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ON A LOW MACH NUCLEAR CORE MODEL

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Abstract. We propose to formally derive a low Mach number model adapted to the modeling of a water nuclear core (*e.g.* of PWR- or BWR-type) in the forced convection regime or in the natural convection regime by filtering out the acoustic waves in the compressible Navier-Stokes system. Then, we propose a monodimensional stationary analytical solution with regular and singular charge loss when the equation of state is a stiffened gas equation. Moreover, we show that this solution may not be admissible from a physical or a mathematical point of view for a particular choice of the mass flux and we study the consistency between this solution and the solution obtained from a Boussinesq approximation. Let us underline that the modeling of the nuclear core is simplified in this paper. For example, the flow is a single-phase flow and we do not model neither the porosity nor the turbulence. Nevertheless, it will be possible to enrich the modeling in the future.

Résumé. On se propose de formellement dériver un modèle bas Mach adapté à la modélisation d'un cœur de réacteur nucléaire à eau (par exemple de type REP ou REB) en régime de convection forcée ou en régime de convection naturelle en filtrant les ondes acoustiques dans un modèle de type Navier-Stokes compressible. On construit ensuite une solution analytique stationnaire monodimensionnelle avec perte de charge régulière et singulière dans le cas où l'équation d'état est de type gaz raidi. Puis, on montre que cette solution peut ne pas être physiquement ou mathématiquement admissible pour un choix particulier du flux de masse et on étudie la cohérence entre cette solution et la solution obtenue à partir d'une approximation de Boussinesq. Soulignons que la modélisation proposée du cœur nucléaire est ici simplifiée. Par exemple, l'écoulement est monophasique et on ne modélise ni la porosité, ni la turbulence. Il sera par contre tout à fait possible d'enrichir la modélisation par la suite.

1. Introduction

The flow in the core of a nuclear reactor whose the coolant fluid is water (*e.g.* in a Pressurised Water Reactor (PWR) or in a Boiling Water Reactor (BWR)) is a low Mach number flow when the situation is nominal or in some accidental situations. As a consequence, there are two different time scales in the compressible Navier-Stokes system used to model the flow in the nuclear core: a first time scale which is the material time scale t_{mat} and a second time scale which is the acoustic time scale t_{ac}. Since $t_{\mathrm{ac}} \ll t_{\mathrm{mat}}$ and since the usefull time scale is t_{mat} in the study of thermal-hydraulic phenomena in a nuclear core whose the flow is at low Mach number, the discretization of the compressible Navier-Stokes system needs to use an implicit scheme in such a way that the time step is of the order or greater than t_{mat}, implicit scheme which may be expensive from a CPU point of view and which may suffer from a lack of robustness and/or of a lack of accuracy [6, 10].

Since the incompressible Navier-Stokes system or the Boussinesq approximation cannot be a good approximation of the flow because of the high heat exchanges in the nuclear core that induces a high thermal expansion of the

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flow, a good approach is to filter out the acoustic waves in the compressible Navier-Stokes system by keeping at the same time compressibility effects due to heat exchanges. This approach is similar to the one proposed by Paolucci [15] and by Majda and Sethian for combustion problems [12, 13]. This technique was also used to propose a low Mach number model modeling the thermal expansion of bubbles at low Mach number [4, 5, 16].

Thus, we propose to formally derive a low Mach number model that we name *Low Mach Nuclear Core* (LMNC) model. The LMNC model takes into account the power density due to fission reactions and the inlet and outlet boundary conditions in the nuclear core. The LMNC model is interesting not only because there are only one time scale (the material time scale $t_{\rm mat}$) but also because we are able to construct monodimensional analytical solutions when the equation of state is a stiffened gas equation. These analytical solutions are obtained:

- in the stationary case or in the unstationary case,
- in the single-phase case or in the two-phase case with phase change phenomenon,
- with or without regular and singular charge loss.

In the present paper, we focus on the monodimensional stationary single-phase case. The monodimensional unstationary single-phase case is studied in [1] and the monodimensional unstationary two-phase case with phase change phenomenon is studied in [2]. Let us underline that the choice of the stiffened gas equation of state is interesting. Indeed, it is possible to approach with a quite good accuracy the tabulated water equation of state with the stiffened gas equation [14]. As a consequence, the proposed analytical solutions are usefull:

- to validate industrial nuclear thermal-hydraulic codes based on the discretization of the compressible Navier-Stokes system whose the coolant fluid is water when the flow is at low Mach number,
- to understand with the help of analytical formulae the behaviour of a water nuclear core when the flow is at low Mach number. For example, we can obtain the minimum inlet mass flux for which there is no phase change when we take into account phase change phenomenon (this point is not studied in the present paper but is studied in [2]).

We underline that if the LMNC model can model low Mach number accidental transient regimes (*e.g.* a main coolant pump trip which is a Loss Of Flow Accident (LOFA) as at the beginning of the Fukushima accident in the reactors 1, 2 and 3), the LMNC model cannot model accidental situations whose the Mach number is not low. As a consequence, the LMNC model is complementary to the compressible Navier-Stokes system and should replace this compressible model if and only if the flow in the nuclear core is at low Mach number.

Let us note that the modeling used in this paper to model the nuclear core is simplified. For example, the flow is a single-phase flow and we do not take into account neither the porosity nor the turbulence. Nevertheless, it will be possible to enrich the LMNC model. For example, the two-phase case with phase change phenomenon is taken into account in [2]. Moreover, this paper focus on the derivation and on the basic properties of the LMNC model and not on existence, uniqueness and regularity results in the unstationary case: this point is important for example when we try to model a power excursion due to a reactivity insertion in the nuclear core with a singular or regularized power density of Dirac-type; it will have to be studied in the future with the tools used in [8, 9] in the case of the low Mach combustion model proposed in [12, 13], and used in [16] in the case of the low Mach bubble model proposed in [4, 5]. At last, we underline that some analytical solutions of the LMNC model proposed in this paper are generalized in some cases in [1, 2] to the unstationary single phase and two-phase case by using again tools used in [16] to obtain unstationary analytical solutions of the abstract bubble vibration model proposed in [7]. These results show that the LMNC model should be useful – both from a physical and a mathematical point of view – for nuclear safety studies in the thermal-hydraulic field applied to a water nuclear core of PWR- or BWR-type.

The outline of this paper is the following. In §2, we introduce the dimensionless compressible Navier-Stokes system used to model the nuclear core. In §3, we formally derive the *Low Mach Nuclear Core* (LMNC) model by filtering out the acoustic waves in the compressible Navier-Stokes system. We also give dimensionless parameters which allow to estimate the magnitude of the compressibility of the flow. At last, we formally obtain the Boussinesq approximation. In §4, we construct a monodimensional stationary analytical solution of the LMNC model when the equation of state is a stiffened gas equation with or without regular and singular charge loss, we study the validity of this solution from a physical and a mathematical point of view in function of the sign of the mass flux, and we show the consistency of this solution with the solution obtained from the Boussinesq approximation. At last, we conclude the paper in §5.

2. The compressible model

We describe in this section a compressible Navier-Stokes system adapted to the modeling of a water nuclear core and we give the dimensionless version of this model. The proposed modeling is simplified in order to simplify the derivation of the low Mach number model in §3 and the study of the stationary analytical solutions in §4. For example, the flow is a single-phase flow and we do not model neither the porosity nor the turbulence. Nevertheless, it will be possible to enrich the modeling in the future. Let us note that we take into account regular and singular charge loss in §4.5 and that the two-phase case with phase change phenomenon is taken into account in [2].

2.1. **The model**

The compressible Navier-Stokes system used to model the water nuclear core contained in the cubic domain $\Omega := [0, L_1] \times [0, L_2] \times [0, L_3]$ is given by

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, & \text{(a)} \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \rho \mathbf{g} + \nabla \cdot \tau(\mathbf{u}), & \text{(b)} \\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \partial_t p + \mathbf{u} \cdot \nabla p + \tau(\mathbf{u}) : \nabla \mathbf{u} + \nabla \cdot (\lambda \nabla T) + \Phi(t, x) & \text{(c)} \end{cases} \tag{1}$$

where t is the time, x is the space variable, ρ is the density, $\mathbf{u}$ is the velocity, h is the internal enthalpy, T is the temperature, p is the pressure, λ is the heat conductivity, $\mathbf{g} := (0, 0, -g)$ is the gravity field ($g > 0$) and $\Phi \geq 0$ is the power density due to the fission reactions in the nuclear core. The function Φ is a given function which depend on the time t and on the space variable x. Let us note that in a more accurate modeling, Φ should also depend on the thermal-hydraulic quantities such that ρ and T. The tensor τ is the newtonian viscosity tensor that is to say

$$\tau(\mathbf{u}) = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \eta(\nabla \cdot \mathbf{u})\mathbf{I} \tag{2}$$

where μ is the viscosity and where η is such that $2\mu + 3\eta = 0$ (Stokes hypothesis). System (1)(2) is closed as soon as we know the equations of state (EOS) $\rho = \rho(h, p)$ and $T = T(h, p)$, and the positive transport coefficients $\mu(h, p)$ and $\lambda(h, p)$. The boundary conditions on $\partial\Omega_a := [0, L_1] \times [0, L_2] \times \{0\}$ are given by

$$\begin{cases} \forall x \in \partial\Omega_a : & h(t, x) & = & h_e(t), & \text{(a)} \\ & \rho \mathbf{u}(t, x) & = & (D_e(t), 0, 0) & \text{(b)} \end{cases} \tag{3}$$

for the forced convection regime ($D_e(t)$ is the mass flux), or are given by

$$\begin{cases} \forall x \in \partial\Omega_a : & h(t, x) & = & h_e(t), & \text{(a)} \\ & p(t, x) & = & P_e(t) & \text{(b)} \end{cases} \tag{4}$$

for the natural convection regime. The boundary condition on $\partial\Omega_b := [0, L_1] \times [0, L_2] \times \{L_3\}$ is given by

$$\forall x \in \partial\Omega_b : \quad p(t, x) = P_0(t). \tag{5}$$

When the mass flux $\rho\mathbf{u}(t, 0)$ is positive, the boundary conditions (3) (or (4)) and (5) define respectively the inlet and outlet boundary conditions in the nuclear core. When the mass flux $\rho\mathbf{u}(t, 0)$ is negative, the boundary conditions (3) (or (4)) and (5) define respectively the outlet and inlet boundary conditions in the nuclear core, and may define an ill-posed problem (see §4.2). Of course, we can replace (3)(a) and (3)(b) respectively by the equivalent boundary conditions $\forall x \in \partial\Omega_a : T(t, x) = T_e(t)$ and $\forall x \in \partial\Omega_a : \mathbf{u}(t, x) = (u_e(t), 0, 0)$. Let us also note that h_e (or T_e) and D_e (or P_e, or u_e) may also depend on $x \in \partial\Omega_a$; in the same way, P_0 may also depend on $x \in \partial\Omega_b$: these variables do not depend on x in this paper for the sake of simplicity. We also apply on the boundary $\partial\Omega - (\partial\Omega_a \cup \partial\Omega_b)$ the no-slip boundary condition $\mathbf{u} = 0$ and (for example) the non-homogeneous Neumann boundary condition $\nabla T \cdot \mathbf{n} = q$ ($\mathbf{n}$ is the outer normal on $\partial\Omega$) where q models the thermal transfers between the nuclear core and the exterior of the core (q is given by experimental correlations). At last, we define the initial conditions $\mathbf{u}^0(x)$, $h^0(x)$ and $p^0(x)$ (or $\rho^0(x)$, or $T^0(x)$).

