ESAIM: PROCEEDINGS, **March 2012, Vol. 35, p. 210-215**
Fédération Denis Poisson (Orléans-Tours) et E. Trélat (UPMC), Editors

HYBRID FINITE VOLUME SCHEME FOR A TWO-PHASE FLOW IN HETEROGENEOUS POROUS MEDIA [*]

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Abstract. We propose a finite volume method on general meshes for the numerical simulation of an incompressible and immiscible two-phase flow in porous media. We consider the case that can be written as a coupled system involving a degenerate parabolic convection-diffusion equation for the saturation together with a uniformly elliptic equation for the global pressure. The numerical scheme, which is implicit in time, allows computations in the case of a heterogeneous and anisotropic permeability tensor. The convective fluxes, which are non monotone with respect to the unknown saturation and discontinuous with respect to the space variables, are discretized by means of a special Godunov scheme. We prove the existence of a discrete solution which converges, along a subsequence, to a solution of the continuous problem. We present a number of numerical results in space dimension two, which confirm the efficiency of the numerical method.

Résumé. Nous proposons un schéma de volumes finis hybrides pour la discrétisation d'un problème d'écoulement diphasique incompressible et immiscible en milieu poreux. On suppose que ce problème a la forme d'une équation parabolique dégénérée de convection-diffusion en saturation couplée à une équation uniformément elliptique en pression. On considère un schéma implicite en temps, où les flux diffusifs sont discrétisés par la méthode des volumes finis hybride, ce qui permet de pouvoir traiter le cas d'un tenseur de perméabilité anisotrope et hétérogène sur un maillage très général, et l'on s'appuie sur un schéma de Godunov pour la discrétisation des flux convectifs, qui peuvent être non monotones et discontinus par rapport aux variables spatiales. On démontre l'existence d'une solution discrète, dont une sous-suite converge vers une solution faible du problème continu. On présente finalement des cas test bidimensionnels.

Introduction

We study the simplified *dead-oil* model in so-called global pressure or fractional flow formulation [5]. Let Ω be a polyhedral open bounded connected subset of $\mathbb{R}^d$ and $T > 0$; we consider the following system

$$\mathbf{u} = -\mathbf{K}\left(\lambda(s)\nabla p - \xi(s)\mathbf{g}\right), \qquad -\nabla\cdot\mathbf{u} = q_w + q_o \ \text{ in } \ \Omega\times(0,T), \tag{1}$$

$$\omega\frac{\partial s}{\partial t} + \nabla\cdot(\mathbf{u}f(s) + \gamma(s)\mathbf{K}\mathbf{g}) - \nabla\cdot(\mathbf{K}\nabla\varphi(s)) = q_o \ \text{ in } \ \Omega\times(0,T), \tag{2}$$

together with some initial condition and homogeneous Dirichlet boundary conditions

$$s(\cdot,0) = s_0 \text{ in } \Omega, \quad p = 0 \text{ and } s = 0 \text{ on } \partial\Omega\times(0,T). \tag{3}$$

[*] *This work was supported by the GdR MoMaS (PACEN/CNRS, ANDRA, BRGM, CEA, EdF, IRSN), France.*

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The unknown functions are the global pressure p and the saturation of the oil-phase s. The usual assumption $\lambda_o + \lambda_w \geq \underline{\lambda} > 0$ implies that the first equation is uniformly elliptic in p, whereas the second one is parabolic degenerate with respect to s.

The heterogeneity and anisotropy of porous media is a numerical challenge even when studying the elliptic problem derived from Darcy's law for one-phase problem. Many schemes were proposed and analyzed in the last decades for its discretization. See [9] and [7] for more references and for the detailed description and comparison of those numerical methods. In this paper we propose an implicit fully coupled hybrid finite volume scheme for the problem (1)-(3). The discretization of the diffusion terms is based upon a hybrid finite volume method [6], which allows the tensor $\mathbf{K}$ to be anisotropic and highly variable in space. Remark that heterogeneity also affects the first order terms $\mathbf{K}(\mathbf{x})\mathbf{g}\xi(\cdot)$ and $\mathbf{u}f(\cdot) + \mathbf{K}(\mathbf{x})\mathbf{g}\gamma(\cdot)$, which may be discontinuous with respect to the space variable $\mathbf{x}$; in that case they require a suitable treatment see e.g. [8, 12, 13]. We apply the Godunov scheme proposed in [10], which seems to be a natural choice, since the hybrid (interface) unknowns are used.

