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A PROBLEM OF OPTIMAL CONTROL WITH FREE INITIAL STATE.

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Abstract. We are studying an optimal control problem with free initial condition. The initial state of the optimized system is not known exactly, information on initial state is exhausted by inclusions $x_0 \in X_0$. Accessible controls for optimization of continuous dynamic system are discrete controls defined on quantized axes. The method presented is based on the concepts and operations of the adaptive method [9] of linear programming. The results are illustrated by a fourth order problem, efficiency estimates of proposed methods are given.

1. Introduction

Problems of optimal control (OC) have been intensively investigated in the world literature for over forty years. During this period, series of fundamental results have been obtained, among which should be noted maximum principle [1] and dynamic programming [2]. For many of the problems of the optimal control theory (OCT) adequate solutions are found [4, 5, 7, 8]. Results of the theory were taken up in various fields of science, engineering, and economics.

The aim of this paper is to solve a problem of optimal with free initial state. The problem has the following sense, the initial state of the optimized system is not known exactly, information on initial state is exhausted by inclusion $x_0 \in X_0$, by analogy with the theory of filtration, we say that the set X_0 is a priori distribution of the initial state of the control system.

The paper has the following structure: In section 2, the canonical OC problem is formulated. And the definition of support is introduced. Primal and dual ways of its dynamical identification are given. In section 3, the value of suboptimality is calculated. For this result is deduced, Optimality and ε-Optimality criteria are formulated in section 4. In section 5, Numerical algorithm for solving the problem ; three procedure can be distinguished in iterative process: change of control, change of a support, final procedure.

2. Statement of the problem

Let us consider the optimal control problem for a linear system at the time interval $T = [0, t^*]$:

$$c'x(t^*) \to max \tag{1}$$

$$\dot{x} = Ax + bu, x(0) = z \in X_0 = \{z \in \mathbb{R}^n, \quad Gz = \gamma, \quad d_* \le z \le d^*\}, \tag{2}$$

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$$Hx(t^*) = g, \tag{3}$$

$$f_* \leq u(t) \leq f^*, \quad t \in T = [0, t^*]. \tag{4}$$

Here $x \in \mathbb{R}^n$ is a state of control system (2); $u(.) = (u(t), t \in t), T = [0, t^*]$, is a piecewise continuous function; $A \in \mathbb{R}^{n \times n}$; $b, c \in \mathbb{R}^n$; $g \in \mathbb{R}^{m \times n}, rankH = m \leq n; f_*, f^*$ are scalars; $d_* = (d_{*j}, j \in J), d^* = d^*(J) = (d^*_j, j \in J)$ are $n-$vectors; $G \in \mathbb{R}^{l \times n}, rankG = l \leq n, \gamma \in \mathbb{R}^l, I = \{1, ..., m\}, J = \{1, ..., n\}, L = \{1, ..., l\}$ are sets of indices.

By using the Cauchy formula, we obtain the solution of the system (2) :

$$x(t) = F(t)(z + \int_0^t F^{-1}(\vartheta) bu(\vartheta) d\vartheta), t \in T, \tag{5}$$

where $F(t) = e^{At}, t \in T = [0, t^*]$ is defined by the relations: $\left\{ \begin{array}{ll} \dot{F}(t) = AF(t) & , \\ F(0) = I_n & . \end{array} \right.$

Substituting (5) into $(1) - (4)$, we obtain the following equivalent formulation of the problem:

$$\tilde{c}' z + \int_0^{t^*} c(t) u(t) dt \longrightarrow max, \tag{6}$$

$$D(I, J) z + \int_0^{t^*} \varphi(t) u(t) dt = g, \tag{7}$$

$$G(L, J) z = \gamma, \ d_* \leq z \leq d^*, \tag{8}$$

$$f_* \leq u(t) \leq f^*, t \in T, \tag{9}$$

where $\tilde{c}' = c' F(t^*), \ c(t) = c' F(t^*) F^{-1}(t) b, \ D(I, J) = HF(t^*), \ \varphi(t) = HF(t^*) F^{-1}(t) b.$

A pair $v = (z, u(.))$ formed of an $n-$vector z and a piecewise continuous function $u(.)$ is called a generalized control.