In the sequel, we suppose that $P_0(t) > 0$, $T^0(x) > 0$, $p^0(x) > 0$ (or $\rho^0(x) > 0$), and that $h_e(t)$ is such that $T_e(t) > 0$. When we model a nuclear core of a PWR- or BWR-type reactor, the flow is an upward flow in the forced convection regime: in that case, we apply the boundary conditions (3) and we suppose that $D_e(t) \geq 0$. Nevertheless, we will also study in §4.2 the case of a downward flow since the flow in some research reactors (as in some material testing reactors) is a downward flow; the natural convection regime will be also studied in §4.2. At last, we suppose that $h_e(t)$, $D_e(t)$ (or $P_e(t)$) and $P_0(t)$ depend on the time t to be able to model transient regimes induced by accidental situation. For example, when $D_e(t)$ goes to zero, this models a main coolant pump trip which is a Loss Of Flow Accident (LOFA) as at the beginning of the Fukushima accident in the reactors 1, 2 and 3.

2.2. The dimensionless model

By defining the characteristic orders of magnitude $L_* = (L_1L_2L_3)^{1/3}$, $p_* = \mathcal{O}(P_0)$[1] ($p_* = P_0$ when $P_0'(t) = 0$), $h_* = \mathcal{O}(h_e)$ ($h_* = h_e$ when $h_e'(t) = 0$), $\rho_* = \rho(h_*, p_*)$, $u_* = \mathcal{O}(D_e/\rho_*)$ ($u_* = D_e/\rho_*$ when $D_e'(t) = 0$), $T_* = T(h_*, p_*)$, $t_* = \dfrac{L_*}{u_*}$, $\mu_* = \mu(h_*, p_*)$, $\lambda_* = \lambda(h_*, p_*)$, $g_* = g$ and $\Phi_* = \mathcal{O}(\Phi)$ ($\Phi_* = \int_\Omega \Phi(x)dx / \int_\Omega dx$ when $\partial_t\Phi = 0$), the dimensionless compressible Navier-Stokes system used to model the nuclear core is given by

$$\left\{\begin{aligned}
&\partial_{\overline{t}}\overline{\rho} + \nabla_{\overline{x}} \cdot (\overline{\rho}\,\overline{\mathbf{u}}) = 0, && \text{(a)}\\
&\partial_{\overline{t}}(\overline{\rho}\,\overline{\mathbf{u}}) + \nabla_{\overline{x}} \cdot (\overline{\rho}\,\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = -\left(\frac{\zeta_h}{M}\right)^2 \nabla_{\overline{x}}\overline{p} + \frac{1}{R_e}\nabla_{\overline{x}} \cdot \tau(\mathbf{u}) + \frac{1}{F_r}\overline{\rho}\,\overline{\mathbf{g}}, && \text{(b)}\\
&\partial_{\overline{t}}(\overline{\rho}\overline{h}) + \nabla_{\overline{x}} \cdot (\overline{\rho}\overline{h}\overline{\mathbf{u}}) = \zeta_h^2(\partial_{\overline{t}}\overline{p} + \overline{\mathbf{u}} \cdot \nabla_{\overline{x}}\overline{p}) + \frac{M^2}{R_e}\tau(\overline{\mathbf{u}}) : \nabla_{\overline{x}}\overline{\mathbf{u}} && \text{(c)}\\
&\qquad + \left(\frac{\zeta_h}{\zeta_T}\right)^2 \cdot \frac{1}{R_eP_r}\nabla_{\overline{x}} \cdot (\overline{\lambda}\nabla_{\overline{x}}\overline{T}) + \left(\frac{\zeta_h}{\zeta_T}\right)^2 \cdot C_H\overline{\Phi}(\overline{t}, \overline{x}).
\end{aligned}\right. \tag{6}$$

The notation $\overline{f}$ means that f is a dimensionless variable knowing that $f \in \{t, x, \rho, \mathbf{u}, h, T, p, \tau, \lambda, \mathbf{g}, \Phi\}$. In particular, the notation $\partial_{\overline{t}}$ and $\nabla_{\overline{x}}$ recall that these operators are applied to dimensionless variables. Moreover,

[1]The notation $\mathcal{O}(f)$ means *of the order of f*.

we use the dimensionless number

$$\begin{cases} R_e = \dfrac{u_* L_* \rho_*}{\mu_*} & \text{(Reynolds number)}, \quad \text{(a)} \\ P_r = \dfrac{\mu_* C_{p*}}{\lambda_*} & \text{(Prandtl number)}, \quad \text{(b)} \\ F_r = \dfrac{u_*^2}{L_* g_*} & \text{(Froude number)} \quad \text{(c)} \end{cases} \tag{7}$$

and

$$\begin{cases} \zeta_h = \sqrt{\dfrac{p_*}{\rho_* h_*}}, & \text{(a)} \\ \zeta_T = \sqrt{\dfrac{p_*}{\rho_* C_{p*} T_*}}, & \text{(b)} \\ C_H = \dfrac{\Phi_* L_*}{u_* \rho_* C_{p*} T_*} \quad \text{(heating coefficient)}. & \text{(c)} \end{cases} \tag{8}$$

The constant $C_{p_*} = C_p(T_*, p_*)$ in (7)(b) and in (8)(b,c) where $C_p(T,p) := \partial_T h(T,p)$ is an order of magnitude of the calorific capacity at constant pressure. We also use the dimensionless number

$$M = \frac{u_*}{a_*} \qquad \text{(Mach number)} \tag{9}$$

where

$$a_* = \sqrt{h_*}$$

is an order of magnitude of the sound velocity ($\sqrt{h_*}$ is a better estimate of the sound velocity than $\sqrt{p_*/\rho_*}$ in a PWR-or BWR-type reactor where the fluid is water and not a perfect gas). Let us note that the sound velocity $a(h,p)$ is given by

$$a(h,p) := \frac{1}{\sqrt{\dfrac{\partial_h \rho}{\rho}(h,p) + \partial_p \rho(h,p)}}. \tag{10}$$

Formula (10) implies that we suppose in this paper that the EOS is such that the sound velocity is well defined: this means that $\frac{\partial_h \rho}{\rho}(h,p) + \partial_p \rho(h,p) > 0$, which implies that we can define the Mach number with (9). In other words, the convective part of the compressible Navier-Stokes system (1) – which is the compressible Euler system – is supposed to be hyperbolic.

3. The low Mach number model

We now suppose that:

Hypothesis 3.1. *The flow is at low Mach number that is to say $M \ll 1$.*

This modeling hypothesis is fundamental in the sequel. Indeed, it allows to simplify the compressible Navier-Stokes System (1)-(5) by filtering out the acoustic waves, and then, to obtain a *Low Mach Nuclear Core* (LMNC) model. Let us note that a necessary condition to satisfy Hypothesis 3.1 should be that the characteristic time scale τ_* of the time variation of $\Phi(t,x)$, $h_e(t)$, $D_e(t)$ and $P_0(t)$ is such that

$$\mathcal{O}(\tau_*) \geq \frac{L_*}{u_*}. \tag{11}$$

Let us note that P_0 in the boundary condition (5) may also depend on x. Nevertheless, we will see in the sequel that the formal derivation of the LMNC model implies that the spatial pressure variation of $p(t,x)$ is of the order of M^2 (see (13)). As a consequence, any spatial variation of $P_0(t,x)$ have to be also of the order of M^2 to be compatible with this property. In the same way, h_e and D_e in the boundary condition (3) (or P_e in (4)(b)) may also depend on x as soon as such a spatial dependency implies a spatial variation of the pressure of the order of M^2. To simplify the problem, we suppose in this paper that the boundary conditions do not depend on x.

At last, we underline that the proposed LMNC model will not be able to model accidental situations whose the Mach number is not close to zero. For example, accidental situations induced by a decreasing of $D_e(t)$ or of $P_0(t)$ will be treated by the LMNC model only when the characteristic time scale τ_* of this decreasing is equal or greater than the material time scale L_*/u_* (see Condition (11)).

3.1. **Formal derivation of the LMNC model**

Let us suppose that $\forall \overline{\varphi} \in \{\overline{\rho}, \overline{\mathbf{u}}, \overline{h}, \overline{T}, \overline{p}\}$, the asymptotic development

$$\overline{\varphi}(\overline{t},\overline{x}) = \overline{\varphi}^{(0)}(\overline{t},\overline{x}) + M\varphi^{(1)}(M,\overline{t},\overline{x}) + M^2\overline{\varphi}^{(2)}(M,\overline{t},\overline{x}) + \dots \tag{12}$$

is valid under Hypothesis 3.1. By injecting (12) in (6), we formally obtain:

- **Orders M^{-2} and M^{-1}:**

$$\nabla\overline{p}^{(0)} = \nabla\overline{p}^{(1)} = 0$$

which implies that

$$\overline{p}(\overline{t},\overline{x}) = \overline{p}^{(0)}(\overline{t}) + M^2\overline{p}^{(2)}(M,\overline{t},\overline{x}) + \dots\ . \tag{13}$$

By using the boundary condition (5), we obtain $\overline{p}^{(0)}(\overline{t}) = \overline{P}_0(\overline{t})$. Moreover, to take into account the fact that $P_0(t)$ may be an unstationary function whose the time scale is equal to τ_*, we introduce the function $\mathcal{P}_0$ in such a way $\mathcal{P}_0(t/\tau_*) = P_0(t)$. Then, we define $\overline{\mathcal{P}}_0(\kappa\overline{t}) = \dfrac{\mathcal{P}_0(t/\tau_*)}{p_*}$ with

$$\kappa = \frac{t_*}{\tau_*},$$

which implies that

$$\begin{cases} \overline{P}_0(\overline{t}) = \overline{\mathcal{P}}_0(\kappa\overline{t}), & \text{(a)} \\ \overline{P}'_0(\overline{t}) = \kappa\overline{\mathcal{P}}'_0(\kappa\overline{t}). & \text{(b)} \end{cases} \tag{14}$$

To summarize, we have

$$\forall(t,x):\ \overline{p}^{(0)}(\overline{t},\overline{x}) = \overline{P}_0(\overline{t}) = \overline{\mathcal{P}}_0(\kappa\overline{t}) \qquad i.e. \qquad \partial_{\overline{x}}\overline{p}^{(0)} = 0. \tag{15}$$

Let us underline that Condition (11) implies that $\mathcal{O}(\kappa) \leq 1$.

- **Order M^0:**

$$\begin{cases} \partial_{\bar t}\overline{\rho}^{(0)} + \nabla_{\overline{x}}\cdot(\overline{\rho}^{(0)}\overline{\mathbf{u}}^{(0)}) = 0, & \text{(a)}\\[2ex] \partial_{\bar t}(\overline{\rho}^{(0)}\overline{\mathbf{u}}^{(0)}) + \nabla_{\overline{x}}\cdot(\overline{\rho}^{(0)}\overline{\mathbf{u}}^{(0)}\otimes\overline{\mathbf{u}}^{(0)}) = -\nabla_{\overline{x}}(\zeta_h^2\overline{p}^{(2)}) + \dfrac{\nabla_{\overline{x}}\cdot\overline{\tau}^{(0)}}{R_e} + \dfrac{\overline{\rho}^{(0)}\overline{\mathbf{g}}}{F_r}, & \text{(b)}\\[2ex] \partial_{\bar t}(\overline{\rho}^{(0)}\overline{h}^{(0)}) + \nabla_{\overline{x}}\cdot(\overline{\rho}^{(0)}\overline{h}^{(0)}\overline{\mathbf{u}}^{(0)}) = \zeta_h^2\kappa\overline{\mathcal{P}}_0'(\kappa\bar t) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot\dfrac{1}{R_eP_r}\nabla_{\overline{x}}\cdot(\overline{\lambda}^{(0)}\nabla_{\overline{x}}\overline{T}^{(0)}) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot C_H\overline{\Phi}(\bar t,\overline{x}) & \text{(c)}\end{cases} \tag{16}$$