Assumptions on the data:

$(\mathcal{H}_1)$ $\varphi \in C(\mathbb{R}), \varphi(0) = 0$, is a strictly increasing piecewise continuously differentiable Lipschitz continuous function. We assume that the function φ^{-1} is Hölder continuous, namely that there exists $H_\varphi > 0$ and $\alpha \in (0, 1]$ such that $|s_1 - s_2| \leq H_\varphi |\varphi(s_1) - \varphi(s_2)|^\alpha$. It is also such that φ^{-1} is Lipschitz continuous on $\mathbb{R} \setminus (0, 1)$;
$(\mathcal{H}_2)$ The functions $\lambda, \xi, \gamma, f \in C(\overline{\mathbb{R}})$ are Lipschitz continuous;
$(\mathcal{H}_{2a})$ ξ and γ are convex functions, moreover γ is such that $\gamma(0) = \gamma(1) = 0$; $\mathbf{g}$ is a constant vector from $\mathbb{R}^d$;
$(\mathcal{H}_{2b})$ f is a nondecreasing function and it satisfies $f(0) = 0, f(1) = 1$;
$(\mathcal{H}_3)$ $s_0 \in L^\infty(\Omega)$; $\omega \in L^\infty(\Omega)$;
$(\mathcal{H}_4)$ $q_o, q_w \in L^\infty(0, T; L^2(\Omega))$ and such that $q_o + q_w \geq 0$ a.e. in $\Omega \times (0, T)$.

The physical range of values for the saturation is $[0, 1]$, however we extend the definition of all nonlinear functions in (1)-(2) outside of $[0, 1]$, since the numerical scheme which we study does not preserve neither the maximum principle, nor the positivity of s (nor the bound $s \leq 1$).

Assumptions on the geometry:

$(\mathcal{H}_{5a})$ Ω is a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary.
$(\mathcal{H}_{5b})$ $\mathbf{K}$ is a piecewise constant function from Ω to $\mathcal{M}_d(\mathbb{R})$, where $\mathcal{M}_d(\mathbb{R})$ denotes the set of real $d \times d$ matrices. More precisely we assume that there exist a finite family $(\Omega_i)_{i \in \{1,\ldots,I\}}$ of open connected polyhedral in $\mathbb{R}^d$, such that $\overline{\Omega} = \bigcup_{i \in \{1,\ldots,I\}} \overline{\Omega_i}$, $\Omega_i \bigcap \Omega_j = \emptyset$ if $i \neq j$ and such that $\mathbf{K}(\mathbf{x})|_{\Omega_i} = \mathbf{K}_i \in \mathcal{M}_d(\mathbb{R})$. By $\Gamma_{i,j}$ we denote the interface between the sub-domains i, j, $\Gamma_{i,j} = \overline{\Omega_i} \bigcap \overline{\Omega_j}$. We assume that there exist two positive constants $\underline{\mathrm{K}}$ and $\overline{\mathrm{K}}$ such that the eigenvalues of the symmetric positive definite $\mathbf{K}_i$ are included in $[\underline{\mathrm{K}}, \overline{\mathrm{K}}]$ for all $i \in \{1, \ldots, I\}$.

For the sake of simplicity we have assumed that the heterogeneity of the medium is only expressed through the $\mathbf{x}$-dependence of the absolute permeability tensor $\mathbf{K} = \mathbf{K}(\mathbf{x})$. However it is very simple to extend the analysis to the case where λ, ξ, γ and f depend on the rock type. On the other hand if we suppose that φ is discontinuous in space, this may lead to significant difficulties. The analysis of the case where the capillary pressure field (and also φ) is discontinuous was carried out in [4] and [3]. It is also worth noting that the partitioning of Ω introduced in $\mathcal{H}_{5b}$ is only used in order to provide a control on the gravity terms. In the case that the gravity effects are neglected, one can consider a fully heterogenous permeability field $\mathbf{K}$.

1. The finite volume scheme

1.1. The main definitions

Definition 1.1 (Discretization of Ω)**.** *Let Ω be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$, $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary, and $(\Omega_i)_{i \in \{1,\ldots,I\}}$ is the partition of Ω in the sense of $(\mathcal{H}_{5b})$. A discretization of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

1. $\mathcal{M}$ is a finite family of non empty connex open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K; we define $m(K) > 0$ as the measure of K and h_K as the diameter of K. We also assume that the mesh resolve the structure of the medium, i.e.

for all $K \in \mathcal{M}$ there exist $i \in \{1, \dots, I\}$ such that $K \subset \Omega_i$. The size of the discretization $\mathcal{D}$ is defined by $h_{\mathcal{D}} = \sup_{K\in\mathcal{M}} diam(K)$.

