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ASYMPTOTIC-PRESERVING SCHEME FOR A TWO-FLUID EULER-LORENTZ MODEL.

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Abstract. The present work is devoted to the simulation of a strongly magnetized plasma as a mixture of an ion fluid and an electron fluid. For simplicity reasons, we assume that each fluid is isothermal and is modelized by Euler equations coupled with a term representing the Lorentz force, and we assume that both Euler systems are coupled through a quasi-neutrality constraint of the form $n_i = n_e$. The numerical method which is described in the present document is based on an asymptotic-preserving time semi-discretization of a variant of this two-fluid Euler-Lorentz model which is based on a small perturbation of the quasi-neutrality constraint.

1. Introduction.

This work is devoted to the construction of an asymptotic-preserving scheme for the isothermal two-fluid Euler-Lorentz system coupled with the quasineutrality equation in the gyrofluid regime.

The situation of a single Euler-Lorentz system has been considered in [1] and [6]. In [6], the authors constructed an asymptotic-preserving scheme in the situation where the magnetic field lines are aligned with one cartesian coordinate and where the component of the velocity which is parallel to the magnetic field and satisfies Dirichlet boundary conditions. This situation is generalized in [1] to a variable magnetic field and the component of the velocity which is parallel to the magnetic field lines and satisfies Neumann boundary conditions. The method of resolution of these problems is based on a decomposition of the parallel velocity into a component constant along the magnetic field lines and a fluctuation around this constant. In [1], the parallel velocity is solution of a degenerate anisotropic elliptic problem.

On another hand, the two-fluid approach has been studied in [3], [4], [5]. In these papers, a two-fluid Euler-Poisson model is considered and the goal is to build an asymptotic-preserving scheme which remains efficient when we reach the quasineutral limit regime.

In the present paper, we generalized the Euler-Lorentz approach of [1] and [6] to the situation of a two-fluid system which corresponds to a more relevant physical situation. Indeed, in the ITER context, it corresponds to a population of electrons interacting with a population of ions.

In the following section, we present the model and its drift limit. In section 3 and 4 the numerical scheme presented and in the last section numerical results are given.

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2. The isothermal two-fluid Euler-Lorentz model.

2.1. The model.

Consider the isothermal two-fluid Euler-Lorentz system coupled with the quasi-neutrality equation describing the isothermal flow of a mixture of ions and electrons. The scaled Euler-Lorentz model reads

$$\begin{cases} \partial_t n^\tau + \nabla_{\mathbf{x}} \cdot \mathbf{q}_i^\tau = 0\,, & (2.1a)\\ \tau \left[\partial_t \mathbf{q}_i^\tau + \nabla_{\mathbf{x}} \cdot \left(\dfrac{\mathbf{q}_i^\tau \otimes \mathbf{q}_i^\tau}{n^\tau}\right)\right] + \nabla_{\mathbf{x}} n^\tau = -n^\tau\, \nabla_{\mathbf{x}} \phi^\tau + \mathbf{q}_i^\tau \times \mathbf{B}\,, & (2.1b)\\ \partial_t n^\tau + \nabla_{\mathbf{x}} \cdot \mathbf{q}_e^\tau = 0\,, & (2.1c)\\ \epsilon\, \tau \left[\partial_t \mathbf{q}_e^\tau + \nabla_{\mathbf{x}} \cdot \left(\dfrac{\mathbf{q}_e^\tau \otimes \mathbf{q}_e^\tau}{n^\tau}\right)\right] + T_e\, \nabla_{\mathbf{x}} n^\tau = n^\tau\, \nabla_{\mathbf{x}} \phi^\tau - \mathbf{q}_e^\tau \times \mathbf{B}\,. & (2.1d) \end{cases}$$

n^τ represents the common density of ions and electrons, $\mathbf{q}_i^\tau$ and $\mathbf{q}_e^\tau$ are the momenta of ions and electrons respectively, T_e is the electron temperature and ϕ^τ is the potential giving the electric field. The symbol $\nabla_{\mathbf{x}}$ is the gradient operator and $\nabla_{\mathbf{x}}\cdot$ is the divergence operator. This model is completed with Neumann boundary conditions for $\mathbf{q}_i^\tau$ and $\mathbf{q}_e^\tau$ and with some non-classical boundary conditions for n^τ and ϕ^τ. These last conditions are precised in Section 3.