(we have used Relation (14)(b) in the first term of the right hand side of (16)(c)) which may be written as

$$\begin{cases} \nabla_{\overline{x}}\cdot\overline{\mathbf{u}}^{(0)} = -\dfrac{1}{\overline{\rho}^{(0)}}(\partial_{\bar t}\overline{\rho}^{(0)} + \overline{\mathbf{u}}\cdot\nabla\overline{\rho}^{(0)}), & \text{(a)}\\[2ex] \overline{\rho}^{(0)}\left(\partial_{\bar t}\overline{\mathbf{u}}^{(0)} + \overline{\mathbf{u}}^{(0)}\cdot\nabla_{\overline{x}}\overline{\mathbf{u}}^{(0)}\right) = -\nabla_{\overline{x}}(\zeta_h^2\overline{p}^{(2)}) + \dfrac{\nabla_{\overline{x}}\cdot\overline{\tau}^{(0)}}{R_e} + \dfrac{\overline{\rho}^{(0)}\overline{\mathbf{g}}}{F_r}, & \text{(b)}\\[2ex] \overline{\rho}^{(0)}\left(\partial_{\bar t}\overline{h}^{(0)} + \overline{\mathbf{u}}^{(0)}\cdot\nabla_{\overline{x}}\overline{h}^{(0)}\right) = \zeta_h^2\kappa\overline{\mathcal{P}}_0'(\kappa\bar t) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot\dfrac{1}{R_eP_r}\nabla_{\overline{x}}\cdot(\overline{\lambda}^{(0)}\nabla_{\overline{x}}\overline{T}^{(0)}) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot C_H\overline{\Phi}(\bar t,\overline{x}). & \text{(c)}\end{cases} \tag{17}$$

By taking into account the fact that $d\overline{\rho}(\overline{h},\overline{P}_0) = \partial_{\overline{h}}\overline{\rho}(\overline{h},\overline{P}_0)d\overline{h} + \partial_{\overline{p}}\overline{\rho}(\overline{h},\overline{P}_0)d\overline{p}$, by using (15) and (17)(c), we obtain that (17)(a) is equivalent to

$$\nabla_{\overline{x}}\cdot\overline{\mathbf{u}}^{(0)} = -\frac{\zeta_h^2\kappa}{\overline{\rho}(\overline{h}^{(0)},\overline{P}_0)\overline{a}(\overline{h}^{(0)},\overline{P}_0)^2}\overline{\mathcal{P}}_0'(\kappa\bar t) + \left(\frac{\zeta_h}{\zeta_T}\right)^2\cdot\frac{\overline{\beta}_h(\overline{h}^{(0)},\overline{P}_0)}{\overline{P}_0}\left[\frac{\nabla_{\overline{x}}\cdot(\overline{\lambda}^{(0)}\nabla_{\overline{x}}\overline{T}^{(0)})}{R_eP_r} + C_H\overline{\Phi}(\bar t,\overline{x})\right]$$

where

$$\overline{a}(\overline{h},\overline{P}_0) := \frac{1}{\sqrt{\dfrac{\partial_{\overline{h}}\overline{\rho}}{\overline{\rho}}(\overline{h},\overline{P}_0) + \dfrac{1}{\zeta_h^2}\partial_{\overline{p}}\overline{\rho}(\overline{h},\overline{P}_0)}} \tag{18}$$

and where

$$\overline{\beta}_h(\overline{h},\overline{P}_0) := -\overline{P}_0\frac{\partial_{\overline{h}}\overline{\rho}}{\overline{\rho}^2}(\overline{h},\overline{P}_0). \tag{19}$$

Thus, System (17) is given by (we now omit the subscript $^{(0)}$)

$$\begin{cases} \nabla_{\overline{x}}\cdot\overline{\mathbf{u}} = -\dfrac{\zeta_h^2\kappa}{\overline{\rho}(\overline{h},\overline{P}_0)\overline{a}(\overline{h},\overline{P}_0)^2}\overline{\mathcal{P}}_0'(\kappa\bar t) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot\dfrac{\overline{\beta}_h(\overline{h},\overline{P}_0)}{\overline{P}_0}\left[\dfrac{\nabla_{\overline{x}}\cdot(\overline{\lambda}\nabla_{\overline{x}}\overline{T})}{R_eP_r} + C_H\overline{\Phi}(\bar t,\overline{x})\right], & \text{(a)}\\[2ex] \overline{\rho}(\overline{h},\overline{P}_0)\left(\partial_{\bar t}\overline{\mathbf{u}} + \overline{\mathbf{u}}\cdot\nabla_{\overline{x}}\overline{\mathbf{u}}\right) = -\nabla_{\overline{x}}\overline{\Pi} + \dfrac{\nabla_{\overline{x}}\cdot\overline{\tau}}{R_e} + \dfrac{\overline{\rho}(\overline{h},\overline{P}_0)\ \overline{\mathbf{g}}}{F_r}, & \text{(b)}\\[2ex] \overline{\rho}(\overline{h},\overline{P}_0)\left(\partial_{\bar t}\overline{h} + \overline{\mathbf{u}}\cdot\nabla_{\overline{x}}\overline{h}\right) = \zeta_h^2\kappa\overline{\mathcal{P}}_0'(\kappa\bar t) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot\dfrac{1}{R_eP_r}\nabla_{\overline{x}}\cdot(\overline{\lambda}\nabla_{\overline{x}}\overline{T}) + \left(\dfrac{\zeta_h}{\zeta_T}\right)^2\cdot C_H\overline{\Phi}(\bar t,\overline{x}). & \text{(c)}\end{cases} \tag{20}$$

The dimensioned version of (20) is given by

$$\left\{\begin{array}{ll} \nabla \cdot \mathbf{u} = -\dfrac{P_0'(t)}{\rho(h,P_0)a(h,P_0)^2} + \dfrac{\beta_h(h,P_0)}{P_0}\left[\nabla \cdot (\lambda \nabla T) + \Phi(t,x)\right], & \text{(a)} \\ \rho(h,P_0)\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\Pi + \nabla\cdot\tau(\mathbf{u}) + \rho(h,P_0)\mathbf{g}, & \text{(b)} \\ \rho(h,P_0)\left(\partial_t h + \mathbf{u}\cdot\nabla h\right) = P_0'(t) + \nabla\cdot(\lambda\nabla T) + \Phi(t,x) & \text{(c)} \end{array}\right. \tag{21}$$

where

$$a(h,p) := \frac{1}{\sqrt{\dfrac{\partial_h \rho}{\rho}(h,p) + \partial_p \rho(h,p)}} \tag{22}$$

is the sound velocity ((18) is the dimensionless sound velocity), where

$$\beta_h(h,p) := \frac{p\alpha_h}{\rho}(h,p) \tag{23}$$

and where

$$\alpha_h(h,p) := -\frac{\partial_h \rho}{\rho}(h,p) \tag{24}$$

is the (enthalpic) coefficient of thermal expansion. Let us note that $\beta_h(h,p) = \beta_T[T(h,p),p]$ with

$$\beta_T(T,p) := \frac{p\alpha_T}{\rho C_p}(T,p) \tag{25}$$

and

$$\alpha_T(T,p) := -\frac{\partial_T \rho}{\rho}(T,p) \tag{26}$$

since $\alpha_T[T(h,p),p] = C_p(h,p)\alpha_h(h,p)$. The coefficient α_T is a coefficient of thermal expansion which is more classical than α_h.

We add to System (21) the boundary conditions

$$\left\{\begin{array}{llcll} \forall x \in \partial\Omega_a : & h(t,x) & = & h_e(t), & \text{(a)} \\ & \rho\mathbf{u}(t,x) & = & (D_e(t),0,0) & \text{(b)} \end{array}\right. \tag{27}$$

in the forced convection regime, or the boundary conditions

$$\left\{\begin{array}{llcll} \forall x \in \partial\Omega_a : & h(t,x) & = & h_e(t), & \text{(a)} \\ & p(t,x) & = & P_e(t) & \text{(b)} \end{array}\right. \tag{28}$$

in the natural convection regime. We also add the boundary conditions

$$\forall x \in \partial\Omega_b : \quad \Pi(t,x) = P_0(t). \tag{29}$$

We recall that $\partial\Omega_a := [0,L_1]\times[0,L_2]\times\{0\}$ and $\partial\Omega_b := [0,L_1]\times[0,L_2]\times\{L_3\}$. Let us note that the initial velocity field $\mathbf{u}^0(x)$ and the initial internal enthalpy $h^0(x)$ have to satisfy the compatibility condition

$$\nabla\cdot\mathbf{u}^0 = -\frac{P_0'(0)}{(\rho a^2)^0} + \frac{\beta_h[h^0,P_0(0)]}{P_0(0)}[\nabla\cdot(\lambda\nabla T^0) + \Phi(0,x)] \tag{30}$$

with $(\rho a^2)^0 = \rho[h^0, P_0(0)] \cdot (a[h^0, P_0(0)])^2$ and $T^0(x) = T[h^0(x), P_0(0)]$. System (21)-(30) defines the *Low Mach Nuclear Core* (LMNC) model.

Let us underline that the divergence constraint (21)(a) is equivalent to the mass conservation equation (1)(a) (this may not be the case at the discrete level). Nevertheless, we keep the velocity constraint (21)(a):

- firstly because (21)(a) underlines the elliptic nature of the LMNC model,
- secondly because the dimensionless formulation (20)(a) of (21)(a) shows the dimensionless parameters giving the magnitude of the thermal expansion (see §3.2), which is not the case of the dimensionless formulation of the mass conservation equation (see (6)(a)).

The LMNC model (21)-(30) is interesting compared to the compressible Navier-Stokes System (1)-(5) because:

(1) At low Mach number, the compressible Navier-Stokes System (1)-(5) has two time scales[2] which are the material time scale $t_{\text{mat}} := L_*/u_*$ and the acoustic time scale $t_{\text{ac}} := L_*/a_*$: these time scales are such that $t_{\text{ac}} \ll t_{\text{mat}}$. The LMNC model (21)-(30) is obtained by filtering out the acoustic waves in the compressible Navier-Stokes System (1)-(5). As a consequence, the LMNC model has only one time scale which is the material time scale $t_{\text{mat}} := L_*/u_*$.

(2) In the LMNC model (21)-(30), the density $\rho(t,x)$ is linked to the internal enthalpy $h(t,x)$ (or to the temperature $T(t,x)$) only through the one-to-one function $\rho \mapsto h(\rho, P_0(t))$ (or to the one-to-one function $\rho \mapsto T(\rho, P_0(t))$) since the pressure $P_0(t)$ is a known function. This is not the case for the compressible Navier-Stokes System (1)-(5) since $p(t,x)$ is an unknown variable (we only know that $||p - P_0||_{L^\infty} = \mathcal{O}(M^2)$ at low Mach number).

(3) Both previous points:

- allow to obtain monodimensional *stationary* analytical solutions (see below) when the EOS is a stiffened gas EOS, with or without regular and singular charge loss. We cannot obtain such *stationary* analytical solutions in the case of the compressible Navier-Stokes system,
- allow to obtain monodimensional *unstationary* analytical solutions (with or without *phase change phenomenon*) again when the EOS is a stiffened gas EOS and when $\partial_t \Phi(t,x) = 0$, $h'_e(t) = 0$, $D'_e(t) = 0$ and $P'_0(t) = 0$, with or without regular and singular charge loss [1,2]. We cannot obtain such *unstationary* analytical solutions in the case of the compressible Navier-Stokes system,
- allow to construct in the monodimensional case robust and accurate *explicit* schemes[3], which are not expensive from a CPU point of view, with or without phase change phenomenon and with or without regular and singular charge loss [1,2]. These monodimensional schemes are easier and less costly to implement than algorithms used to simulate the compressible Navier-Stokes system.

[2]We do not take into account the characteristic time scale τ_* of the time variation of $\Phi(t,x)$, $h_e(t)$, $D_e(t)$ and $P_0(t)$ since τ_* is supposed to be equal or greater than t_{mat}: see (11).

[3]We propose in [1,2] an *explicit* monodimensional scheme based on the method of characteristics (MOC) which is *unconditionally* stable. We also propose in [2] an *explicit* monodimensional finite volume type scheme which is more accurate than the MOC-type scheme but which is stable under a classical CFL criterion based on the fluid velocity.