2. $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non empty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$-dimensional measure $m(\sigma)$ is strictly positive. We also assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. For each $\sigma \in \mathcal{E}$, we set $\mathcal{M}_\sigma = \{K \in \mathcal{M} | \sigma \in \mathcal{E}_K\}$. We then assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \in \partial\Omega$ (the set of these interfaces called boundary interfaces, is denoted by $\mathcal{E}_{ext}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces called interior interfaces, is denoted by $\mathcal{E}_{int}$). For all $\sigma \in \mathcal{E}$, we denote by $\mathbf{x}_\sigma$ the barycenter of σ. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the outward normal unit vector.

3. $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (\mathbf{x}_K)_{K\in\mathcal{M}}$, such that for all $K \in \mathcal{M}$, $\mathbf{x}_K \in K$; moreover K is assumed to be $\mathbf{x}_K$-star-shaped, which means that for all $\mathbf{x} \in K$, there holds $[\mathbf{x}_K, \mathbf{x}] \in K$. Denoting by $d_{K,\sigma}$ the Euclidean distance between $\mathbf{x}_K$ and the hyperplane containing σ, one assumes that $d_{K,\sigma} > 0$. We denote by $D_{K,\sigma}$ the cone of vertex $\mathbf{x}_K$ and basis σ.

Definition 1.2 (Time discretization)**.** *We divide the time interval $(0, T)$ into N equal time steps of length $\delta t = T/N$, where δt is the uniform time step defined by $\delta t = t_n - t_{n-1}$.*

Definition 1.3 (The hybrid space $X_{\mathcal{D}}(\Omega)$)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization of Ω. We define*

$$X_{\mathcal{D}} = \{((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}), v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\}, \quad X_{\mathcal{D},0} = \{v \in X_{\mathcal{D}} \text{ such that } (v_\sigma)_{\sigma\in\mathcal{E}_{ext}} = 0\}. \tag{4}$$

Taking into account the time discretization we define $X_{\mathcal{D},\delta t} = X_{\mathcal{D}}^N$ and $X_{\mathcal{D},\delta t,0} = X_{\mathcal{D},0}^N$.

1.2. The numerical scheme

Let us introduce the discrete saturation $\left((s_K^n)_{K\in\mathcal{M}}, (s_\sigma^n)_{\sigma\in\mathcal{E}}\right)_{n\in\{1,\dots,N\}} \in X_{\mathcal{D},\delta t}$ and the discrete global pressure $\left((p_K^n)_{K\in\mathcal{M}}, (p_\sigma^n)_{\sigma\in\mathcal{E}}\right)_{n\in\{1,\dots,N\}} \in X_{\mathcal{D},\delta t}$, which are the main discrete unknowns. Let $\mathfrak{f}$ denote λ, ξ, γ or f, we introduce the following notation $\mathfrak{f}_K^n = \mathfrak{f}(s_K^n)$ and $\mathfrak{f}_\sigma^n = \mathfrak{f}(s_\sigma^n)$. Next, let $q_{i,K}^n$ denote the mean value of the source terms, i.e. $q_{i,K}^n = \frac{1}{m(K)\delta t}\int_{t_{n-1}}^{t_n}\int_K q_i(\mathbf{x}, t)\, d\mathbf{x}dt$ with $i \in \{w, n\}$ and $\omega(K)$ denotes the porous volume of the element K, $\omega(K) = \int_K \omega(\mathbf{x})\, d\mathbf{x}$.

Let $U_{K,\sigma}^n$ be an approximation of the total flux through the interface σ

$$U_{K,\sigma}^n \approx \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}\int_\sigma \mathbf{K}(\lambda(s)\nabla p - \xi(s)\mathbf{g}) \cdot \mathbf{n}_{K,\sigma}\mathbf{d}\nu dt \tag{5}$$

and let $F_{K,\sigma}^n$ be an approximation of the non-wetting phase flux

$$F_{K,\sigma}^n \approx \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}\int_\sigma (\mathbf{u}f(s) + \gamma(s)\mathbf{K}\mathbf{g} - \mathbf{K}\nabla\varphi(s)) \cdot \mathbf{n}_{K,\sigma}\mathbf{d}\nu dt. \tag{6}$$

The numerical fluxes $U_{K,\sigma}^n$ and $F_{K,\sigma}^n$ have to be constructed as functions of the discrete unknowns. Following the ideas of [6] we write the scheme in the variational form.