The scaled model is obtained by introducing characteristic scales for length $\overline{x}$, time $\overline{t}$, momentum $\overline{q}$, ion temperature $\overline{T}$, potential $\overline{\phi}$, and magnetic field $\overline{B}$. We use $\overline{q} = \frac{\overline{x}\,\overline{n}}{\overline{t}}$ and the characteristic electric potential is supposed to satisfy the relation $\overline{q}\overline{B} = \frac{\overline{n}\,\overline{\phi}}{\overline{x}}$. Next consider $\overline{c}$ the characteristic sound speed of the ions, $\overline{M}$ the characteristic Mach number of the ions and $\overline{\omega}$ the characteristic cyclotron frequency of the ions defined by $\overline{c} = \sqrt{\frac{k_B \overline{T}}{m_i}}$, $\overline{M} = \frac{\overline{q}}{\overline{n}\,\overline{c}}$, $\overline{\omega} = \frac{e\,\overline{B}}{m_i}$. Moreover, the ratio $\epsilon = \frac{m_e}{m_i}$ between the electron mass and the ion mass is fixed. Finally, the model (2.1) is obtained by taking $\tau = \overline{M}^2 = \overline{t}\,\overline{\omega}$.

2.2. The limit model

If τ converges to 0, (2.1) leads to the model

$$\begin{cases} \partial_t n^0 + \nabla_{\mathbf{x}} \cdot \mathbf{q}_i^0 = 0\,, & (2.2a)\\ \nabla_{\mathbf{x}} n^0 = -n^0\, \nabla_{\mathbf{x}} \phi^0 + \mathbf{q}_i^0 \times \mathbf{B}\,, & (2.2b)\\ \partial_t n^0 + \nabla_{\mathbf{x}} \cdot \mathbf{q}_e^0 = 0\,, & (2.2c)\\ T_e\, \nabla_{\mathbf{x}} n^0 = n^0\, \nabla_{\mathbf{x}} \phi^0 - \mathbf{q}_e^0 \times \mathbf{B}\,, & (2.2d) \end{cases}$$

in which the parallel part of $\mathbf{q}_i^0$ and $\mathbf{q}_e^0$ are implicit. Indeed, by splitting the parallel and the perpendicular parts of (2.2b) and (2.2d), we have

$$\begin{cases} \partial_t n^0 + \nabla_{\mathbf{x}} \cdot \mathbf{q}_i^0 = 0\,,\\ \mathbf{b} \cdot \nabla_{\mathbf{x}} n^0 = -n^0\, \mathbf{b} \cdot \nabla_{\mathbf{x}} \phi^0\,,\\ (\mathbf{q}_i^0)_\perp = \dfrac{1}{\|\mathbf{B}\|} \mathbf{b} \times (\nabla_{\mathbf{x}} n^0 + n^0\, \nabla_{\mathbf{x}} \phi^0)\,,\\ \partial_t n^0 + \nabla_{\mathbf{x}} \cdot \mathbf{q}_e^0 = 0\,,\\ T_e\, \mathbf{b} \cdot \nabla_{\mathbf{x}} n^0 = n^0\, \mathbf{b} \cdot \nabla_{\mathbf{x}} \phi^0\,,\\ (\mathbf{q}_e^0)_\perp = \dfrac{1}{\|\mathbf{B}\|} \mathbf{b} \times (-T_e\, \nabla_{\mathbf{x}} n^0 + n^0\, \nabla_{\mathbf{x}} \phi^0)\,, \end{cases} \qquad (2.3)$$

where $\mathbf{q}_{||} = (\mathbf{b} \otimes \mathbf{b})\,\mathbf{q}$, $\mathbf{q}_\perp = \mathbf{q} - \mathbf{q}_{||}$, $\mathbf{b} = \frac{B}{\|\mathbf{B}\|}$ and $\|\mathbf{B}\|^2 = B_x^2 + B_y^2 + B_z^2$. Then we deduce from this limit model that

$$\mathbf{b} \cdot \nabla_{\mathbf{x}} n^0 = \mathbf{b} \cdot \nabla_{\mathbf{x}} \phi^0 = 0\,. \qquad (2.4)$$

3. Semi-discrete AP schemes

In this section, we propose a semi-discretization of (2.1) in time which is also consistent with the limit model (2.2).

