurable random variable such that

$$0 < \lambda \leq 1,\ R - a.s.$$

The conditional Average Value at Risk at the stochastic level λ is defined by the representation:

$$AVaR_\lambda(X) = \operatorname{ess\,sup}_{Q\in\mathcal{Q}_\lambda} E_Q[-X \mid \mathcal{G}],\ R - a.s. \tag{27}$$

where $\mathcal{Q}_\lambda$ is the set of all probability measures whose R-density $\frac{dQ}{dR}$ is R-a.s. bounded by $\frac{1}{\lambda}$ and moreover

$$E_R\left[\frac{dQ}{dR} \mid \mathcal{G}\right] = 1,\ R - a.s.$$

The numerical representation (27) involves the minimal penalty function

$$\alpha^{\min}(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{Q}_\lambda, \\ +\infty, & \text{if } Q \notin \mathcal{Q}_\lambda, \end{cases} \tag{28}$$

which is coercive if

$$\frac{1}{\lambda} \in L^q(\mathcal{G}, R).$$

In this case, the conditional Average Value at Risk defines a real-valued conditional risk measure in $L^p(\mathcal{F}, R)$. In particular in $L^1(\mathcal{F}, R)$, if the quotient $\frac{1}{\lambda}$ is essentially bounded.

Example 5.5. For $p > 1$, let l be the convex loss function

$$l(x) := \frac{1}{p}(x^+)^p.$$

The loss function l defines a special case of the Shortfall risk measure introduced by Föllmer and Schied; see [11], Section 4.9, for a systematic presentation. Let us consider the conditional version of this risk measure. Let $g \in L^p(\mathcal{G}, R)$ be a non negative function. We define a family of acceptable positions by

$$\mathcal{A} := \{X \in L^\infty(\mathcal{F}, R) \mid E_R\left[l(-X) \mid \mathcal{G}\right] \leq g,\ R - a.s.\}$$

Then, the acceptance set $\mathcal{A}$ defines a conditional convex risk measure in $L^\infty(\mathcal{F}, R)$ with robust representation:

$$\rho(X) = \operatorname{ess\,sup}_{Q \in \mathcal{Q}} \left\{E_Q[-X \mid \mathcal{G}] - \alpha(Q)\right\};$$

see e.g., Bion-Nadal [3]. The class $\mathcal{Q}$ consists of absolutely continuous probability measures Q with densities $\frac{dQ}{dR}$ in $L^q(\mathcal{F}, R)$ such that

$$E_R\left[\frac{dQ}{dR} \mid \mathcal{G}\right] = 1.$$

The penalty function α is given by:

$$\alpha(Q) = (pg)^{\frac{1}{p}} E_R\left[\left(\frac{dQ}{dR}\right)^q \mid \mathcal{G}\right]^{\frac{1}{q}}.$$

This penalty function is coercive since $g \in L^p(\mathcal{G}, R)$. Thus, the conditional Shortfall risk can be extended from $L^\infty(\mathcal{F}, R)$ to $L^p(\mathcal{F}, R)$.

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