For each $n \in \{1, \dots, N\}$ find $s^n \in X_{\mathcal{D},0}$ and $p^n \in X_{\mathcal{D},0}$ such that for all $v^n, w^n \in X_{\mathcal{D},0}$:

$$\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(v_K^n - v_\sigma^n)U_{K,\sigma}^n = \sum_{K\in\mathcal{M}} m(K)v_K^n(q_{w,K}^n + q_{o,K}^n), \tag{7}$$

$$\sum_{K\in\mathcal{M}}\omega(K)w_K^n\frac{s_K^n - s_K^{n-1}}{\delta t} + \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(w_K^n - w_\sigma^n)F_{K,\sigma}^n = \sum_{K\in\mathcal{M}} m(K)w_K^n q_{o,K}^n, \tag{8}$$

$$s_K^0 = \frac{1}{m(K)}\int_K s_0(\mathbf{x})\, d\mathbf{x}. \tag{9}$$

Remark that opposite to the classical two-point flux approximation, the discrete fluxes $U^n_{K,\sigma}$ and $F^n_{K,\sigma}$ (which still remain to be constructed) are not *a priori* continuous across the element's interfaces, so that the continuity is prescribed in the scheme by (7) and (8). In order to complete the scheme we have to define the numerical fluxes $U^n_{K,\sigma}$ and $F^n_{K,\sigma}$. Let $\mathbf{K}_K$ denote the mean value of $\mathbf{K}(\mathbf{x})$ over a cell K,

$$\mathbf{K}_K = \frac{1}{m(K)} \int_K \mathbf{K}(\mathbf{x})\, d\mathbf{x} \text{ and let } g_{K,\sigma} = m(\sigma)\mathbf{K}_K \mathbf{g} \cdot \mathbf{n}_{K,\sigma}. \tag{10}$$

Note that $g_{K,\sigma}$ satisfies

$$\sum_{\sigma \in \mathcal{E}_K} g_{K,\sigma} = 0 \text{ for all } K \in \mathcal{M}, \tag{11}$$

but not necessarily

$$g_{K,\sigma} + g_{L,\sigma} = 0 \text{ with } \{K, L\} = \mathcal{M}_\sigma.$$

The above equality remains true for the interfaces which are "interior" with respect to some sub-domain Ω_i, that is to say

$$g_{K,\sigma} + g_{L,\sigma} = 0 \text{ with } \{K, L\} = \mathcal{M}_\sigma \text{ for all } \sigma \notin \Gamma_{i,j}, \tag{12}$$

for any i, j. Next, we define $U^n_{K,\sigma}$ and $F^n_{K,\sigma}$ by

$$U^n_{K,\sigma} = \lambda^n_K \mathcal{F}_{K,\sigma}(p^n) + \mathcal{G}(\xi(\cdot)g_{K,\sigma}; s^n_K, s^n_\sigma), \tag{13}$$

$$F^n_{K,\sigma} = \mathcal{G}(U^n_{K,\sigma} f(\cdot) + \gamma(\cdot)g_{K,\sigma}; s^n_K, s^n_\sigma) + \mathcal{F}_{K,\sigma}(\varphi(s^n)), \tag{14}$$

The terms $\mathcal{F}_{K,\sigma}(\cdot)$ correspond to the diffusive fluxes which are defined below using the SUSHI scheme [6]. The terms $\mathcal{G}(\cdot)$ stand for the discretization of the convective fluxes using the Godunov scheme.

The Godunov scheme and the convection term

Let $a, b \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R})$ we define the Godunov (see [10, p. 3]) flux by

$$\mathcal{G}(f; a, b) = \begin{cases} \min_{s \in [a,b]} f(s) & \text{if } a \le b, \\ \max_{s \in [b,a]} f(s) & \text{if } b \le a. \end{cases} \tag{15}$$

The discrete gradient and the diffusion term

Let us recall a construction of the discrete gradient and of the numerical flux $\mathcal{F}_{K,\sigma}(\cdot)$ proposed in [6]. Let $u \in X_\mathcal{D}$, for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$ we first define

$$\nabla_K u = \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)(u_\sigma - u_K)\mathbf{n}_{K,\sigma} \quad \text{and} \quad R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}}(u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)). \tag{16}$$