3.1. "Usual" semi-discretization

Consider the following semi-discretization

$$\begin{cases} \dfrac{n^{\tau,m+1}-n^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_i^{\tau,m+1})_{||}^{m+1}\big) + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_i^{\tau,m})_{\perp}^{m+1}\big) = 0\,, & (3.1a)\\ \dfrac{\mathbf{q}_i^{\tau,m+1}-\mathbf{q}_i^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\Big(\dfrac{\mathbf{q}_i^{\tau,m}\otimes\mathbf{q}_i^{\tau,m}}{n^{\tau,m}}\Big) + \dfrac{1}{\tau}\nabla_{\mathbf{x}} n^{\tau,m+1} = \dfrac{1}{\tau}\big[-n^{\tau,m+1}\,\nabla_{\mathbf{x}}\phi^{\tau,m+1} + \mathbf{q}_i^{\tau,m+1}\times\mathbf{B}^{m+1}\big]\,, & (3.1b)\\ \dfrac{n^{\tau,m+1}-n^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_e^{\tau,m+1})_{||}^{m+1}\big) + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_e^{\tau,m})_{\perp}^{m+1}\big) = 0\,, & (3.1c)\\ \dfrac{\mathbf{q}_e^{\tau,m+1}-\mathbf{q}_e^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\Big(\dfrac{\mathbf{q}_e^{\tau,m}\otimes\mathbf{q}_e^{\tau,m}}{n^{\tau,m}}\Big) + \dfrac{T_e}{\epsilon\,\tau}\nabla_{\mathbf{x}} n^{\tau,m+1} = \dfrac{1}{\epsilon\,\tau}\big[n^{\tau,m+1}\,\nabla_{\mathbf{x}}\phi^{\tau,m+1} - \mathbf{q}_e^{\tau,m+1}\times\mathbf{B}^{m+1}\big]\,, & (3.1d) \end{cases}$$

where $(\mathbf{q})_{||}^{m+1} = (\mathbf{b}^{m+1}\otimes\mathbf{b}^{m+1})\,\mathbf{q}$ and $(\mathbf{q})_{\perp}^{m+1} = \mathbf{q} - (\mathbf{q})_{||}^{m+1}$. By separating the parallel part of (3.1b) and (3.1d) according to $\mathbf{b}^{m+1}$, and by plugging these equations into (3.1a) and (3.1c), we obtain two diffusions equations for $n^{\tau,m+1}$ and $\phi^{\tau,m+1}$ of the form

$$\begin{cases} -\nabla_{\mathbf{x}}\cdot\big((\mathbf{b}^{m+1}\otimes\mathbf{b}^{m+1})\,\nabla_{\mathbf{x}} n^{\tau,m+1}\big) + \lambda\,\tau\,n^{\tau,m+1} = \tau\,R^{\tau,m}\,, & (3.2a)\\ -\nabla_{\mathbf{x}}\cdot\big(n^{\tau,m+1}\,(\mathbf{b}^{m+1}\otimes\mathbf{b}^{m+1})\,\nabla_{\mathbf{x}}\phi^{\tau,m+1}\big) = \tau\,S^{\tau,m+1}\,, & (3.2b) \end{cases}$$

where $\lambda = \frac{1+\epsilon}{\Delta t^2\,(1+T_e)}$ and where $R^{\tau,m}$ depends on $n^{\tau,m}$, $\phi^{\tau,m}$, $\mathbf{q}_i^{\tau,m}$ and $\mathbf{q}_e^{\tau,m}$, and $S^{\tau,m+1}$ depends on $n^{\tau,m+1}$, $n^{\tau,m}$, $\phi^{\tau,m}$, $\mathbf{q}_i^{\tau,m}$ and $\mathbf{q}_e^{\tau,m}$.

In order to preserve the asymptotic behaviour of the scheme, it is necessary to take into account the following boundary conditions (see [1]):

$$\begin{cases} \mathbf{b}^{m+1}\cdot\nabla_{\mathbf{x}} n^{\tau,m+1} = 0\,, \quad \text{on } \partial\Omega, & (3.3a)\\ \mathbf{b}^{m+1}\cdot\nabla_{\mathbf{x}}\phi^{\tau,m+1} = 0\,, \quad \text{on } \partial\Omega. & (3.3b) \end{cases}$$

Remark that the diffusion problem (3.2b)-(3.3b) is ill posed for any value of $\tau \geq 0$ because of a lack of uniqueness of the solution. In order to avoid this difficulty, we consider instead a perturbed version of the model (2.1) for which the time semi-discretization technique used in (3.1) leads to two diffusion problems for $n^{\tau,m+1}$ and $\phi^{\tau,m+1}$ which are well-posed for any value of $\tau > 0$. This is the subject of the following subsection.

