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SPLINE DISCRETE DIFFERENTIAL FORMS.

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Abstract. The equations of physics are mathematical models consisting of geometric objects and relationships between then. There are many methods to discretize equations, but few maintain the physical nature of objects that constitute them. To respect the geometrical nature elements of physics, it is necessary to change the point of view and using differential geometry, including the numerical study. We propose to construct discrete differential forms using B-splines and a formulation discrete for different operators acting on differential forms. Finally, we apply this theory on the Maxwell equations.

Résumé. Les équations de la Physique sont des modèles matématiques qui mettent en relation des objets géométriques. Il y a beaucoup de méthodes de discrétisation mais peu conservent la nature géométrique des différents objets qui constituent une équation. Afin de respecter cette nature géométrique, il est ncéssaire de changer de point de vue et d'utiliser la géométrie différentielle également pour l'étude numérique. On propose, ici, de construire des formes différentielles discrètes à l'aide des fonctions B-splines ainsi qu'une discrétisation des différents opérateurs présent en géométrie différentielle. On appliquera ensuite cette technique sur les équations de Maxwell.

Introduction

The first who used this point of view to discretize equations is Alain Bossavit [5]. Since, many researches work on it because there are many open problems such as the discretization of Hodge star operator [3,4,10] (an important notion which contains all the metric of our domain), and the interpolation of differential forms [2,5–7]. We propose to construct discrete differential forms using B-splines. This new approach proves to be have many advantages: the higher degree of interpolation is computing by recurrence with de Boor algorithm [9] so its easy to implement them; discrete differential forms verify the same properties than " continuous" differential forms especially they preserve the de Rham diagram; moreover, it appeared that this discretization converge to isogeometric analysis [8,11].

In what follows, we remind what are constructed B-splines and their properties (for this part the reader can be refer to the book of de Boor [9]) then we explain how construct discrete differential forms based on B-splines for uniform. We detail in the $1D$ case and we can expand in the $3D$ case by tensor product using the $1D$ forms. We also adapt the technic of T. Tarhasaari, L. Kettunen and A. Bossavit [4] for discretize the Hodge star operator but using B-splines. Finally, we apply this theory on the Maxwell equations and we test them on uniform mesh with periodic conditions.

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1. A short overview of B-splines

B-splines on a non uniform set of knots $x_0 < x_1 < \cdots < x_{N-1} < x_N$ can be defined recursively. Some kind of boundary conditions need also be defined. In particular natural boundary conditions (vanishing second derivative), Hermite boundary condition (given derivative), or periodic boundary conditions can be used. Let us denote by B_i^α the B-spline of degree α with support in the interval $[x_i, x_{i+\alpha+1}]$. Then B_i^α is defined recursively by

$$B_i^0(x) = \left| \begin{array}{ll} 1 & \text{if } x_i \le x < x_{i+1} \\ 0 & \text{else,} \end{array} \right.$$

and for $\alpha \ge 1$

$$B_i^\alpha(x) = \frac{x - x_i}{x_{i+\alpha} - x_i} B_i^{\alpha-1}(x) + \frac{x_{i+\alpha+1} - x}{x_{i+\alpha+1} - x_{i+1}} B_{i+1}^{\alpha-1}(x). \tag{1}$$

The B-splines verify the following properties:

(1) The B-spline B_i^α is a polynomial of degree α between two consecutive knots,
(2) The B-spline B_i^α is of class $C^{\alpha-1}$,
(3) Partition of unity: for any point x, we have $\sum_i B_i^\alpha(x) = 1$.

We shall also need the recursion formula for the derivatives:

$$B_i^{\alpha\prime}(x) = \alpha \left(\frac{B_i^{\alpha-1}(x)}{x_{i+\alpha} - x_i} - \frac{B_{i+1}^{\alpha-1}(x)}{x_{i+\alpha+1} - x_{i+1}} \right). \tag{2}$$

For details, the reader is refer to the book of de Boor [9]

2. Construction of discrete differential forms based on B-splines

2.1. The 1D case: Uniform periodic mesh

The primal 1D mesh of our periodic domain will be $x_0 < x_1 < \cdots < x_{N-1} < x_N$ and the dual mesh will consist of the points $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$. We shall assume $x_N - x_0$ periodicity, so that all functions will be equal at x_0 and x_N. Then x_N will not be part of the primal mesh and both meshes will have N points.