These three points are induced by the fact that in the *monodimensional* case, the mass conservation equation and the energy conservation equation can be solved *independently* of the momentum conservation equation. This property is not valid in the bidimensional and tridimensional cases. As a consequence, it will be much more difficult (or impossible) to obtain analytical solutions in the bidimensional and tridimensional cases. And, in the bidimensional or tridimensional cases, the LMNC model will have to be solved with a partially implicit scheme (because of its elliptic nature). Nevertheless, this scheme should remain more robust, more accurate and less costly than any scheme used to discretize the compressible Navier-Stokes system (this point will have to be studied carefully).

3.2. Parameters giving the order of magnitude of the thermal expansion

By noting that $\mathcal{O}(\overline{\beta}_h) = \alpha_{h_*} h_*$, the dimensionless Equation (20)(a) shows that

$$\begin{cases} \zeta_h^2 \kappa \ll 1 \quad \text{and} \quad \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \dfrac{\alpha_{h_*} h_*}{R_e P_r} \ll 1 \quad \text{and} \quad \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \alpha_{h_*} h_* C_H \ll 1 \quad \Longrightarrow \quad \nabla \cdot \mathbf{u} = 0, \quad \text{(a)} \\ \zeta_h^2 \kappa = \mathcal{O}(1) \quad \text{or} \quad \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \dfrac{\alpha_{h_*} h_*}{R_e P_r} = \mathcal{O}(1) \quad \text{or} \quad \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \alpha_{h_*} h_* C_H = \mathcal{O}(1) \quad \Longrightarrow \quad \nabla \cdot \mathbf{u} \neq 0. \quad \text{(b)} \end{cases} \tag{31}$$

In other words, we have to take into account the thermal expansion in the nuclear core as soon as one of the three conditions in (31)(b) is satisfied, which is the case in a nuclear core. Let us note that when the fluid is a perfect gas, we can replace (31) by

$$\begin{cases} \kappa \ll 1 \quad \text{and} \quad \dfrac{1}{R_e P_r} \ll 1 \quad \text{and} \quad C_H \ll 1 \quad \Longrightarrow \quad \nabla \cdot \mathbf{u} = 0, \\ \kappa = \mathcal{O}(1) \quad \text{or} \quad \dfrac{1}{R_e P_r} = \mathcal{O}(1) \quad \text{or} \quad C_H = \mathcal{O}(1) \quad \Longrightarrow \quad \nabla \cdot \mathbf{u} \neq 0 \end{cases}$$

since $\mathcal{O}(\zeta_h) = \mathcal{O}(\zeta_T) = 1$ and $\alpha_* = 1/h_*$ in that case.

3.3. Boussinesq approximation

Let us suppose that:

Hypothesis 3.2. *The flow is such that*

$$\begin{cases} \zeta_h^2 \kappa \ll 1, & (a) \\ \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \dfrac{\alpha_{h_*} h_*}{R_e P_r} \ll 1, & (b) \\ \left(\dfrac{\zeta_h}{\zeta_T}\right)^2 \cdot \alpha_{h_*} h_* C_H \ll 1. & (c) \end{cases} \tag{32}$$

We also suppose that:

Hypothesis 3.3. *The flow is such that*

$$\begin{cases} ||\alpha_{h_*}(h - h_*)||_{L^\infty} =: \epsilon \ll 1, & (a) \\ \dfrac{||\alpha_{h_*}(h - h_*)||_{L^\infty}}{F_r} \gg \mathcal{O}(\epsilon). & (b) \end{cases} \tag{33}$$

Thus, we suppose that the Froude number is such that

$$F_r = \mathcal{O}(\epsilon^q) \ll 1 \qquad \text{with} \qquad q \in \mathbb{R}_*^+. \tag{34}$$

Let us note that $\frac{||\alpha_{h_*}(h-h_*)||_{L^\infty}}{F_r}$ is similar to the Richardson number[4].

Hypothesis 3.2 allows to obtain the free divergence constraint (see (31)(a)) and to neglect the pressure variation $P_0'(t)$ in (21)(c). Hypothesis 3.3, firstly (by using (33)(a)), allows to approximate $\overline{\rho}(\overline{h}, \overline{P}_0)$ in the left hand sides of (20)(b,c) by $\overline{\rho}_*$ and, secondly (by using (33)(b)), *imposes* to approximate $\overline{\rho}(\overline{h}, \overline{P}_0)$ in the right hand side of (20)(b) by $\overline{\rho}_*[1 - \overline{\alpha}_{\overline{h}_*}(\overline{h} - \overline{h}_*)]$ (and not by $\overline{\rho}_*$ as in the left hand side of (20)(b,c)). Thus, under Hypothesis 3.2 and 3.3, by defining $\rho_* := \rho(h_*, P_0(t=0))$, and by using the dimensionless LMNC model (20), we formally obtain that the LMNC model (21)-(30) can be approximated by

$$\left\{\begin{array}{ll} \nabla \cdot \mathbf{u} = 0, & \text{(a)} \\ \rho_*(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla \Pi + \nabla \cdot [\mu(h, P_0)(\nabla \mathbf{u} + \nabla \mathbf{u}^T)] + \rho_*[1 - \alpha_{h_*}(h - h_*)]\mathbf{g}, & \text{(b)} \\ \rho_*(\partial_t h + \mathbf{u} \cdot \nabla h) = \nabla \cdot (\lambda \nabla T) + \Phi(t, x) & \text{(c)} \end{array}\right. \tag{35}$$

with the boundary conditions (27) (ρ is replaced by ρ_* in (27)(b)) and (29), the initial velocity field being such that $\nabla \cdot \mathbf{u}^0 = 0$. Let us note that the pressure Π in (35) can be defined with

$$\Pi = \Pi_{\text{th}} + \Pi_{\text{hyd}}$$

with

$$\Pi_{\text{hyd}} := P_0(t) + \rho_* g(L_3 - x_3)$$

where Π_{th} is the pressure induced by the *thermal transfers* and where Π_{hyd} is the hydrodynamic pressure *directly* due to the *gravity field*. Let us underline that it is not possible to decompose in the same way the pressure Π solution of the LMNC model (21)-(30) (see for example the analytical solution (44)(d) in Lemma 4.1).

Let us remark that we keep the time and space variations of $\mu(h(t,x), P_0(t))$ in (35)(b). Nevertheless, we can eventually simplify (35)(b) by approximating $\mu(h(t,x), P_0(t))$ by $\mu_* := \mu(h_*, P_0(t=0))$. With this approximation and by replacing Π by Π_{th} which satisfies the boundary condition

$$\forall x \in [0, L_1] \times [0, L_2] \times \{L_3\} : \quad \Pi_{\text{th}}(t, x) = 0,$$

(35) is given by

$$\left\{\begin{array}{l} \nabla \cdot \mathbf{u} = 0, \\ \rho_*(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla \Pi_{\text{th}} + \mu_* \Delta \mathbf{u} - \rho_* \alpha_{h_*}(h - h_*)\mathbf{g}, \\ \rho_*(\partial_t h + \mathbf{u} \cdot \nabla h) = \nabla \cdot (\lambda \nabla T) + \Phi(t, x) \end{array}\right.$$

which is a Boussinesq-type approximation.

[4]The Richardson number is written in function of the temperature. Here, the dimensionless number $\frac{||\alpha_{h_*}(h-h_*)||_{L^\infty}}{F_r}$ is written in function of the internal enthalpy.

4. Monodimensional stationary analytical solution for a stiffened gas

Let us suppose that the EOS is a stiffened gas EOS that is to say

$$\rho(h,p) = \frac{\gamma}{\gamma-1}\cdot\frac{p+P_\infty}{h-q} \tag{36}$$

where γ (adiabatic coefficient), P_∞ (molecular attraction) and q (binding energy) are three constants [14]. We easily obtain that

$$\begin{cases} \beta_h(h,p) = \dfrac{p}{\rho\,(h-q)} = \dfrac{\gamma-1}{\gamma}\cdot\dfrac{p}{p+P_\infty}, & \text{(a)} \\ \alpha_h(h,p) = \alpha_h(h) = \dfrac{1}{h-q} & \text{(b)} \end{cases} \tag{37}$$

and we can prove that the temperature T is such that [14]

$$\begin{cases} p = \dfrac{\gamma-1}{\gamma}\rho C_p T - P_\infty, & \text{(a)} \\ h = C_p T + q. & \text{(b)} \end{cases} \tag{38}$$

The constant C_p is the calorific capacity at constant pressure[5]. By using (22), (36) and (38), we obtain that

$$\begin{aligned} a &= \sqrt{\frac{\gamma(p+P_\infty)}{\rho}} \\ &= \sqrt{(\gamma-1)(h-q)}. \end{aligned}$$

And by using (25) and (26), we obtain that

$$\begin{cases} \beta_T(T,p) = \beta_T(p) = \dfrac{\gamma-1}{\gamma}\cdot\dfrac{p}{p+P_\infty}, & \text{(a)} \\ \alpha_T(T,p) = \alpha_T(T) = \dfrac{1}{T}. & \text{(b)} \end{cases} \tag{39}$$

Let us note that we suppose that $\gamma > 1$, $C_p > 0$, $p + P_\infty > 0$ and $h > q$: as a consequence, the density ρ and the temperature T are positive, and the sound velocity a is real.

Moreover, we neglect the thermal conduction – *i.e.* $\lambda = 0$ – and we suppose that the geometry is monodimensional – *i.e.* $\Omega := [0, L]$. Of course, to be able to define the stationary LMNC model, we have to suppose that $\Phi(t,x)$, $h_e(t)$, $D_e(t)$ and $P_0(t)$ do not depend on the time t and, thus, are respectively equal to $\Phi(x)$, h_e, D_e

[5] In the case of a perfect gas, we have $C_p = \dfrac{\gamma k}{m(\gamma-1)}$ where k and m are respectively the Boltzmann constant and the atomic mass of the gas.

and P_0. In that case, the stationary LMNC model is given by

$$\begin{cases} \partial_x u = \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \Phi(x), & \text{(a)} \\ \rho u \partial_x u = -\partial_x \Pi + \dfrac{4}{3} \partial_x [\mu(h, P_0) \partial_x u] - \rho g, & \text{(b)} \\ \rho u \partial_x h = \Phi(x). & \text{(c)} \end{cases} \tag{40}$$

The boundary conditions (27)(29) are respectively given by

$$\begin{cases} h(0) = h_e, & \text{(a)} \\ \rho u(0) = D_e & \text{(b)} \end{cases} \tag{41}$$

and

$$\Pi(L) = P_0. \tag{42}$$

Let us underline that the divergence constraint (40)(a) is equivalent to the stationary mass conservation equation

$$\partial_x(\rho u) = 0 \tag{43}$$

(we recall that (21)(a) may be replaced by (1)(a)). And, let us remark that the 4/3 coefficient in (40)(b) comes from the formulation of the newtonian viscosity tensor (2) in the monodimensional case and from the Stokes hypothesis.