3.2. Perturbed Euler-Lorentz two-fluid model.

In order to bypass the difficulty linked with the semi-discretization (3.1) and its reformulation, we choose to include in the model (2.1) a small perturbation of the form $c\,\partial_t\phi$. More specifically, we modify the mass conservation equations as follows:

$$\begin{cases} \partial_t n^\tau + C_1\,\partial_t\phi^\tau + \nabla_{\mathbf{x}}\cdot\mathbf{q}_i^\tau = 0\,, & (3.4a)\\ \tau\left[\partial_t\mathbf{q}_i^\tau + \nabla_{\mathbf{x}}\cdot\Big(\dfrac{\mathbf{q}_i^\tau\otimes\mathbf{q}_i^\tau}{n^\tau}\Big)\right] + \nabla_{\mathbf{x}} n^\tau = -n^\tau\,\nabla_{\mathbf{x}}\phi^\tau + \mathbf{q}_i^\tau\times\mathbf{B}\,, & (3.4b)\\ \partial_t n^\tau + C_2\,\partial_t\phi^\tau + \nabla_{\mathbf{x}}\cdot\mathbf{q}_e^\tau = 0\,, & (3.4c)\\ \epsilon\tau\left[\partial_t\mathbf{q}_e^\tau + \nabla_{\mathbf{x}}\cdot\Big(\dfrac{\mathbf{q}_e^\tau\otimes\mathbf{q}_e^\tau}{n^\tau}\Big)\right] + T_e\,\nabla_{\mathbf{x}} n^\tau = n^\tau\,\nabla_{\mathbf{x}}\phi^\tau - \mathbf{q}_e^\tau\times\mathbf{B}\,, & (3.4d) \end{cases}$$

where C_1, C_2 are two fixed parameters which are close to 0. Then we consider the same semi-discretization method as in the previous paragraph:

$$
\left\{
\begin{aligned}
&\frac{n^{\tau,m+1}-n^{\tau,m}}{\Delta t} + C_1\,\frac{\phi^{\tau,m+1}-\phi^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_i^{\tau,m+1})_{||}^{m+1}\big) + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_i^{\tau,m})_{\perp}^{m+1}\big) = 0\,, && (3.5a)\\
&\frac{\mathbf{q}_i^{\tau,m+1}-\mathbf{q}_i^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\Big(\frac{\mathbf{q}_i^{\tau,m}\otimes\mathbf{q}_i^{\tau,m}}{n^{\tau,m}}\Big) + \frac{1}{\tau}\nabla_{\mathbf{x}} n^{\tau,m+1} \\
&\qquad\qquad = \frac{1}{\tau}\big[-n^{\tau,m+1}\,\nabla_{\mathbf{x}}\phi^{\tau,m+1} + \mathbf{q}_i^{\tau,m+1}\times\mathbf{B}^{m+1}\big]\,, && (3.5b)\\
&\frac{n^{\tau,m+1}-n^{\tau,m}}{\Delta t} + C_2\,\frac{\phi^{\tau,m+1}-\phi^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_e^{\tau,m+1})_{||}^{m+1}\big) + \nabla_{\mathbf{x}}\cdot\big((\mathbf{q}_e^{\tau,m})_{\perp}^{m+1}\big) = 0\,, && (3.5c)\\
&\frac{\mathbf{q}_e^{\tau,m+1}-\mathbf{q}_e^{\tau,m}}{\Delta t} + \nabla_{\mathbf{x}}\cdot\Big(\frac{\mathbf{q}_e^{\tau,m}\otimes\mathbf{q}_e^{\tau,m}}{n^{\tau,m}}\Big) + \frac{T_e}{\epsilon\,\tau}\nabla_{\mathbf{x}} n^{\tau,m+1} \\
&\qquad\qquad = \frac{1}{\epsilon\,\tau}\big[n^{\tau,m+1}\,\nabla_{\mathbf{x}}\phi^{\tau,m+1} - \mathbf{q}_e^{\tau,m+1}\times\mathbf{B}^{m+1}\big]\,. && (3.5d)
\end{aligned}
\right.
$$