A generalized control $v = (z, u(.))$ is said to be an admissible control if it satisfied the constraints (2)-(4).

An admissible control $v^0 = (z^0, u^0(.))$ is said to be an optimal open-loop control if a control criterion reaches its maximal value $J(v^0) = \max_v J(v)$.

For a given $\varepsilon \geq 0$, an $\varepsilon-$optimal control $v^\varepsilon = (z^\varepsilon, u^\varepsilon(.))$ is defined by the inequality $J(v^0) - J(v^\varepsilon) \leq \varepsilon$.

Choose an arbitrary subset $T_B \subset T$ of $k \leq m$ elements and an arbitrary subset $J_B \subset J$ of $m + l - k$ elements. Form the matrix

$$P_B = \begin{pmatrix} D(I, J_B) & \varphi(t), t \in T_B \\ & \\ G(L, J_B) & 0 \end{pmatrix} \tag{10}$$

A set $S_B = \{T_B, J_B\}$ is said to be a support of problem $(1) - (4)$ if $det P_B \neq 0$.

A pair $\{v, S_B\}$ of an admissible control $v = (z, u(.))$ and a support S_B is said to be a support control. A support control $\{v, S_B\}$ is said to be primally not degenerate if $d_{*j} < z_j < d^*_j, j \in J_B, f_* < u(t) < f^*, t \in T_B$.

Let us consider another admissible control $\overline{v} = (\overline{z}, \overline{u}(.)) = v + \Delta v$, where
$\overline{z} = z + \Delta z, \overline{u}(t) = u(t) + \Delta u(t), t \in T$, and let us calculate the increment of the cost functional
$\Delta J(v) = J(\overline{v}) - J(v) = \tilde{c}' \Delta z + \int_{t \in T} c(t) \Delta u(t)$.
Since
$D(I, J) \Delta z + \int_{t \in T} \varphi(t) \Delta u(t) = 0$, and $G(L, J) \Delta z = 0$,
then the increment of the functional equals:
$\Delta J(v) = (\tilde{c}' - \nu' \begin{pmatrix} D(I, J) \\ G(L, J) \end{pmatrix}) \Delta z + \int_{t \in T} (\varphi(t) - \nu' c(t)) \Delta u(t)$

where $\nu = \begin{pmatrix} \nu_u \\ \nu_z \end{pmatrix} \in R^{m+l}, \nu_u \in R^m, \nu_z \in R^l$ is a function of the Lagrange multipliers called potentials, is calculated as a solution to the equation: $\nu' = q_B' Q$, where $Q = P_B^{-1}, q_B = (\tilde{c}_j, j \in J_B, c(t), t \in T_B)$. Introduce an n-vector of estimates $\Delta' = \nu' \begin{pmatrix} D(I,J) \\ G(L,J) \end{pmatrix} - \tilde{c}'$, and a function of cocontrol $\Delta(.) = (\Delta(t) = \nu_u' \varphi(t) - c(t), t \in T)$. By using these notions, the value of the cost of functional increment takes the form:

$$\Delta J(v) = \Delta' \Delta z - \int_{t \in T} \Delta(t) \Delta u(t). \tag{11}$$

A support control $\{v, S_B\}$ is dually not degenerate if $\Delta(t) \neq 0, t \in T_H, \Delta_j \neq 0, j \in J_H$, where $T_H = T/T_B,\ J_H = J/J_B$.

3. Calculation of the value of suboptimality

The new control $\overline{v}(t)$ is admissible, if it satisfies the constraints:

$$d_* - z \leq \Delta z \leq d^* - z;\ f_* - u(t) \leq \Delta u(t) \leq f^* - u(t), t \in T. \tag{12}$$