4.1. **Construction of the analytical solution**

We have the following lemma:

Lemma 4.1. *Let us suppose that the monodimensional stationary LMNC model (40)-(42) admits a solution. Then, when the EOS is the stiffened gas EOS (36), this solution is given by*

$$\begin{cases} \rho(x) = \dfrac{D_e}{u(x)} = \dfrac{\gamma(P_0 + P_\infty)}{\gamma - 1} \cdot \dfrac{D_e}{(h_e - q)D_e + \Psi(x)}, & (a) \\ u(x) = \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \cdot [(h_e - q)D_e + \Psi(x)], & (b) \\ h(x) = h_e + \dfrac{\Psi(x)}{D_e}, & (c) \\ \Pi(x) = P_0 - \dfrac{\gamma - 1}{\gamma} \cdot \dfrac{D_e}{P_0 + P_\infty} \cdot [\Psi(x) - \Psi(L)] - \dfrac{g}{h_e - q} \cdot \dfrac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot [\varphi(x) - \varphi(L)] + \pi_\mu(x) & (d) \end{cases} \tag{44}$$

where the functions $\Psi(x)$ and $\varphi(x)$ are defined with

$$\begin{cases} \Psi(x) := \displaystyle\int_0^x \Phi(y) dy, & (a) \\ \varphi(x) := \displaystyle\int_0^x \dfrac{dy}{1 + \dfrac{\Psi(y)}{(h_e - q)\, D_e}} & (b) \end{cases} \tag{45}$$

and where the charge loss $\pi_\mu(x)$ due to viscous effects is given by

$$\pi_\mu(x) = \frac{4}{3} \cdot \frac{\gamma - 1}{\gamma(P_0 + P_\infty)} \left\{ \mu[h(x), P_0]\Phi(x) - \mu[h(L), P_0]\Phi(L) \right\}. \tag{46}$$

Let us underline that we suppose in Lemma 4.1 that $\Phi(x)$ is enough regular to be able to define $\Psi(x)$ and $\varphi(x)$ with (45) and that the stationary solution given by (44) is admissible from a physical point of view. We will show in §4.2.1 and §4.2.2 that a necessary condition is that the condition

$$D_e \in]-\infty, D_e^c[\cup]0, +\infty[$$

is satisfied, D_e^c being a critical mass flux which belongs to $\mathbb{R}_*^-$. More generally, the influence of the regularity of $\Phi(t, x)$ and of the EOS on the solution of the LMNC model (21)-(30) will have to be studied carefully in a future work.

Let us note that (44) is equivalent to

$$\begin{cases} \rho(x) = \dfrac{\rho_e u_e}{u(x)} = \dfrac{\rho_e u_e}{u_e + \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \Psi(x)}, & \text{(a)} \\[2ex] u(x) = u_e + \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \Psi(x), & \text{(b)} \\[2ex] h(x) = h_e + (h_e - q) \cdot \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \cdot \dfrac{\Psi(x)}{u_e}, & \text{(c)} \\[2ex] \Pi(x) = P_0 - \dfrac{u_e}{h_e - q} \cdot [\Psi(x) - \Psi(L)] - \dfrac{g}{h_e - q} \cdot \dfrac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot [\widetilde{\varphi}(x) - \widetilde{\varphi}(L)] + \pi_\mu(x) & \text{(d)} \end{cases} \tag{47}$$

with

$$\begin{cases} \rho_e = \dfrac{\gamma}{\gamma - 1} \cdot \dfrac{P_0 + P_\infty}{h_e - q}, & \text{(a)} \\[2ex] u_e = \dfrac{\gamma - 1}{\gamma} \cdot \dfrac{h_e - q}{P_0 + P_\infty} \cdot D_e & \text{(b)} \end{cases} \tag{48}$$

and

$$\widetilde{\varphi}(x) := \int_0^x \frac{dy}{1 + \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \cdot \dfrac{\Psi(y)}{u_e}}. \tag{49}$$

Formulations (44) are well adapted when we impose the mass flux D_e in the nuclear core (see (41)(b)); formulations (47) are well adapted when we impose the velocity u_e at $x = 0$ instead of D_e.

We also deduce from Lemma 4.1 that:

Corollary 4.2. *Let us suppose that* $\max_\Omega \Psi > 0$*. Then, when the (positive) inlet mass flux* D_e *goes to zero*[6]*:*

i) $||h||_{L^\infty}$ *goes to the infinity,*

ii) $||T||_{L^\infty}$ *goes to the infinity.*

[6]This condition models a main coolant pump trip in a water nuclear reactor.

Let us note that $\max_\Omega \Psi > 0$ when $\Phi(x)$ is (for example) almost everywhere positive, which is the case when $\Phi(x)$ is the power density in a nuclear core. Let us also note that the internal enthalpy given by (44)(c) and that *Point i)* of Corollary 4.2 are satisfied for any EOS. Nevertheless, the other formulae and *Point ii)* of Corollary 4.2 are obtained in the case of the stiffened gas EOS (36)[7].

Proof of Lemma 4.1: The inlet density ρ_e given by (48)(a) is a consequence of the boundary condition (41)(a) and of the stiffened gas EOS (36) with $p = P_0$ (we recall that the LMNC model is such that the thermodynamic pressure p does not depend on the space variable and is thus equal to the outlet pressure P_0). The inlet velocity u_e given by (48)(b) is deduced from the boundary condition (41)(b), from $D_e := \rho_e u_e$ and from (48)(a). The velocity $u(x)$ given by (44)(b) is obtained by integrating (40)(a) and by taking into account (48)(b). By using the mass conservation Equation (43), we obtain that $\rho u = D_e$ which gives (44)(a). Thus, Equation (40)(c) is equivalent to $D_e \partial_x h = \Phi(x)$ which allows to obtain (44)(c) by taking into account (41)(a). By taking into account (40)(a) and (44)(a), Equation (40)(b) is given by

$$\begin{aligned}\partial_x \Pi \quad = \quad & -\frac{\gamma-1}{\gamma(P_0+P_\infty)} D_e \Phi(x) + \frac{4}{3} \cdot \frac{\gamma-1}{\gamma(P_0+P_\infty)} \partial_x [\mu(h, P_0)\Phi(x)] \\ & -\frac{\gamma(P_0+P_\infty)}{\gamma-1} \cdot \frac{D_e}{(h_e-q)D_e+\Psi(x)} \cdot g\end{aligned}$$

that is to say

$$\partial_x \Pi = -\frac{\gamma-1}{\gamma(P_0+P_\infty)} D_e \Phi(x) + \partial_x \pi_\mu(x) - \frac{g}{h_e-q} \cdot \frac{\gamma(P_0+P_\infty)}{\gamma-1} \cdot \frac{1}{1+\dfrac{\Psi(x)}{(h_e-q)\,D_e}}. \tag{50}$$

By integrating (50) and by taking into account the boundary condition (42), we obtain (44)(d).□

Proof of Corollary 4.2: This is a direct consequence of (38)(b) and (44)(c).□

4.2. Downward critical mass flux, inversion flow and natural convection regime

We show in this section that the monodimensional analytical solution (44) of the LMNC model (40)-(42) is physically relevent under the necessary condition

$$D_e \in]-\infty, D_e^c[\,\cup\,]0, +\infty[$$

where D_e^c is a critical mass flux which belongs to $\mathbb{R}_*^-$ and which strongly depends on the power density $\Phi(x)$. Thus, we suppose in the sequel that $D_e \in \mathbb{R}$ to be able to take into account the case $D_e \in \mathbb{R}^-$ (let us note that when $D_e < 0$, $x = 0$ and $x = L$ define respectively the outlet and the inlet of the nuclear core). The critical mass flux D_e^c defines the area $[D_e^c, 0[$ where the (outlet) boundary condition $\rho u(x = 0) = D_e$ (see (41)(b)) is not compatible with the (outlet) boundary condition $h(x = 0) = h_e$ (see (41)(a)). Let us note that when we impose $\rho u(x = 0) = D_e$ with $D_e < 0$, we can see the computation of the (inlet) internal enthalpy $h(x = L)$ through the resolution of the LMNC model (40)-(42) as the resolution of the inverse problem: "*Find the (inlet) internal enthalpy $h(x = L)$ which gives the (outlet) internal enthalpy $h(x = 0)$ for a given negative mass flux*". We will see that when $D_e \in [D_e^c, 0[$, this inverse problem may have a solution but which is not physically relevent or does not have any solution.

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[7] In fact, we just have to suppose that the EOS is such that $\lim_{h\to+\infty} T(h, P_0) = +\infty$ when $P_0 > 0$ to obtain *Point ii)* of Corollary 4.2.

Then, we study the natural convection regime by replacing the boundary condition $\rho u(x=0) = D_e$ on the mass flux by a boundary condition on the pressure $\Pi(x=0) = P_e$ (see (60)).

4.2.1. *Downward critical mass flux*

The density $\rho(x)$ and the temperature $T(x)$ have to be positive for physical reasons. Formula (44)(a) shows that when $D_e \neq 0$, the density $\rho(x)$ is positive on $[0, L]$ if and only if

$$\text{sign}\,(u(x)) = \text{sign}(D_e). \tag{51}$$

Moreover, the density ρ_e (given by (48)(a)) and the temperature T_e (deduced from (38)(a)) are supposed to be positive (also for physical reasons). This implies that

$$\frac{\gamma}{\gamma-1} \cdot \frac{P_0 + P_\infty}{h_e - q} > 0 \tag{52}$$

and

$$P_0 + P_\infty > 0 \tag{53}$$

(we recall that $\gamma > 1$ and $C_p > 0$). Thus, we have also

$$\frac{\gamma-1}{\gamma(P_0 + P_\infty)} > 0 \qquad \text{and} \qquad h_e > q. \tag{54}$$

As a consequence, we deduce from (44)(b) and (51) that the mass flux D_e has to satisfy[8]

$$D_e \in]-\infty, D_e^c[\cup]0, +\infty[\tag{55}$$

with

$$D_e^c = -\frac{\Psi(L)}{h_e - q} \tag{56}$$

since $\Psi(x)$ given by (45)(a) is a positive increasing function on [0,L] ($\Psi(0) = 0$) (the power density $\Phi(x)$ is positive since we model a nuclear core). Let us note that Formula (56) is equivalent to

$$\begin{aligned} D_e^c &= -\alpha_h(h_e) \cdot \frac{\mathcal{P}_{\text{th}}(\Phi)}{S_e} \\ &= -\frac{\alpha_T(T_e)}{C_p} \cdot \frac{\mathcal{P}_{\text{th}}(\Phi)}{S_e} \end{aligned}$$

where α_h and α_T are the coefficients of thermal expansion (see Definitions (24) and (26), and Formulae (37)(b) and (39)(b) obtained in the case of the stiffened gas EOS), and where $\mathcal{P}_{\text{th}}(\Phi) := S_e \int_0^L \Phi(x)dx$ and S_e are respectively the thermal power and the inlet area[9] of the nuclear core. This result shows that when the flow in the nuclear core is a downward flow and not an upward flow, it is possible to impose the boundary condition

$$\rho u(0) = D_e < 0$$

if and only if the absolute value of the mass flux $|D_e|$ is strictly greater than the critical mass flux $|D_e^c|$.

[8]We do not allow $D_e = 0$ in (55) (and in the sequel of the paper) to exclude the pathological case studied in Corollary 4.2.

[9]The quantity $S_e D_e$ is the mass flow.

4.2.2. *Inversion flow*

When $D_e \in [D_e^c, 0[$, there exists $X \in [0, L]$ such that

$$\begin{cases} \forall x \in [0, X[: \ u(x) < 0, \\ u(X) = 0, \\ \forall x \in]X, L]: \ u(x) > 0 \end{cases} \tag{57}$$

which means that there is an *inversion flow* in the nuclear core at the point $x = X$. As a consequence, we have

$$\begin{cases} \forall x \in [0, X[: \ \rho(x) > 0, \\ \lim\limits_{x \to X^-} \rho(x) = +\infty, \\ \lim\limits_{x \to X^+} \rho(x) = -\infty, \\ \forall x \in]X, L]: \ \rho(x) < 0 \end{cases} \tag{58}$$

and

$$\begin{cases} \forall x \in [0, X[: \ T(x) > 0, \\ \lim\limits_{x \to X} T(x) = 0, \\ \forall x \in]X, L]: \ T(x) < 0. \end{cases} \tag{59}$$

Thus, the density $\rho(x)$ and the temperature $T(x)$ are not physically relevent. Nevertheless, we can wonder if the LMNC model (40)-(42) is well-posed when $D_e \in [D_e^c, 0[$ even if the solution is not physically relevent. There are two cases:

• **When** $D_e \in [D_e^c, 0[$ **and** $g = 0$**:** We can solve (40)(b) and the pressure $\Pi(x)$ is given by

$$\Pi(x) = P_0 - \frac{u_e}{h_e - q} \cdot [\Psi(x) - \Psi(L)] + \pi_\mu(x).$$

Thus, (44) remains *formally* valid in the sense that we just have to put $g = 0$ in (44)(b). In other words, the LMNC model (40)-(42) is well-posed.