In the 1D case, we need to define discrete 0-forms and 1-forms on both meshes that will be constructed using basis functions denoted respectively by $w_i^{0,\alpha}$ and $w_i^{1,\alpha}$, for the primal mesh, and $\tilde{w}_{i+1/2}^{0,\alpha}$ and $\tilde{w}_{i+1/2}^{1,\alpha}$ for the dual mesh. Those will be defined using the B-splines of degree α, B_i^α.

Let us start with the discrete 0-form on the primal mesh. Any function $C^0 \in \mathcal{S}_0^\alpha$ writes

$$C^0(x) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x),$$

with the c_j^0 defined by the interpolation conditions $C^0(x_i) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x_i)$ for $0 \le i \le N-1$ which is a linear system that can be written in matrix form $M_\alpha^0 c^0 = \mathbb{C}^0$, with $\mathbb{C}^0 = (C^0(x_0), \ldots, C^0(x_{N-1}))^T$, $c^0 = (c_0^0, \ldots, c_{N-1}^0)^T$ and M_α^0 the square matrix whose components are $m_{ij}^0 = B_j^\alpha(x_i)$ for $i = 0 \ldots N-1$ and $j = -\alpha + 2 \ldots N - \alpha + 2$ with $B_j^\alpha = B_{N+j}^\alpha$ when $j < 0$.

In order to define the 1-forms we shall need the notation

$$D_i^\alpha(x) = \frac{\alpha}{x_{i+\alpha} - x_i} B_i^{\alpha-1}(x),$$

for this function linked to the derivative of the B-spline $B_i^{\alpha-1}$. We can now define the basis functions for the discrete 1-forms by

$$w_i^{1,\alpha}(x) = D_i^\alpha(x)\, dx.$$

The space of linear spline 1-forms $\mathcal{S}_1^\alpha$ will be the vector space generated by these basis functions. Any 1-form $C^1 \in \mathcal{S}_1^\alpha$ writes

$$C^1(x) = \sum_{j=0}^{N-1} c_j^1 D_j^\alpha(x)\, dx,$$

the coefficients c_j^1 being defined by the relations

$$\int_{x_i}^{x_{i+1}} C^1(x) = \sum_{j=0}^{N-1} c_j^1 \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx \qquad \text{for } 0 \le i \le N-1,$$

this also defines a linear system that can be written in matrix form $M_\alpha^1 c^1 = \mathbb{C}^1$, with $\mathbb{C}^1 = (\int_{x_0}^{x_1} C^1(x), \ldots, \int_{x_{N-1}}^{x_N} C^1(x))^T$, $c^1 = (c_0^1, \ldots, c_{N-1}^1)^T$ and M_α^1 the square matrix whose components are $m_{ij}^1 = \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx$.

Denoting by $B_{j+1/2}^\alpha$ the splines whose knots are based on the dual mesh, a discrete p-form on the dual mesh is defined by the same manner than previously.

The discrete Hodge operator: Having defined discrete 0-forms and 1-forms on both grids, we can now define in a natural way the discrete Hodge operators [3, 4], mapping primal 0-forms to dual 1-forms, primal 1-forms to dual 0-forms and the other way round.