Let $D_{K,\sigma}$ be a cone with vertex $\mathbf{x}_K$ and basis σ and $u = (u^n)_{n \in \{1,\dots,N\}} \in X_{\mathcal{D},\delta t}$ We define the discrete gradient $\nabla_\mathcal{D} u$ as the following piecewise constant function

$$\nabla_{\mathcal{D},\delta t} u(\mathbf{x}, t)|_{(\mathbf{x},t) \in D_{K,\sigma} \times (t_{n-1}, t_n]} = \nabla_K u^n + R_{K,\sigma} u^n \cdot \mathbf{n}_{K,\sigma}, \tag{17}$$

Note that $R_{K,\sigma} u$ is a stabilizing second order error term, which vanishes for piecewise linear functions. For an arbitrary $u \in X_D$ the numerical flux $\mathcal{F}_{K,\sigma}(u)$ can be defined through the following discrete integration by parts formula

$$\sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma)\mathcal{F}_{K,\sigma}(u) = \int_K \mathbf{K} \nabla_D u \cdot \nabla_\mathcal{D} v \, d\mathbf{x} \text{ for all } v \in X_\mathcal{D}. \tag{18}$$

The explicit form of $\mathcal{F}_{K,\sigma}$ can be obtained by setting $v_K - v_\sigma = 1$ and $v_K - v_{\sigma'} = 0$ for all $\sigma' \neq \sigma$. We refer to [6] for more details on construction of $\mathcal{F}_{K,\sigma}$ and its practical implementation.

Theorem 1.1 (Existence of a discrete solution)**.** *The problem* (7)-(9) *has at least one solution.*

2. The main result

Definition 2.1 (Approximate solution)**.** *Let $\mathcal{D}$ be a discretization of Ω, $N \in \mathbb{N}^*$ and $\delta t = T/N > 0$. Let $(s^n, p^n)_{n\in\{1,\ldots,N\}} \in X^2_{\mathcal{D},\delta t,0}$ be a solution to (7)-(9). We say that the pair of functions $s_{\mathcal{D},\delta t}$ and $p_{\mathcal{D},\delta t}$ is an approximate solution of problem (1)-(3) if*

$$s_{\mathcal{D},\delta t}(\mathbf{x},0) = s^0_K \text{ for all } \mathbf{x} \in K, \tag{19}$$

$$s_{\mathcal{D},\delta t}(\mathbf{x},t) = s^n_K \quad \text{and} \quad p_{\mathcal{D},\delta t}(\mathbf{x},t) = p^n_K \text{ for all } (\mathbf{x},t) \in K \times (t_{n-1}, t_n], \tag{20}$$

Theorem 2.1 (Main result)**.** *Let $\mathfrak{D}$ be a sequence of sufficiently regular discretizations of Ω and such that $h_{\mathcal{D}}$ tends to zero along $\mathfrak{D}$. Let $\delta\mathfrak{t}$ be a sequence of real positif numbers, such that $T/\delta t \in \mathbb{N}$ for all $\delta t \in \delta\mathfrak{t}$ and such that δt tends to zero along $\delta\mathfrak{t}$. Let $(s_{\mathfrak{D},\delta\mathfrak{t}}, p_{\mathfrak{D},\delta\mathfrak{t}}) = (s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t})_{\mathcal{D}\in\mathfrak{D},\delta t\in\delta\mathfrak{t}}$ be a sequence of approximate solutions corresponding to $\mathfrak{D}$ and $\delta\mathfrak{t}$. Then there exists a subsequence of $(s_{\mathfrak{D},\delta\mathfrak{t}}, p_{\mathfrak{D},\delta\mathfrak{t}})$, which we denote again by $(s_{\mathfrak{D},\delta\mathfrak{t}}, p_{\mathfrak{D},\delta\mathfrak{t}})$, such that $s_{\mathcal{D},\delta t} \to s$ strongly in $L^2(Q_T)$ and $p_{\mathcal{D},\delta t} \to p$ weakly in $L^2(Q_T)$ as $h_{\mathcal{D}}, \delta t \to 0$, where (s,p) is a weak solution of the problem $(1)-(2)$.*

Remark 2.1. *The proof of Theorem 2.1 and Theorem 1.1 is given in [2]. See also [1], where a single parabolic degenerate equation was considered.*