• **When** $D_e \in [D_e^c, 0[$ **and** $g \neq 0$**:** We cannot solve (40)(b) because of the singularity of the density $\rho(x)$ at the point $x = X$ (see (58)). In that case, Formula (44)(d) is not valid. This means that the LMNC model (40)-(42) is ill-posed.

To summarize, $[D_e^c, 0[$ is a *forbidden area* in the sense that when we do not take into account the gravity, the solution of the LMNC model (40)-(42) is not physically relevent and when we take into account the gravity, the LMNC model (40)-(42) is ill-posed.

4.2.3. *Natural convection regime*

We model the natural convection regime instead of the forced convection regime by replacing the boundary condition

$$\rho u(0) = D_e$$

(see (41)(b)) by the boundary condition

$$\Pi(0) = P_e \tag{60}$$

where P_e is a given positive constant defining the inlet pressure. Thus, D_e is now an unknow which is solution of

$$\Pi_e(D_e, h_e, P_0) = P_e \tag{61}$$

where $\Pi_e(D_e, h_e, P_0)$ is deduced from (44)(d) with $x = 0$ that is to say

$$\Pi_e(D_e, h_e, P_0) = P_0 + \frac{\gamma - 1}{\gamma} \cdot \frac{D_e}{P_0 + P_\infty} \cdot \Psi(L) + \frac{g}{h_e - q} \cdot \frac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot \varphi(L) + \pi_\mu(0). \tag{62}$$

We have the following lemma:

Lemma 4.3. *Let us define the function* $\beta(D_e) := \Pi_e(D_e, h_e, P_0) - P_e$*. We have*

$$\forall D_e \in \mathbb{R}_* : \quad \beta'(D_e) > 0. \tag{63}$$

As a consequence, for any P_e *such that*

$$P_e > P_0 + \pi_\mu(0), \tag{64}$$

the equation

$$\beta(D_e) = 0 \tag{65}$$

admits an unique solution D_e^{conv}*. Moreover,* D_e^{conv} *belongs to* $\mathbb{R}_*^+$*.*

This lemma shows that under Condition (64), the LMNC model (40)-(42) admits the stationary solution (44) where D_e is replaced by $D_e^{conv} > 0$ when the boundary condition (41)(b) is replaced by (60).

Proof of Lemma 4.3: By using (62), we obtain that

$$\beta'(D_e) = \frac{\gamma - 1}{\gamma} \cdot \frac{\Psi(L)}{P_0 + P_\infty} + \frac{g}{h_e - q} \cdot \frac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot \partial_{D_e} \varphi(L).$$

Moreover, we have

$$\partial_{D_e} \varphi(L) = \int_0^L \frac{\Psi(y)}{D_e^2 \, (h_e - q)} \cdot \frac{1}{\left[1 + \dfrac{\Psi(y)}{(h_e - q) \, D_e}\right]^2} dy.$$

Thus, by using (54), and since $g > 0$ and $\Psi(L) > 0$, we obtain that $\beta'(D_e) > 0$ on $\mathbb{R}$. We conclude by noting that $\beta(0) = P_0 + \pi_\mu(0) - P_e < 0$ under Condition (64) and that $\beta(+\infty) = +\infty$.□

4.3. Study in the case of two specific power density profiles

For some specific choices of the power density $\Phi(x)$, we obtain explicit formulae for $\Psi(x)$ and $\varphi(x)$ defined with (45) (and, thus, for $\widetilde{\varphi}(x)$ defined with (49)). As a consequence, we obtain explicit formulae for the analytical solutions (44), (46) and (62) (and, thus, for (47)).

To obtain these formulae, we study two power density profiles: a constant power density profile and an exponential power density profile. These two cases will allow us to study in detail the influence of the power density profile on the downward critical mass flux.

4.3.1. *Constant power density profile*

Let us suppose that

$$\forall x \in [0, L]: \quad \Phi(x) = \Phi_0 \tag{66}$$

where Φ_0 is a positive constant.

Construction of the analytical solution:

We deduce from (45) and (66) that

$$\begin{cases} \Psi(x) = \Phi_0 x, & \text{(a)} \\ \varphi(x) = \dfrac{(h_e - q)\, D_e}{\Phi_0} \ln \left[1 + \dfrac{\Phi_0 x}{(h_e - q)\, D_e} \right]. & \text{(b)} \end{cases} \tag{67}$$

Thus, by using (44) and (46), we obtain

$$\begin{cases} \rho(x) = \dfrac{\gamma(P_0 + P_\infty)}{\gamma - 1} \cdot \dfrac{D_e}{(h_e - q) D_e + \Phi_0 x}, & \text{(a)} \\ u(x) = \dfrac{\gamma - 1}{\gamma(P_0 + P_\infty)} \cdot [(h_e - q) D_e + \Phi_0 x], & \text{(b)} \\ h(x) = h_e + \dfrac{\Phi_0}{D_e} x, & \text{(c)} \\ \Pi(x) = P_0 - \dfrac{\gamma - 1}{\gamma} \cdot \dfrac{D_e}{P_0 + P_\infty} \cdot \Phi_0 (x - L) - \dfrac{g D_e}{\Phi_0} \cdot \dfrac{\gamma(P_0 + P_\infty)}{\gamma - 1} \ln \left[\dfrac{1 + \dfrac{\Phi_0 x}{(h_e - q)\, D_e}}{1 + \dfrac{\Phi_0 L}{(h_e - q)\, D_e}} \right] + \pi_\mu(x) & \text{(d)} \end{cases} \tag{68}$$

where the charge loss $\pi_\mu(x)$ is given by

$$\pi_\mu(x) = \frac{4}{3} \cdot \frac{(\gamma - 1)\Phi_0}{\gamma(P_0 + P_\infty)} \left\{ \mu[h(x), P_0] - \mu[h(L), P_0] \right\}. \tag{69}$$

We deduce from (68)(d) that the inlet pressure $\Pi_e(D_e, h_e, P_0)$ is given by

$$\Pi_e(D_e, h_e, P_0) = P_0 + \frac{\gamma - 1}{\gamma} \cdot \frac{D_e \Phi_0 L}{P_0 + P_\infty} + \frac{g D_e}{\Phi_0} \cdot \frac{\gamma(P_0 + P_\infty)}{\gamma - 1} \ln \left[1 + \frac{\Phi_0 L}{(h_e - q)\, D_e} \right] + \pi_\mu(0) \tag{70}$$

with a charge loss $\pi_\mu(0)$ equal to

$$\pi_\mu(0) = \frac{4}{3} \cdot \frac{(\gamma - 1)\Phi_0}{\gamma(P_0 + P_\infty)} \left\{ \mu(h_e, P_0) - \mu[h(L), P_0] \right\}. \tag{71}$$

The previous formulae allow to study for example the sign of the charge loss due to viscous effects $\pi_\mu(0)$ in function of the sign of $\partial_h \mu(h, P_0)$ (the constant $\frac{\gamma-1}{\gamma(P_0+P_\infty)}$ is positive: see (54)). There are two cases[10]:

[10]Although we only consider $\Phi_0 \geq 0$ in this paper, we verify that these two cases do not depend on the sign of Φ_0.

- when $\Phi_0 \neq 0$ and $\partial_h \mu(h, P_0) < 0$ (which is the case when the fluid is a liquid):

$$\mathrm{sign}(\pi_\mu) = \mathrm{sign}(D_e),$$

- when $\Phi_0 \neq 0$ and $\partial_h \mu(h, P_0) > 0$ (which is the case when the fluid is a gas):

$$\mathrm{sign}(\pi_\mu) = -\mathrm{sign}(D_e),$$

- when $\Phi_0 = 0$: $\pi_\mu(x) = 0$.

To obtain this result, we just have to note that $h'(x) = \frac{\Phi_0}{D_e}$ (see (44)(c)).

Downward critical mass flux:

By using (55) and (56), we obtain that the mass flux D_e has to satisfy

$$D_e \in]-\infty, D_e^{c,0}[\cup]0, +\infty[\tag{72}$$

with

$$D_e^{c,0} = -\frac{\Phi_0 L}{h_e - q}. \tag{73}$$

4.3.2. *Exponential power density profile*

Let us suppose that

$$\forall x \in [0, L]: \qquad \Phi(x) = \Phi_\lambda(x) \qquad \text{where} \qquad \Phi_\lambda(x) = \Phi_0 e^{\lambda x} \tag{74}$$

with $\Phi_0 \in \mathbb{R}_*^+$ and $\lambda \in \mathbb{R}_*$. Let us note that $\lambda \in \mathbb{R}_*$ since $\lambda = 0$ makes (74) equivalent to the constant case (66).

Construction of the analytical solution:

We deduce from (45)(a) and (74) that

$$\Psi_\lambda(x) = \frac{\Phi_0}{\lambda}\left(e^{\lambda x} - 1\right).$$

Thus, we deduce from (45)(b) that

$$\varphi(x) = \varphi_\lambda(x)$$

with

$$\varphi_\lambda(x) = \int_0^x \frac{dy}{1 + \frac{\lambda_0}{\lambda}\left(e^{\lambda y} - 1\right)}.$$

Let us define $z = e^{\lambda y} - 1$. Since $dz = \lambda(1+z)dy$, we can write that

$$\varphi_\lambda(x) = \frac{1}{\lambda}\int_0^{e^{\lambda x}-1} \frac{dz}{(1+z)\left(1 + \frac{\lambda_0}{\lambda} z\right)}.$$

There are two cases:

(1) Let us suppose that

$$\lambda = \lambda_0.$$

Then

$$\varphi_{\lambda_0}(x) = \frac{1}{\lambda_0} \int_0^{e^{\lambda_0 x}-1} \frac{dz}{(1+z)^2}$$

that is to say

$$\varphi_{\lambda_0}(x) = \frac{1}{\lambda_0}\left(1 - e^{-\lambda_0 x}\right). \tag{75}$$

Let us note that $x \mapsto \varphi_{\lambda_0}(x)$ is well defined for any $\lambda_0 \in \mathbb{R}$.

(2) Let us suppose that

$$\lambda \neq \lambda_0.$$

Then

$$\varphi_\lambda(x) = \frac{1}{\lambda} \cdot \frac{1}{1 - \dfrac{\lambda_0}{\lambda}} \int_0^{e^{\lambda x}-1} \left(\frac{1}{1+z} - \frac{\lambda_0}{\lambda} \cdot \frac{1}{1 + \dfrac{\lambda_0}{\lambda} z} \right) dz$$

which implies that

$$\varphi_\lambda(x) = \frac{1}{1 - \dfrac{\lambda_0}{\lambda}} \left\{ x - \frac{1}{\lambda} \ln \left[1 + \frac{\lambda_0}{\lambda}\left(e^{\lambda x} - 1\right) \right] \right\}. \tag{76}$$

By using (75) and (76), we obtain formulae similar to (68)-(71). Let us remark that $\lambda = \lambda_0$ is not a critical value since the function $\lambda \mapsto \varphi_\lambda$ is continuous for the $L^\infty([0, L])$ norm at the point $\lambda = \lambda_0$. Indeed, by noting that

$$1 + \frac{\lambda_0}{\lambda}\left(e^{\lambda x} - 1\right) = e^{\lambda x}\left[1 + \frac{\lambda_0 - \lambda}{\lambda}(1 - e^{-\lambda x})\right],$$

we obtain that

$$\begin{aligned} \varphi_\lambda(x) &= \frac{1}{1 - \dfrac{\lambda_0}{\lambda}} \left\{ x - \frac{1}{\lambda}\left[\lambda x + \frac{\lambda_0 - \lambda}{\lambda}(1 - e^{-\lambda x}) + \mathcal{O}((\lambda - \lambda_0)^2) \right] \right\} \\ &= \frac{1}{\lambda_0}\left(1 - e^{-\lambda_0 x}\right) + \mathcal{O}(\lambda - \lambda_0). \end{aligned}$$

Downward critical mass flux:

By using (55) and (56), we obtain that the mass flux D_e has to satisfy

$$D_e \in]-\infty, D_e^{c,\lambda}[\cup]0, +\infty[\tag{77}$$

with

$$D_e^{c,\lambda} = -\frac{\Phi_0\left(e^{\lambda L} - 1\right)}{\lambda\,(h_e - q)} \qquad (\text{with } \lambda \neq 0). \tag{78}$$

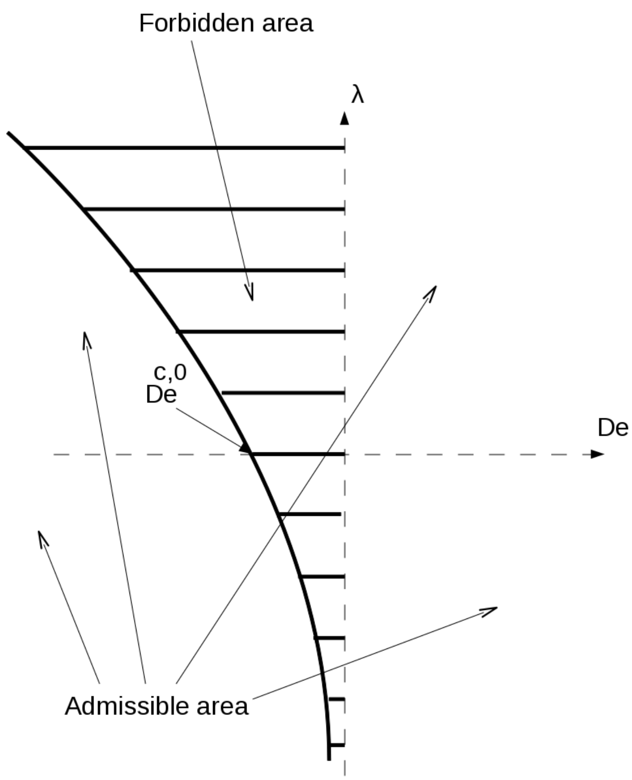

Fig. 1

Let us note that $\lim_{\lambda \to 0} D_e^{c,\lambda} = D_e^{c,0}$ where $D_e^{c,0}$ is given by (73). Thus, we define $\lambda \mapsto D_e^{c,\lambda}$ on $\mathbb{R}$. It is easy to prove that the function $\lambda \mapsto D_e^{c,\lambda}$ is an increasing function from $\mathbb{R}$ to $\mathbb{R}_*^-$ and that

$$\begin{cases} \lim_{\lambda \to -\infty} D_e^c = 0^-, \\ \\ \lim_{\lambda \to +\infty} D_e^c = -\infty. \end{cases}$$

This allows to easily represent on Figure 1 the area where Condition (77) is satisfied (admissible area) and the area where Condition (77) is not satisfied (forbidden area). This result shows that *the more the power density is, the more the downward critical mass flux* $|D_e^c|$ *is.*

4.4. Consistency with the Boussinesq approximation

The monodimensional stationary solution of the Boussinesq approximation (35) without thermal conduction is solution of

$$\begin{cases} \partial_x u = 0, \\ \rho_e u \partial_x u = -\partial_x \Pi + 2\partial_x(\mu_e \partial_x u) - \rho_e g[1 - \alpha_{h_e}(h - h_e)], \\ \rho_e u \partial_x h = \Phi(x) \end{cases}$$

and, thus, is solution of

$$\begin{cases} \partial_x u = 0, & \text{(a)} \\ \partial_x \Pi + \rho_e g[1 - \alpha_{h_e}(h - h_e)] = 0, & \text{(b)} \\ \rho_e u \partial_x h = \Phi(x) & \text{(c)} \end{cases} \tag{79}$$

with the boundary conditions

$$\begin{cases} h(0) = h_e, & \text{(a)} \\ \rho_e u(0) = D_e & \text{(b)} \end{cases} \tag{80}$$

and

$$\Pi(t, L) = P_0 \tag{81}$$

where ρ_e is given by (48)(a).

We recall that the Boussinesq approximation is valid under Hypothesis 3.2 and 3.3. To simplify the analysis, we now suppose that

$$\forall x \in [0, L]: \qquad \Phi(x) = \Phi_0.$$

In the present case, Hypothesis 3.2 (see (32)(c)) is equivalent to impose that

$$\epsilon := \frac{\Phi_0 L \alpha_{h_e}}{D_e} = \frac{\Phi_0 L}{(h_e - q)\, D_e} = \frac{\gamma - 1}{\gamma} \cdot \frac{\Phi_0 L}{u_e(P_0 + P_\infty)} \ll 1. \tag{82}$$

And, Hypothesis 3.3 imposes that the Froude number (defined with (7)(c)) has to be such that (see (34))

$$F_r := \frac{u_e^2}{Lg} = \mathcal{O}(\epsilon^q) \ll 1 \qquad \text{with} \qquad q \in \mathbb{R}_*^+. \tag{83}$$

We have the following lemma:

Lemma 4.4. *When the EOS is the stiffened gas EOS (36), the solution of the Boussinesq model (79)-(81) is given by*

$$\begin{cases} u(x) = u_e := \dfrac{D_e}{\rho_e}, & \text{(a)} \\ h(x) = h_e + \dfrac{\Phi_0}{D_e} x, & \text{(b)} \\ \Pi(x) = P_0 - \rho_e g(x - L) \left[1 - \dfrac{\Phi_0}{2\,(h_e - q)\, D_e}(x + L)\right] & \text{(c)} \end{cases} \tag{84}$$

where ρ_e is given by (48)(a).

It is important to note that Solution (84) of the Boussinesq model exists for any $D_e \neq 0$ (we suppose that $\Phi_0 > 0$) which is not true in the case of the LMNC model whose the solution exists if and only if $D_e \in]-\infty, D_e^c[\cup]0, +\infty[$ where D_e^c is the downward critical mass flux (see §4.2.1). This is due to the fact that ϵ defined with (82) is also given by

$$\epsilon = -\frac{D_e^c}{D_e}$$

where D_e^c is given by (73) when the power density is constant. As a consequence, we have $\lim_{\epsilon \to 0} D_e^c = 0$ which explains why there is no downward critical mass flux in the Boussinesq model.

By noting that (84) is equivalent to

$$\begin{cases} u(x) = u_e := \dfrac{D_e}{\rho_e}, & \text{(a)} \\[2ex] h(x) = h_e + \dfrac{\epsilon}{\alpha_{h_e}} \cdot \dfrac{x}{L}, & \text{(b)} \\[2ex] \Pi(x) = P_0 - \rho_e g L \left(\dfrac{x}{L} - 1\right)\left[1 - \dfrac{\epsilon}{2}\left(\dfrac{x}{L} + 1\right)\right] & \text{(c)} \end{cases} \tag{85}$$

where ϵ is defined with (82), we prove that Solution (84) of the Boussinesq model is coherent with Solution (68) of the LMNC model with constant power density. Indeed, we have the following result:

Lemma 4.5. *Under Conditions (82) and (83):*

i) the solution $\rho(x)$ of the LMNC model given by (68)(a) is equal to ρ_e up to an error of order $\mathcal{O}(\varepsilon)$,

ii) the solution $u(x)$ and $h(x)$ of the LMNC model given by (68)(b,c) is equal to the solution $u(x)$ and $h(x)$ of the Boussinesq model given by (84)(a,b) up to an error of order $\mathcal{O}(\epsilon)$,

iii) the solution $\Pi(x)$ of the LMNC model given by (68)(d) is equal to the solution $\Pi(x)$ of the Boussinesq model given by (84)(c) up to an error of order $\mathcal{O}[\min(\epsilon^2, \epsilon^{q+1})]$.

It is important to recall that $q \in \mathbb{R}_*^+$ and, thus, that $q \neq 0$. We can understand this rectriction by noting that the pressure $\Pi(x)$ given by (85)(c) can be written with

$$\Pi(x) = \Pi_{\text{th}}(x) + \Pi_{\text{hyd}}(x) \tag{86}$$

where

$$\begin{cases} \Pi_{\text{th}}(x) = -\epsilon \rho_e g \dfrac{L^2 - x^2}{2L}, & \text{(a)} \\[2ex] \Pi_{\text{hyd}}(x) = P_0 + \rho_e g (L - x). & \text{(b)} \end{cases} \tag{87}$$

If $q = 0$ was admissible, the error deduced from *Point iii)* of Lemma 4.5 would be of the same order than the order of the pressure Π_{th} given by (87)(a). At last, let us note that we deduce from (82) and (84) that

$$||\alpha_{h_e}(h - h_e)||_{L^\infty} = \epsilon. \tag{88}$$

Thus, Condition (33)(a) is a consequence of Condition (32)(c) when the EOS is the stiffened gas EOS (36).

Proof of Lemma 4.4: Relation (84)(a) is a direct consequence of (79)(a) and (80)(b). In the same way, by using (84)(a), (84)(b) is a direct consequence of (79)(c) and (80)(a). We obtain (84)(c) by taking into account (79)(b), (81) and (84)(a,b).□

Proof of Lemma 4.5: The velocity $u(x)$ (solution of the LMNC model) given by (68)(b) can be written with

$$u(x) = u_e \cdot \left(1 + \epsilon \frac{x}{L}\right)$$

where ϵ is defined with (82) and where u_e is given by (48)(b), which proves that $u(x) = u_e + \mathcal{O}(\epsilon)$. The internal enthalpy $h(x)$ (solution of the LMNC model) given by (68)(c) is identical to the internal enthalpy $h(x)$ (solution of the Boussinesq model) given by (84)(b). To establish the consistency between the pressure $\Pi(x)$ (solution of the LMNC model) given by (68)(d) and the pressure $\Pi(x)$ (solution of the Boussinesq model) given by (84)(c), we have to compute the asymptotic development of the pressure $\Pi(x)$ given by (68)(d) with respect to the parameter ϵ. We deduce from (46), (82) and (85)(b) that the charge loss due to viscous effects π_μ is such that

$$\begin{aligned}
\pi_\mu(x) &= \frac{4}{3} \cdot \frac{(\gamma-1)\Phi_0}{\gamma(P_0+P_\infty)} \left\{[h(x)-h(L)]\partial_h \mu[h(L),P_0] + \mathcal{O}[(h(x)-h(L))^2]\right\} \\
&= \frac{4}{3} \cdot \frac{\gamma-1}{\gamma} \cdot \frac{\Phi_0 L}{u_e(P_0+P_\infty)} \left\{ \frac{\epsilon}{\alpha_{h_e}} \cdot \left(\frac{x}{L}-1\right) \cdot \frac{u_e}{L} \partial_h \mu[h(L),P_0] + \mathcal{O}(\epsilon^2) \right\} \\
&= \frac{4}{3} \cdot \left\{ \frac{\epsilon^2}{\alpha_{h_e}} \cdot \left(\frac{x}{L}-1\right) \cdot \frac{u_e}{L} \partial_h \mu[h(L),P_0] + \mathcal{O}(\epsilon^3) \right\}
\end{aligned}$$

which allows to obtain that $||\pi_\mu||_{L^\infty} = \mathcal{O}(\epsilon^2)$. Then, we deduce from (68)(d) and by taking into account (48)(a) that

$$\begin{aligned}
\Pi(x) &= P_0 - \frac{\gamma-1}{\gamma} \cdot \frac{D_e}{P_0+P_\infty} \cdot \Phi_0 L \left(\frac{x}{L}-1\right) - \frac{gD_e}{\Phi_0} \cdot \frac{\gamma(P_0+P_\infty)}{\gamma-1} \ln\left[\frac{1+\epsilon\frac{x}{L}}{1+\epsilon}\right] + \mathcal{O}(\epsilon^2) \\
&= P_0 - \epsilon\rho_e u_e^2 \left(\frac{x}{L}-1\right) - \frac{\rho_e g L}{\epsilon}\left[\epsilon\left(\frac{x}{L}-1\right) - \frac{\epsilon^2}{2}\left(\left(\frac{x}{L}\right)^2-1\right) + \mathcal{O}(\epsilon^3)\right] + \mathcal{O}(\epsilon^2) \\
&= P_0 - \epsilon\rho_e u_e^2 \left(\frac{x}{L}-1\right) - \rho_e g L\left[\left(\frac{x}{L}-1\right) - \frac{\epsilon}{2}\left(\left(\frac{x}{L}\right)^2-1\right) + \mathcal{O}(\epsilon^2)\right] \\
&= P_0 - \left(\epsilon\rho_e u_e^2 + \rho_e g L\right)\left(\frac{x}{L}-1\right) + \frac{\rho_e g L\epsilon}{2}\left(\left(\frac{x}{L}\right)^2-1\right) + \mathcal{O}(\epsilon^2) \\
&= P_0 - \rho_e g L\left(\epsilon F_r + 1\right)\left(\frac{x}{L}-1\right) + \frac{\rho_e g L\epsilon}{2}\left(\left(\frac{x}{L}\right)^2-1\right) + \mathcal{O}(\epsilon^2) \\
&= P_0 - \rho_e g L\left(\frac{x}{L}-1\right) + \frac{\rho_e g L\epsilon}{2}\left(\left(\frac{x}{L}\right)^2-1\right) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon F_r) \\
&= P_0 - \rho_e g L\left(\frac{x}{L}-1\right)\left[1 - \frac{\epsilon}{2}\left(\frac{x}{L}+1\right)\right] + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon F_r)
\end{aligned}$$

which corresponds to Formula (85)(c) up to an error of order $\mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon F_r)$.□