As discrete differential forms are defined by their coefficients in the appropriate basis, the discrete Hodge operator should map those coefficients to those on the image basis. Let us start with the discrete Hodge mapping primal 0-forms to dual 1-forms. Given a discrete 0-form on the primal mesh

$$C^0(x) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x),$$

we can apply the continuous Hodge operator to it, as $\star 1 = dx$, we get

$$\star C^0(x) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x)\, dx.$$

Now, as B_j^α are not splines on the dual mesh, this does not define a discrete differential form on the dual mesh. We need an additional projection step. Denoting by πC^0 the projection of $\star C^0$ on the space of discrete differential forms of the same order on the dual mesh, we can write

$$\pi C^0(x) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^1 D_{j+1/2}^\alpha(x)\, dx,$$

with

$$\int_{x_{i+1/2}}^{x_{i+3/2}} \star C^0(x) = \sum_{j=0}^{N-1} \tilde{c}_{j+1/2}^1 \int_{x_{i+1/2}}^{x_{i+3/2}} D_{j+1/2}^\alpha(x)\, dx \qquad \text{for } 0 \le i \le N-1.$$

Now defining $\tilde{S}^1$ the matrix whose i, j coefficient is $\int_{x_{i+1/2}}^{x_{i+3/2}} B_j^\alpha(x)\, dx$, this relation becomes in matrix form

$$\tilde{S}^1 c^0 = \tilde{M}_\alpha^1 \tilde{c}^1,$$

so that the discrete Hodge operator mapping c^0 to $\tilde{c}^1$ is

$$(\tilde{M}_\alpha^1)^{-1} \tilde{S}^1 \qquad \text{with } \tilde{S}_{i,j}^1 = \int_{x_{i+1/2}}^{x_{i+3/2}} B_j^\alpha(x)\, dx.$$

2.2. The 3D case

We are now going to define the 3D discrete differential forms on a periodic cartesian grid, which will be needed for Maxwell's equations, by tensor product using the 1D form. This procedure can be generalized in a natural way to any number of dimensions.

The set of 3D discrete differential forms will be defined as the span of the following basis functions:

- The basis functions for the 0-forms are

$$^0w^{\alpha}_{i,j,k}(x,y,z) = B^{\alpha}_i(x)B^{\alpha}_j(y)B^{\alpha}_k(z).$$

- The basis functions for the 1-forms are

$$^1\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) = D^{\alpha}_i(x)B^{\alpha}_j(y)B^{\alpha}_k(z)\,dx,\;\; ^1\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) = B^{\alpha}_i(x)D^{\alpha}_j(y)B^{\alpha}_k(z)\,dy,\;\; ^1\mathbf{w}^{\alpha,z}i,j,k(x,y,z) = B^{\alpha}_i(x)B^{\alpha}_j(y)D^{\alpha}_k(z)\,dz.$$

- The basis functions for the 2-forms are

$$^2\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) = B^{\alpha}_i(x)D^{\alpha}_j(y)D^{\alpha}_k(z)\,dy\wedge dz,\;\; ^2\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) = D^{\alpha}_i(x)B^{\alpha}_j(y)D^{\alpha}_k(z)\,dz\wedge dx,\;\; ^2\mathbf{w}^{\alpha,z}_{i,j,k}(x,y,z) = D^{\alpha}_i(x)D^{\alpha}_j(y)B^{\alpha}_k(z)\,dx\wedge dy.$$

- The basis functions for the 3-forms are

$$^3w^{\alpha}_{i,j,k}(x,y) = D^{\alpha}_i(x)D^{\alpha}_j(y)D^{\alpha}_k(z)\,dx\wedge dy\wedge dz.$$

3. Application to the Maxwell equations

3.1. General case

The 3D Maxwell equations can be written in terms of differential forms [5] in the following way

$$-\partial_t\,{}^2\mathbf{D} + d\,{}^1\mathbf{H} = {}^2\mathbf{J}, \tag{3}$$

$$\partial_t\,{}^2\mathbf{B} + d\,{}^1\mathbf{E} = 0, \tag{4}$$

$$d\,{}^2\mathbf{D} = {}^3\rho, \tag{5}$$

$$d\,{}^2\mathbf{B} = 0, \tag{6}$$

where ${}^2\mathbf{D}$, ${}^2\mathbf{B}$, ${}^2\mathbf{J}$ are 2-forms, ${}^1\mathbf{E}$, ${}^1\mathbf{H}$ are 1-forms and ${}^3\rho$ is a 3-form.