$\beta = \beta(v, S_B) = \sum_{j \in J_H^+} \Delta_j (z_j - d_{*j}) + \sum_{j \in J_H^-} \Delta_j (z_j - d_j^*) + \int_{t \in T^+} \Delta(t)(u(t) - f_*) + \int_{t \in T^-} \Delta(t)(u(t) - f^*)$
where
$T^+ = \{t \in T_H, \Delta(t) > 0\},\ T^- = \{t \in T_H, \Delta(t) < 0\},\ J_H^+ = \{j \in J_H, \Delta_j > 0\},\ J_H^- = \{j \in J_H, \Delta_j < 0\}$.
The number $\beta(v, S_B)$ is called a value of suboptimality of the support control $\{v, S_B\}$.
From there, $J(\overline{v}) - J(v) \leq \beta(v, S_B)$. Of this last inequality, the following result is deduced:

4. Optimality and ε-Optimality criterion [5]

Theorem 4.1. *Following relations:*

$$\begin{cases} u(t) = f_*, & \text{if } \Delta(t) > 0 \\ u(t) = f^*, & \text{if } \Delta(t) < 0 \\ f_* \leq u(t) \leq f^*, & \text{if } \Delta(t) = 0,\ t \in T \\ z_j = d_{*j}, & \text{if } \Delta_j > 0 \\ z_j = d_j^*, & \text{if } \Delta_j < 0 \\ d_{*j} \leq z_j \leq d_j^*, & \text{if } \Delta_j = 0,\ j \in J. \end{cases}$$

are sufficient, and in the cases of non-degeneracy, they are necessary for the optimality of support control $\{v, S_B\}$.

Theorem 4.2. *For any $\varepsilon \geq 0$, the admissible control v is $\varepsilon-$optimal if and only if there exists a support S_B so that $\beta(v, S_B) \leq \varepsilon$.*

5. Numerical algorithm for solving the problem

Supposing $\varepsilon > 0$ is a given number and $\{v, S_B\}$ is a known support control that does not satisfy optimality and $\varepsilon-$ optimality criterion. The method suggested is iterative, its aim is to construct an $\varepsilon-$ solution of problem $(1)-(4)$. As the support will be changing during the iterations together with an admissible control it is natural to consider them as a pair. The iteration of the method is to change initial support control $\{v, S_B\}$ for the "new" $\{\overline{v}, \overline{S}_B\}$ so that $\beta(v, S_B) \leq \beta(\overline{v}, \overline{S}_B)$. Three procedures can be distinguished in iterative process:

1. Change of an admissible control $v \to \overline{v}$.
2. Change of support $S_B \to \overline{S}_B$.
3. Final procedure.

5.1. **Change of control.**

Let us consider $\alpha_1 > 0, \alpha_2 > 0, h > 0, \mu > 0$ parameters of the method, and we set up the following sets: $J_0 = \{j \in J : |\Delta_j| \le \alpha_2\}, J_* = \{j \in J : |\Delta_j| > \alpha_2\},\ T_0 = \{t \in T : |\Delta(t)| \le \alpha_1\}, T_* = \{t \in T : |\Delta(t)| > \alpha_1\}$, where $|J_0| = K$, and subdivide T_0 into subintervals $[\tau_i, \tau^i[, i = \overline{1,N}; \tau_i < \tau^i, T_0 = \bigcup_{i=1}^{N}[\tau_i, \tau^i], \tau^i - \tau_i \le h, T_B \subset \{\tau_i, i = \overline{1,N}\}, u(t) = u_i = const, t \in [\tau_i, \tau^i[, i = \overline{1,N}$.
A new admissible control $\overline{v} = (\overline{z}, \overline{u}(t), t \in T)$ so that:

$$\begin{cases} \overline{z}_j = z_j + \kappa\Delta z_j, & j \in J \\ \overline{u}(t) = u(t) + \theta\Delta u(t), & t \in T, \end{cases} \tag{13}$$