4.5. **On regular and singular charge loss**

Let us suppose that we take into account regular and singular charge loss in the monodimensional stationary LMNC model. Here, a regular charge loss models the friction of the fluid on regular walls inside the nuclear core[11] and is taking into account by substracting $\zeta(\rho, u, P_0)$ in the right hand side of (40)(b) where ζ is a given positive function (supposed to be regular). A singular charge loss models the friction of the fluid on singular walls inside the nuclear core[12] which are located at $x = X$ ($X \in [0, L]$) and is taken into account by substracting the singular charge loss $\chi(\rho, u, p)\delta_X(x)$ in the right hand side of (40)(b) where χ is a given positive function (supposed to be regular), $\delta_X(x)$ being the Dirac distribution located at $x = X$. The functions $\zeta(\rho, u, p)$ and $\chi(\rho, u, p)$ depend on the type of nuclear core.

In that case, the pressure $\Pi(x)$ is solution of (50) minus the regular term $\zeta(\rho, u, P_0)$ and minus the singular term $\chi(\rho, u, P_0)\delta_X(x)$ that is to say

$$\begin{aligned} \partial_x \Pi \quad &= \quad -\frac{\gamma - 1}{\gamma(P_0 + P_\infty)} D_e \Phi(x) + \partial_x \pi_\mu(x) - \frac{g}{h_e - q} \cdot \frac{\gamma(P_0 + P_\infty)}{\gamma - 1} \cdot \frac{1}{1 + \dfrac{\Psi(x)}{(h_e - q)\, D_e}} \\ &\quad -\zeta(\rho, u, P_0) - \chi(\rho, u, P_0)\delta_X(x). \end{aligned} \tag{89}$$

As a consequence, the pressure $\Pi(x)$ is given by

$$\begin{aligned} \Pi(x) \quad &= \quad P_0 - \frac{\gamma - 1}{\gamma} \cdot \frac{D_e}{P_0 + P_\infty} \cdot [\Psi(x) - \Psi(L)] - \frac{g}{h_e - q} \cdot \frac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot [\varphi(x) - \varphi(L)] \\ &\quad +\pi_\mu(x) + \pi_r(x) + \pi_s(x) \end{aligned} \tag{90}$$

where π_μ is the charge loss due to viscous effects given by (46), where π_r is the regular charge loss given by

$$\pi_r(x) = \int_x^L \zeta[\rho(y), u(y), P_0] dy \tag{91}$$

and where π_s is the singular charge loss given by

$$\left\{ \begin{aligned} \pi_s(x) \quad &= \quad \chi[\rho(X), u(X), P_0] \quad \text{if} \quad x \in [0, X[, \\ &= \quad 0 \qquad\qquad\qquad\qquad \text{if} \quad x \in]X, L], \end{aligned} \right. \tag{92}$$

$\rho(X)$ and $u(X)$ are given by (see (44)(a,b))

$$\left\{ \begin{aligned} \rho(X) &= \frac{\gamma(P_0 + P_\infty)}{\gamma - 1} \cdot \frac{D_e}{(h_e - q)D_e + \Psi(X)}, \\ u(X) &= \frac{\gamma - 1}{\gamma(P_0 + P_\infty)} \cdot [(h_e - q)D_e + \Psi(X)] \end{aligned} \right.$$

with $\Psi(X) = \int_0^X \Phi(y) dy$ (see (45)(a)). We deduce from (90) with $x = 0$ that:

$$\Pi_e(D_e, h_e, P_0) = P_0 + \frac{\gamma - 1}{\gamma} \cdot \frac{D_e}{P_0 + P_\infty} \cdot \Psi(L) + \frac{g}{h_e - q} \cdot \frac{\gamma}{\gamma - 1} \cdot (P_0 + P_\infty) \cdot \varphi(L) + \pi_\mu(0) + \pi_r(0) + \pi_s(0),$$

and we have $\pi_s(0) = \chi[\rho(X), u(X), P_0]$. This formula gives the hydraulic power (here, we suppose that $D_e > 0$)

$$\mathcal{P}_{\text{hyd}}(D_e, h_e, P_0) := [\Pi_e(D_e, h_e, P_0) - P_0] \frac{D_e S_e}{\rho_e}$$

[11]These regular walls are for example the walls of the fuel rods.

[12]These singular walls are due to a technological device (as a mixing grid or a support grid) which is geometrically complex and located in a small volume.

of the main coolant pump (S_e is the inlet area in the nuclear core) that is to say

$$\mathcal{P}_{\text{hyd}}(D_e, h_e, P_0) = \left\{ \left[\frac{\gamma - 1}{\gamma(P_0 + P_\infty)} \right]^2 (h_e - q)\, D_e \Psi(L) + g\varphi(L) + \eta(D_e, h_e, P_0) \right\} D_e S_e \tag{93}$$

with

$$\eta(D_e, h_e, P_0) := \frac{\gamma - 1}{\gamma} \cdot \frac{h_e - q}{P_0 + P_\infty} \cdot [\pi_\mu(0) + \pi_r(0) + \pi_s(0)].$$

The first term, the second term and the third term in the right hand side of (93) are the part of the hydraulic power directly linked to the thermal transfers, to the gravity field and to the charge loss. Let us note that in a nuclear core, we can neglect $\pi_\mu(0)$ in $\eta(D_e, h_e, P_0)$.

The previous calculus show that it is easy to take into account a regular and a singular charge loss in the LMNC model. Let us remark that it is immediate to extend Formula (92) when $\chi(\rho, u, p)\delta_X(x)$ is replaced by $\sum_{i=1}^{i_{\max}} \chi_i(\rho, u, p)\delta_{X_i}(x)$ where the subscript $i \in \{1, \ldots, i_{\max}\}$ defines singular walls located at $X_i \in [0, L]$ (we may have $\chi_i \neq \chi_j$ when $i \neq j$).

We wish to underline that the main property used to obtain (91) and (92), and also to obtain (46), is that the density $\rho(x)$ and the velocity $u(x)$ are obtained in the monodimensional case *independently* of the pressure $\Pi(x)$ and that the thermodynamic pressure $p(x)$ is a given constant P_0. Of course, this is not at all the case for the monodimensional compressible Navier-Stokes system which makes impossible to obtain any analytical formula similar to (46), (91) and (92). This property implies also that it is easy to take into account a regular or a singular charge loss at the monodimensional *discrete* level in the LMNC model, with or without phase change phenomenon [1, 2].

At last, let us note that a necessary condition to be able to introduce from a modeling point of view regular or singular charge loss in the LMNC model through the regular term $\zeta(\rho, u, P_0)$ and the singular term $\chi(\rho, u, P_0)\delta_X(x)$ in (89) is that

$$\mathcal{O}\left(\frac{\zeta_* L}{P_0}\right) \leq M^2 \qquad \text{and} \qquad \mathcal{O}\left(\frac{\chi_*}{P_0}\right) \leq M^2$$

where ζ_* and χ_* are an order of magnitude respectively of $\zeta(\rho, u, P_0)$ and of $\chi(\rho, u, P_0)$. In particular, this is the case when $\zeta(\rho, u, p) = \frac{\eta_1}{2L}\rho u|u|$ and when $\chi(\rho, u, p) = \frac{\eta_2}{2}\rho u|u|$ where η_k is a positive modeling parameter such that $\mathcal{O}(\eta_k) \leq 1$.

4.6. **A remark on the Ledinegg instability**

The Ledinegg instability is an hydrodynamic instability which may exist only when $\partial_{D_e}\Pi_e(D_e, h_e, P_0) < 0$ [3,11]. When $\pi_r = 0$ and $\pi_s = 0$, we obtain that $\partial_{D_e}\Pi_e(D_e, h_e, P_0) > 0$ (see Inequality (63) in Lemma 4.3). As a consequence, in the studied monophasic case, the Ledinegg instability cannot exist when we do not take into account regular and singular charge loss. But, when we take into account regular or singular charge loss, the sign of $\partial_{D_e}\pi_r(0)$ or the sign of $\partial_{D_e}\pi_s(0)$ may be negative (it will depend on the functions ζ and χ which respectively define π_r and π_s), which may induce $\partial_{D_e}\Pi_e(D_e, h_e, P_0) < 0$. Nevertheless, when ζ and χ are functions of the type $\eta\rho u|u|$ (η being a positive constant), by noting that $\eta\rho u|u| = \frac{\eta(\gamma-1)}{\gamma(P_0 + P_\infty)} \cdot D_e \cdot [(h_e - q)D_e + \Psi(x)]$, we obtain that $\partial_{D_e}\pi_r(0) > 0$ and $\partial_{D_e}\pi_s(0) > 0$, that is to say $\partial_{D_e}\Pi_e(D_e, h_e, P_0) > 0$: again, the Ledinegg instability cannot exist.

5. Conclusion

By filtering out the acoustic waves in a compressible Navier-Stokes system, we proposed a low Mach number model – named *Low Mach Nuclear Core* (LMNC) model – that is able to model a nuclear core when the flow is at low Mach number. The LMNC model is usefull, firstly, because it has only one time scale (the material time scale) and because the thermodynamic pressure does not depend on the space variable: a direct consequence of these properties is that it is possible to propose in the monodimensional case robust and accurate *explicit* schemes which are not expensive from a CPU point of view, with or without phase change phenomenon [1,2]. The LMNC model is usefull, secondly, because it is possible to construct monodimensional analytical solutions when the equation of state is a stiffened gas equation in the stationary single-phase case but also in the unstationary two-phase case with phase change phenomenon, and with or without regular and singular charge loss [1, 2].

Let us note that we do not propose any existence, uniqueness and regularity result in the present paper. More explicitly, all quantities are supposed to be enough regular to be able to apply any differential operators. Nevertheless, this point will have to be studied carefully in a future work. For example, it will be interesting to study the influence of the phase change phenomenon on the regularity of the solution of the LMNC model since phase change phenomenon induces a loss of regularity in the solution. A second example concerns the influence of the regularity of the power density $\Phi(t, x)$ on the regularity of the solution of the LMNC model, *e.g.* when $\Phi(t, x)$ is a time Heaviside function modeling an emergency stop or when $\Phi(t, x)$ is a time and spatial Dirac distribution modeling a power excursion due to a reactivity insertion (moreover, in such cases the proposed formal derivation of the LMNC model may not be valid). At last, we expect to be able to study the Ledinegg instability [3, 11] with the LMNC model when phase change phenomenon is taken into account.

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