3.2. The 2D case

The mesh we shall consider will be a the cartesian product of a 2D mesh with one cell of length one in the z direction. The primal 2D grid is based on the points $x_i = i/\Delta x$, $y_j = j/\Delta y$, with $(i,j) \in [0,N_x]\times[0,N_y]$ and $N_x\Delta x = N_y\Delta y$. In case of periodic boundary conditions in the x direction, the point x_{N_x} corresponds to x_0 and is omitted from the grid. Periodic boundary conditions in the other direction are dealt with in the same manner.

The points of the dual grid are $x_{i+1/2} = (i+1/2)/\Delta x$, $y_{j+1/2} = (j+1/2)/\Delta y$, with $(i,j,k) \in [0,N_x-1]\times[0,N_y-1]$.

The discrete differential forms on the dual mesh are defined in the same way with their indices on the dual mesh.

Let us now express the relevant components of our electromagnetic field in the appropriate basis of discrete differential forms

$$
\begin{aligned}
{}^2\mathbf{D}_h^x(t,x,y) &= \sum_{i,j} d^x_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,x}_{i+1/2,j+1/2}(x,y), &\quad {}^1\mathbf{E}_h^x(t,x,y) &= \sum_{i,j} e^x_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,x}_{i,j}(x,y),\\
{}^2\mathbf{D}_h^y(t,x,y) &= \sum_{i,j} d^y_{i+1/2,j+1/2}(t)\,{}^2\tilde{\mathbf{w}}^{\alpha,y}_{i+1/2,j+1/2}(x,y) &\quad {}^1\mathbf{E}_h^y(t,x,y) &= \sum_{i,j} e^y_{i,j}(t)\,{}^1\mathbf{w}^{\alpha,y}_{i,j}(x,y),\\
{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\,{}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(x,y), &\quad {}^2\mathbf{B}_h(t,x,y) &= \sum_{i,j} b^z_{i,j}(t)\,{}^2\mathbf{w}^{\alpha,z}_{i,j}(x,y).
\end{aligned}
$$

In order to obtain equations relating the coefficients of this discrete differential forms, we inject these expressions into the Maxwell equations (3)-(6), and take the De Rahm maps for two forms on each facet of the the dual mesh for (3) and (5) and of primal mesh for (4) and (6) .

Let us first compute the exterior derivatives of ${}^1\mathbf{H}_h$ and ${}^1\mathbf{E}_h$:

$$
\begin{aligned}
d\,{}^1\mathbf{H}_h^z(t,x,y) = &\sum_{i,j}(h^z_{i-1/2,j+1/2}(t) - h^z_{i+1/2,j+1/2}(t))D^\alpha_{i+1/2}(x)B^\alpha_{j+1/2}(y)\,dz\wedge dx\\
&+\sum_{i,j}(h^z_{i+1/2,j+1/2}(t) - h^z_{i+1/2,j-1/2}(t))B^\alpha_{i+1/2}(x)D^\alpha_{j+1/2}(y)\,dy\wedge dz.
\end{aligned}
\tag{7}
$$

and

$$
d\,{}^1\mathbf{E}_h^x(t,x,y) = -\sum_{i,j}(e^x_{i,j}(t) - e^x_{i,j-1}(t))D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy,\; d\,{}^1\mathbf{E}_h^y(t,x,y) = \sum_{i,j}(e^y_{i,j}(t) - e^y_{i-1,j}(t))D^\alpha_i(x)D^\alpha_j(y)\,dx\wedge dy, \tag{8}
$$

Ampere's law (3), without current, for the first two components can be written

$$
\partial_t\,{}^2\mathbf{D}^x + \partial_t\,{}^2\mathbf{D}^y - d\,{}^1\mathbf{H} = 0.
$$

Then using expression (7) and identifying the components on the basis vectors of the discrete differential forms we get the following relation between the spline coefficients

$$
\begin{aligned}
{d^x_{i+1/2,j+1/2}}'(t) + h^z_{i+1/2,j+1/2}(t) - h^z_{i+1/2,j-1/2}(t) = 0,\\
{d^y_{i+1/2,j+1/2}}'(t) - h^z_{i+1/2,j+1/2}(t) + h^z_{i-1/2,j+1/2}(t) = 0.
\end{aligned}
$$