Following the same methodology as in the previous paragraph, we isolate the parallel part of (3.5b) and (3.5d) and we plug them into (3.5a) and (3.5c). We obtain two diffusions equations of the form

$$
\begin{aligned}
-\nabla_{\mathbf{x}}\cdot\big((\mathbf{b}^{m+1}\otimes\mathbf{b}^{m+1})\,\nabla_{\mathbf{x}} n^{\tau,m+1}\big) + \tau\,\lambda_1\,n^{\tau,m+1} &= \tau\,R^{\tau,m}\,,\\
-\nabla_{\mathbf{x}}\cdot\big(n^{\tau,m+1}\,(\mathbf{b}^{m+1}\otimes\mathbf{b}^{m+1})\,\nabla_{\mathbf{x}}\phi^{\tau,m+1}\big) + \tau\,\lambda_2\,C\,\phi^{\tau,m+1} &= \tau\,S^{\tau,m+1}\,,
\end{aligned}
\tag{3.6}
$$

where $\lambda_1 = \frac{1+\epsilon}{\Delta t^2\,(1+T_e)}$, $\lambda_2 = \frac{T_e}{\Delta t^2\,(1+T_e)}$, $C>0$ is a fixed constant linked with C_1 and C_2 by

$$
C_1 = \frac{T_e\,C}{1+T_e}, \qquad C_2 = -\frac{T_e\,C}{\epsilon\,(1+T_e)},
\tag{3.7}
$$

and where $R^{\tau,m}$ depends on $n^{\tau,m}$, $\phi^{\tau,m}$, $\mathbf{q}_i^{\tau,m}$ and $\mathbf{q}_e^{\tau,m}$ and $S^{\tau,m+1}$ depends on $n^{\tau,m+1}$, $n^{\tau,m}$, $\phi^{\tau,m}$, $\mathbf{q}_i^{\tau,m}$ and $\mathbf{q}_e^{\tau,m}$.

As it has been suggested in the previous paragraph, we couple the diffusion equations (3.6) with the Neumann type boundary conditions given by

$$
\left\{
\begin{aligned}
&\mathbf{b}^{m+1}\cdot\nabla_{\mathbf{x}} n^{\tau,m+1} = 0\,, \quad \text{on } \partial\Omega,\\
&\mathbf{b}^{m+1}\cdot\nabla_{\mathbf{x}} \phi^{\tau,m+1} = 0\,, \quad \text{on } \partial\Omega.
\end{aligned}
\right.
\tag{3.8}
$$

Since the problem (3.6)-(3.8) is well-posed for any value of $\tau>0$ and remains ill-posed when $\tau=0$, we use Brull *et al.*'s method proposed in [1] to compute $n^{\tau,m+1}$ and $\phi^{\tau,m+1}$.

4. Numerical results

The semi-discrete approach described in (3.5) in currently used in a 2D finite volume scheme where the explicit part of the fluxes are computed with Rusanov's method. Concerning the diffusion equations (3.6), a three points method is used and is based on the decomposition method introduced by Brull *et al.* in [1]. At present time, this full-discrete scheme is producing the first numerical results which need to be rigourously analyzed. This is one of the main goals of [2].

References

[1] Brull, S., Degond, P., Deluzet, F., *Numerical degenerate elliptic problems and their applications to magnetized plasma simulations.*, to appear in Comm. Comput. Phys.

[2] Brull, S., Degond, P., Deluzet, F., Mouton, A., *Asymptotic-Preserving scheme for a bi-fluid Euler-Lorentz model*, submitted.

[3] Crispel, P., Degond, P., Vignal, M.-H., *A plasma expansion model based on the full Euler-Poisson system*, Math. Models Methods Appl. Sci., **17**-7 (2007), pp. 1129-1158.

[4] Crispel, P., Degond, P., Vignal, M.-H., *An asymptotic-preserving scheme for the two-fluid Euler-Poisson model in the quasi-neutral limit*, J. Comput. Phys., **223**-1 (2007), pp. 208-234.

[5] Crispel, P., Degond, P., Vignal, M.-H., *Quasi-neutral fluid models for current-carrying plasmas*, J. Comput. Phys., **205**-2 (2005), pp. 408-438.

[6] Degond, P., Deluzet, F., Sangam, A., Vignal, M.-H., *An asymptotic preserving scheme for the Euler equations in a strong magnetic field.* J. Comput. Phys., **228** (2009), pp. 3540-3558.

[7] LeVeque, R., *Finite volume methods for hypoerbolic problems*, Cambridge texts in Applied mathematics (2002).

[8] Rusanov, V. V., *The calculation of the iteraction of non-stationary shock waves and obstacles*, J. Comp. Math. Phys. **1** (1961), pp 267-279.