3. Numerical experiments

We consider two test cases proposed in [11], which allow to investigate the qualitative behavior of the scheme in the case of anisotropic and heterogeneous absolute permeability tensor $\mathbf{K}$. More precisely we consider a two dimensional horizontal (gravity is neglected) domain $\Omega = (0,1)^2$, which is initially saturated in oil ($s_0 = 1$ in Ω). The oil-phase is displaced by water, which is injected at the lower left corner; the production well is placed at the upper right corner. The effects of wells are modeled by terms on the right-hand side of (1) and (2)

$$q_o(s) = f(\overline{s})q^+ + f(s)q^- \qquad q_w(s) = (1 - f(\overline{s}))\, q^+ + (1 - f(s))\, q^-,$$

where $q^+ = q^+(\mathbf{x},t)$ and $q^- = q^-(\mathbf{x},t)$ denote the production and the injection rates. The value $\overline{s}$ is the oil saturation of an injected fluid, which is set to $\overline{s} = 0$. We assume that q^+ and q^- are Dirac functions $q^+ = \delta(\mathbf{x})$ and $q^- = \delta(\mathbf{x} - (1,1))$.

First we assume that the porous medium is homogeneous. The absolute permeability tensor $\mathbf{K}$ is defined by $\mathbf{K} = \mathbf{R}_\theta \mathbf{D} \mathbf{R}'_\theta$, where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} \text{ with } \mathbf{R}_\theta = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix},$$

for all real θ. We set $\theta = \pi/4$ and we present on Figure 1 the approximate water-phase saturation and the global pressure fields at time $t = 0.02$. Note that due to the anisotropy the solution forms a relatively fine strip propagating along the direction $(cos\theta, sin\theta)$, which is an eigenvector of $\mathbf{K}$ corresponding to the value 1. Next

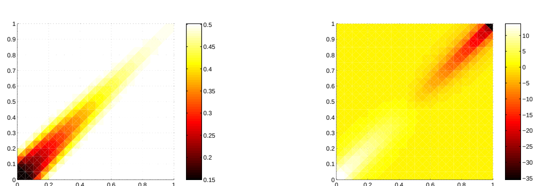

Figure 1. Oil saturation (left) and global pressure field (right) at time $t = 0.02$ (homogeneous case)

let us consider a more complex geometry. We assume that the computational domain $\Omega = (0,1)^2$ is composed of four layers

$$\Omega_1 = \Omega \cap \{|x_1| + |x_2| < 0.5\}, \qquad \Omega_2 = \Omega \cap \{0.5 < |x_1| + |x_2| < 1\},$$
$$\Omega_3 = \Omega \cap \{1 < |x_1| + |x_2| < 1.5\}, \quad \Omega_4 = \Omega \cap \{|x_1| + |x_2| > 1.5\},$$

which we represent at left of Figure 2 by different colors. The piecewise constant absolute permeability field is defined by $\mathbf{K}(\mathbf{x})|_{\mathbf{x}\in\Omega_i} = \mathbf{K}_i$, with $i \in \{1, \ldots, 4\}$ and

$$\mathbf{K}_1 = \mathbf{K}_4 = \mathbf{R}_{\pi/4}\mathbf{D}\mathbf{R}'_{\pi/4}, \quad \mathbf{K}_2 = \mathbf{D}, \quad \mathbf{K}_3 = \mathbf{R}_{\pi/2}\mathbf{D}\mathbf{R}'_{\pi/2}.$$

On Figures 2 we present the approximate water-phase saturation and global pressure field at time $t = 0.07$. As in the previous case water propagates in the most permeable direction, which changes at the interface between two different layers. It seems that the scheme gives fairly correct results in the case of a highly anisotropic and heterogeneous absolute permeability fields. In future it would be interesting to compare our scheme with others methods both in the cases with and without gravity.

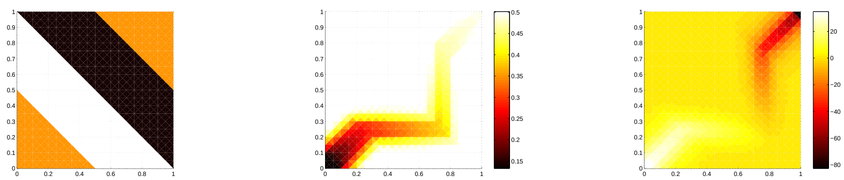

FIGURE 2. Absolute permeability field (left), oil saturation (center) and global pressure field (right) at time $t = 0.07$ (heterogeneous case)

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