On the other hand, Faraday's law (4), for the third component can be written

$$
\partial_t\,{}^2\mathbf{B}^z + d\,{}^1\mathbf{E} = 0.
$$

This becomes using (8) and identifying the components on the basis vectors of the discrete differential forms

$$
{b^z_{i,j}}'(t) + (e^y_{i,j}(t) - e^y_{i-1,j}(t)) - (e^x_{i,j}(t) - e^x_{i,j-1}(t)) = 0.
$$

Discrete Hodge operators: Let us denote by

$$
\mathbf{d}^x = ((d^x_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1}, \quad \mathbf{d}^y = ((d^y_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},
$$

the matrices of spline coefficients on the discrete grids.

We now need to define the discrete Hodge operators mapping $\mathbf{d}^x$ to $\mathbf{e}^x$, $\mathbf{d}^y$ to $\mathbf{e}^y$ and $\mathbf{b}^z$ to $\mathbf{h}^z$. The same procedure as for the 1D case will be used. We have

$$^2\mathbf{D}_h^x(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(x) D^\alpha_{j+1/2}(y)\, dy \wedge dz,$$

defines a 2-form to which we can apply the continuous Hodge operator, yielding the one form

$$\star\, ^2\mathbf{D}_h^x(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(x) B^\alpha_{j+1/2}(y)\, dx.$$

We now define the image of $^2\mathbf{D}$ by the discrete Hodge operator as the projection of this 1-form onto the primal grid. Denoting $^1\mathbf{E}_h = \sum_{i,j} e^x_{i,j}(t) D^\alpha_i(x) B^\alpha_j(y)\, dx$ this image. Then we have for any $(k,l) \in [0, N_x - 1] \times [0, N_y - 1]$

$$\int_{x_k}^{x_{k+1}} \star\, ^2\mathbf{D}_h^x(t,x,y_l) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) \int_{x_k}^{x_{k+1}} B^\alpha_{i+1/2}(x)\, dx\, D^\alpha_{j+1/2}(y_l) = \sum_{i,j} e^x_{i,j}(t) \int_{x_k}^{x_{k+1}} D^\alpha_i(x)\, dx\, B^\alpha_j(y_l).$$

4. Numerical results: Test case in $2D$ with periodic boundary conditions

In 2 dimension with periodic boundary condition, let us consider the following solution of Maxwell's equations: the electric field, a 1-form, is given by $^1\mathbf{E} = -k_y \sin(k_x\, x + k_y\, y - \omega\, t) dx - k_x \sin(k_x\, x + k_y\, y - \omega\, t) dy$, and the magnetic field, a 2- form, is given by $^2\mathbf{B} = \omega \cos(k_x\, x + k_y\, y - \omega\, t) dx\, dy$. Then, with help of the Hodge star operator we obtain the formula for the electric displacement field: $^1\mathbf{D} = -k_y \sin(k_x\, x + k_y\, y - \omega\, t) dy - k_x \sin(k_x\, x + k_y\, y - \omega\, t) dx$ and for the magnetizing field: $^0\mathbf{H} = \omega \cos(k_x\, x + k_y\, y - \omega\, t)$. Constants are given by $k_x = \frac{2\pi}{L_x}, k_y = \frac{2\pi}{L_y}$, where L_x and L_y are the length of our domain in x and y respectively and $\omega = \sqrt{k_x^2 + k_y^2}$. We test our code with $L_x = L_y = 1$ and with a time scheme of order 4 and we observe that the order of our scheme is given by the spline order.

degree of spline $\alpha = 1$		
Number of points	Errors L^2 for Dx	conv. order in Dx
10	0.963466687612	
20	0.244961611114	1.97567910882
40	0.0617111800439	1.98895188936
80	0.015439432949	1.99891211506

degree of spline $\alpha = 3$		
Number of points	Errors L^2 for Dx	conv. order in Dx
10	0.0346451065418	
20	0.00221920090229	3.96453940841
40	0.000139584011258	3.99083467768
80	8.73723793614e-06	3.99781260726

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