Here
$$\Delta z_j = \begin{cases} d_j^* - z_j, & if \Delta_j < -\alpha_2 \\ d_{*j} - z_j, & if \Delta_j > \alpha_2, j \in J_* \\ 0, & if \Delta_j = 0, j \in J_0, \end{cases} \qquad \Delta u(t) = \begin{cases} f^* - u(t), & if \Delta(t) < -\alpha_1 \\ f_* - u(t), & if \Delta(t) > \alpha_1, t \in T_* \\ u_i = const, & if t \in [\tau_i, \tau^i[, i = \overline{1,N}, t \in T_0. \end{cases}$$
We introduce the parameter vector: $l_i = \theta u_i, i = \overline{1,N}, h_j = \kappa\Delta z_j, j \in J_0, h_{K+1} = \kappa$, and define these quantities:
$g_i = -\int_{\tau_i}^{\tau^i} \Delta(t)dt, i = \overline{1,N}, g_{N+1} = -\int_{T_*} \Delta(t)\Delta u(t)dt$,
$\phi_i = -\int_{\tau_i}^{\tau^i} \varphi(t)dt, i = \overline{1,N}, \phi_{N+1} = -\int_{T_*} \varphi(t)\Delta u(t),\ q_j = -\Delta_j, j \in J_0, q_{K+1} = \sum_{j\in J_*} -\Delta_j\Delta z_j, j \in J_*$, $D_j = D(I,j), j \in J_0, D_{K+1} = \sum_{j\in J_*} D(I,j)\Delta z_j$,
$f_{*i} = f_* - u_i, f_i^* = f^* - u_i, i = \overline{1,N}, f_{*N+1} = 0, f_{N+1}^* = 1,\ d_{*j} = d_* - z_j, d_j^* = d^* - z_j, j = \overline{1,K}, d_{*K+1} = 0, d_{K+1}^* = 1$.
In order to find $(h_j, l_i), j = \overline{1,K+1}, i = \overline{1,N+1}$, we formulate the mathematical programming problem:

$$\begin{cases} \Delta J(v) = \sum_{j\in J_0\cup\{K+1\}} q_j h_j + \sum_{i=1}^{N+1} g_i l_i \to max_{h_j,l_i}, \\ \sum_{J_0\cup\{K+1\}} D(I,j)h_j + \sum_{i=1}^{N+1} \phi_i l_i = 0, \\ \sum_{j\in J_0\cup\{K+1\}} G(l,j)h_j = 0, \\ f_{*i} \le l_i \le f_i^*, & i = \overline{1,N+1} \\ d_{*j} \ge h_j \ge d_j^*, & j = \overline{1,K+1}. \end{cases} \tag{14}$$

Problem (14) is solved by adaptive method. As a result, we obtain an $\varepsilon-$ optimal support plan $(h_j^\varepsilon, l_i^\varepsilon, \overline{J}_B, \overline{T}_B)$.
The new control $(\overline{z}, \overline{u}(t), t \in T)$ are constructed according to the rules:

$$\overline{z}_j = \begin{cases} z_j + h_{K+1}\Delta z_j, & j \in J_* \\ z_j + h_j, & j \in J_0. \end{cases} \tag{15}$$

Here

$$\overline{u}(t) = \begin{cases} u(t) + l_{N+1}\Delta u(t), & t \in T_* \\ u(t) + l_i, & t \in [\tau_i, \tau^i[, i = \overline{1,N}. \end{cases} \tag{16}$$

It is clear that $J(\overline{v}) \ge J(v)$.

- If $K+1 \notin \overline{J}_B$ and $t_{N+1} \notin \overline{T}_B$, then we put:

$$\tilde{S}_B = \{\tilde{J}_B = \overline{J}_B, \tilde{T}_B = \overline{T}_B\}.$$

- If not, we would have the following cases:
  (1) If $K+1 \notin \overline{J}_B$ and $t_{N+1} \in \overline{T}_B$, we exclude index $N+1$ from the support in the following way: Let be: $\overline{\Delta}(t) = \Delta(t) + \sigma\delta(t)$, where σ is the maximal dual step and $\delta(t)$ the direction . Let us determine i_* so that: $\sigma(t_{i*}) = min\sigma(t_i), t_i \in T_H$, with

$$\sigma(t_i) = \begin{cases} -\Delta(t_i)/\delta(t_i), & \text{if } \Delta(t_i)\times\delta(t_i) \le 0, \delta(t_i) \neq 0 \\ +\infty, & \text{otherwise.} \end{cases} \qquad \delta(t) = \begin{cases} 0, & \text{on } T_B/\{t_{N+1}\}; \\ 1, & \text{if } \overline{u}(t) = f_*; \\ -1, & \text{if } \overline{u}(t) = f^*. \end{cases}$$

$$\delta(t) = \delta'_B P_B^{-1}\phi(t), t \in T.$$

Then a new support is: $\tilde{J}_B = \overline{J}_B; \tilde{T}_B = (\overline{T}_B/\{t_{N+1}\}) \cup \{t_{i*}\}$.

(2) if $K+1 \in \overline{J}_B$ and $t_{N+1} \notin \overline{T}_B$, we exclude index $K+1$ from the support in the following way: Let be: $\overline{\Delta}_j = \Delta_j + \sigma_j\delta_j$, where σ_j is the maximal dual step and δ_j the direction
Let us determine j_* so that: $\sigma_{j*} = min\sigma_j, j \in J_H$,
with $\sigma_j = \begin{cases} -\Delta_j/\delta_j, & \text{if } \Delta_j \times \delta_j \le 0, \delta_j \neq 0 \\ +\infty, & \text{otherwise.} \end{cases} \qquad \delta_j = \begin{cases} 0, & \text{on } J_B/\{K+1\}; \\ 1, & \text{if } \overline{z}_j = d_*; \\ -1, & \text{if } \overline{z}_j = d^*. \end{cases}$

$$\delta_j = \delta'_B P_B^{-1} \begin{pmatrix} D(I,J) \\ G(L,J) \end{pmatrix}, j \in J.$$

Then a new support is: $\tilde{J}_B = (\overline{J}_B/\{K+1\}) \cup \{j_*\}; \tilde{T}_B = \overline{T}_B$.

(3) The last case will be if $K+1 \in \overline{J}_B, t_{N+1} \in \overline{T}_B$,
the new support will be: $\tilde{J}_B = (\overline{J}_B/\{K+1\}) \cup \{j_*\}; \tilde{T}_B = (\overline{T}_B/\{t_{N+1}\}) \cup \{t_{i*}\}$.

At this stage, let us denote the new support $\tilde{S}_B$, construct the support matrix $P(\tilde{S}_B)$ and check that it is not singular. Let us calculate the new suboptimality estimate $\beta(\tilde{v}, \tilde{S}_B)$.

- If $\beta(\tilde{v}, \tilde{S}_B) = 0$, then $\overline{v}$ is an optimal control.
- If $\beta(\tilde{v}, \tilde{S}_B) \le \varepsilon$, then $\overline{v}$ is an $\varepsilon-$ optimal control.
- otherwise, we perform either a new iteration with $\{\overline{v}, \tilde{S}_B\}, \overline{\alpha}_1 < \alpha_1, \overline{\alpha}_2 < \alpha_2, \overline{h} < h$ or the procedure change of support.

5.2. **Change of support.**

let us assume that for the new control $\overline{v}$, we have $\beta(\overline{v}, \tilde{S}_B) > \varepsilon$, then we perform change of support. By using support $\tilde{S}_B$, let us construct the quasi-control $\tilde{v} = (\tilde{z}, \tilde{u}(t), t \in T)$:

$$\tilde{z}_j = \begin{cases} d_{j*} & \text{if } \tilde{\Delta}_j > 0 \\ d_j^* & \text{if } \tilde{\Delta}_j < 0 \\ \in [d_{j*}, d_j^*] & \text{if } \tilde{\Delta}_j = 0,\ j \in J \end{cases} \qquad \tilde{u}(t) = \begin{cases} f_*, & \text{if } \tilde{\Delta}(t) < 0 \\ f^*, & \text{if } \tilde{\Delta}(t) > 0, \\ \in [f_* f^*] & \text{if } \tilde{\Delta}(t) = 0,\ t \in T, \end{cases}$$

where: $\tilde{\Delta}(t) = -\tilde{\psi}'(t)b, t \in T, \tilde{\Delta}' = (\tilde{\Delta}_j, j \in J)' = \nu' \begin{pmatrix} D(I,J) \\ G(L,J) \end{pmatrix} - \tilde{c}'$.

Here, $\tilde{\psi}(t), t \in T$, the solution to the adjoint system corresponding to $\tilde{S}_B$. Let us the quasi trajectory corresponding $\chi = (\chi(t), t \in T), \chi(0) = z \in X_0$ of the system $\dot{\chi} = A\chi + b\tilde{u}, \chi(0) = z \in X_0$.
If $D(I,J)\tilde{z} + \int_0^{t^*} \varphi(t)\tilde{u}(t)dt = g,\ G(L,J)\tilde{z} = \gamma$,
then $\overline{v}$ is optimal control, and if
$D(I,J)\tilde{z} + \int_0^{t^*} \varphi(t)\tilde{u}(t)dt \neq g, G(L,J)\tilde{z} \neq \gamma$,
then construct a vector $\lambda(\tilde{J}_B, \tilde{T}_B)$ as follows:

$$P(\tilde{S}_B)\cdot\lambda(\tilde{J}_B, \tilde{T}_B) = \begin{pmatrix} D(I,J)\tilde{z} + \int_0^{t^*} \tilde{u}(t)dt - g \\ G(L,J)\tilde{z} - \gamma \end{pmatrix} \qquad \lambda(\tilde{J}_B, \tilde{T}_B) = P_B^{-1}(\tilde{S}_B) \begin{pmatrix} D(I,J)\tilde{z} + \int_0^{t^*} \tilde{u}(t)dt - g \\ G(L,J)\tilde{z} - \gamma \end{pmatrix}.$$

Now, we studies the following cases:

- If $\parallel \lambda(\tilde{J}_B, \tilde{T}_B) \parallel = 0$, then the quasi-control $\tilde{v}$ is optimal for the problem $(1)-(4)$.
- If $\parallel \lambda(\tilde{J}_B, \tilde{T}_B) \parallel > \mu$, then let us change a support $\tilde{S}_B$ to $\overline{S}_B$ by dual method.
- if $\parallel \lambda(\tilde{J}_B, \tilde{T}_B) \parallel < \mu$, then we perform final procedure.

Let us calculate the new suboptimality estimate $\beta(\overline{v}, \hat{S}_B)$:
1. If $\beta(\overline{v}, \hat{S}_B) = 0$, then the control $\overline{v}$ is optimal for problem (1)-(4) .
2. If $\beta(\overline{v}, \hat{S}_B) < \varepsilon$, then the control $\overline{v}$ is ε-optimal for problem (1)-(4) .
3. If $\beta(\overline{v}, \hat{S}_B) > \varepsilon$, then we perform the next iteration starting from the support control $\{\overline{v}, \hat{S}_B\}$.

5.3. **final procedure.**

Let us assume that for the new control $\overline{v}$, we have $\beta(\overline{v}, \hat{S}_B) > \varepsilon$. With the use of the support $\overline{S}_B$ we construct a quasicontrol $\widehat{v} = (\widehat{z}, \widehat{u}(t), t \in T)$:

$$\widehat{z}_j = \begin{cases} d_{j*} & \text{if } \Delta_j > 0 \\ d_j^* & \text{if } \Delta_j < 0 \\ \in [d_{j*}, d_j^*] & \text{if } \Delta_j = 0,\ j \in J \end{cases} \qquad \widehat{u}(t) = \begin{cases} f_*, & \text{if } \Delta(t) < 0 \\ f^*, & \text{if } \Delta(t) > 0, \quad t \in T. \end{cases}$$

If $D(I, J)\widehat{z} + \int_0^{t^*} \varphi(t)\widehat{u}(t)dt = g,\ G(L, J)\widehat{z} = \gamma$,
then $\widehat{v}$ is optimal, and if
$D(I, J)\widehat{z} + \int_0^{t^*} \varphi(t)\widehat{u}(t)dt \neq g,\ G(L, J)\widehat{z} \neq \gamma$,
then denote $T^0 = \{t_i,\ i = \overline{1, s}\},\ s = |T_B|$. Here, $t_i,\ i = \overline{1, s}$ are zeroes of the optimal cocontrol $\Delta(t) = 0, t \in T;\ t_0 = 0,\ t_{s+1} = t^*$. Suppose $\dot{\Delta}(t_i) \neq 0,\ i = \overline{1, s}$.
Let us construct the following function:

$$f(\Theta) = \begin{pmatrix} D(I, J_B)z(J_B) + D(I, J_H)z(J_H) + \sum_{i=0}^{s}(\frac{f^*+f_*}{2} - \frac{f^*-f_*}{2} sign\dot{\Delta}(t_i)) \int_{t_i}^{t_{i+1}} \varphi(t)dt - g \\ G(L, J_B)z(J_B) + G(L, J_H)z(J_H) - \gamma \end{pmatrix}$$

where
$z_j = \frac{d_j^*+d_{j*}}{2} - \frac{d_j^*-d_{j*}}{2} sign\Delta_j,\ j \in J_H$.
$\Theta = (t_i,\ i = \overline{1, s}; z_j, j \in J_B)$.
The final procedure consists in finding the solution $\Theta^0 = (t_i^0, i = \overline{1, s}; z_j^0, j \in J_B)$ of the system of $m+l$ nonlinear equations

$$f(\Theta) = 0. \tag{17}$$

We solve this system by the Newton method using as an initial approximation the vector $\Theta^{(0)} = (\overline{t}_i,\ i = \overline{1, s}; \overline{z}_j, j \in J_B)$.
The $(k+1)^{th}$ approximation $\Theta^{(k+1)}$,equal: $\Theta^{(k+1)} = \Theta^{(k)} + \Delta\Theta^{(k)}, \quad \Delta\Theta^{(k)} = -\frac{\partial f^{-1}(\Theta^{(k)})}{\partial \Theta^{(k)}} \cdot f(\Theta^{(k)})$.
Let us compute the Jacobi matrix for equation (17)

$$\frac{\partial f(\Theta^{(k)})}{\partial \Theta^{(k)}} = \begin{pmatrix} D(I, J_B) & (f_* - f^*) sign\dot{\Delta}(t_i^{(k)})\varphi(t_i^{(k)}),\ i = \overline{1, s} \\ & \\ G(L, J_B) & 0 \end{pmatrix}.$$

As $detP_B \neq 0$, we can easily show that

$$det\frac{\partial f(\Theta^{(0)})}{\partial \Theta^{(0)}} \neq 0. \tag{18}$$

For instants $t \in T_B$ there exists a small $\mu > 0$ that for any $\tilde{t}_i \in [t_i - \mu, t_i + \mu], i = \overline{1, s}$, the matrix $(\varphi(\tilde{t}_i), i = \overline{1, s})$ is not degenerate and the matrix $\frac{\partial f(\Theta^{(k)})}{\partial \Theta^{(k)}}$ is also not degenerate, if elements $t_i^{(k)}, i = \overline{1, s}, k = 1, 2, ...$ do not leave the μ-vicinity of $t_i,\ i = \overline{1, s}$. Vector $\Theta^{(k^*)}$ is taken as solution of equation (17) if $\| f(\Theta^{(k^*)}) \| \leq \eta$, for a given $\eta > 0$. So we put $\theta^0 = \theta^{(k^*)}$. The suboptimal control for problem (1)-(4) is computed as

$$z_j^0 = \begin{cases} z_j^0, & j \in J_B \\ \widehat{z}_j, & j \in J_H; \end{cases} \qquad u^0(t) = \frac{f^*+f_*}{2} - \frac{f^*-f_*}{2} sign\dot{\Delta}(t_i^0),\ t \in [t_i^0, t_{i+1}^0[,\ i = \overline{1, s}.$$

If the Newton method does not converge, we decrease the parameter $\alpha_1 > 0, \alpha_2 > 0, h > 0$ and perform the iterative process again.

6. Conclusion

An optimal control problem with free initial condition has been considered. Sufficient and necessary conditions are derived to characterize the optimality of the current solution and an algorithm called the adaptive method is described. This direct method of interior points allows in a finite number of iterations obtaining the approximate or optimal solution. The innovation is the introduction of the final procedure based on the Newton method's, which converges quickly.

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