ESAIM: PROCEEDINGS, April 2012, Vol. 36, p. 15-25
Danièle Fournier-Prunaret, Laura Gardini, & Ludwig Reich, Editors

EXAMPLE OF THE SMOOTH SKEW PRODUCT IN THE PLANE WITH THE ONE-DIMENSIONAL RAMIFIED CONTINUUM AS THE GLOBAL ATTRACTOR *

L.S. Efremova [1]

Dedicated to the memory of R.V. Plykin

Abstract. The example is constructed of the C^1-smooth skew product of interval maps possessing the one-dimensional ramified continuum (containing no arcs homeomorphic to the circle) with an infinite set of ramification points as the global attractor.

AMS (2000) subject classification. 37Exx, 34D45.
Keywords. skew product, attractor, ramified continuum.

Résumé. L'exemple est construit à partir d'un produit biaisé lisse de classe C^1 de transformations d'un intervalle, qui a un continuum unidimensionnel ramifié (ne contenant pas d'arcs homéomorphes à un cercle) avec un ensemble infini de points de branchement comme attracteur global.

Mots clefs. Produit biaisé, attracteur, continuum ramifié.

Introduction

One of the features of modern physics is the increasing of the interest to complicated (usually, unstable) structures (see e.g. [1] – [4]). At the same time the hyperbolic theory of dynamic systems (see [5] – [6]) gives different examples of complicated structures connected with interesting dynamic phenomena. Let us refer to Smale's horseshoe [6], indecomposable continua connected with homoclinic tangencies [7] – [8], Plykin's attractor [9] – [10], dendrites as the limit sets of some Kleinian groups in hyperbolic 3-manifolds [11] – [12] and others. Nonhyperbolic dynamic systems, such as dynamic systems of skew product class, demonstrate new examples of complicated structures. The nonchaotic attractor is one of these structures (see e.g. [13] – [16]).

In this paper the example of the C^1-smooth skew product of interval maps is given such that this skew product possesses the one-dimensional ramified continuum (with an infinite set of ramification points) as the global one-dimensional attractor.

Constructed map demonstrates both nonchaotic and chaotic dynamics on the attractor. There coexist non-degenerate closed intervals of the following types in the attractor: the countable set of closed intervals in the vertical fibers over the periodic points of the quotient map is presented as the union of mutually disjoint closed

* *This paper was supported in the part by the Federal Target Program "Scientific and Scientific-Pedagogical Personnel of Innovative Russia" (2009 – 2011) of the Federal Education Agency (Project No. NK -13/9).*

[1] Lobachevsky Nizhny Novgorod State University (Research University), Nizhny Novgorod, Russia.

invariant sets consisting of periodic orbits of the same (least) period (the least period of these periodic orbits coincides with the number of vertical closed intervals in the invariant set and depends on this set); the unique closed interval of the second type (the horizontal interval) contains the residual set of everywhere dense trajectories in this interval. The construction of this example is based on the ideas of the papers [17] – [19].

The paper is organized as follows. In Sec.1 we formulate the main definitions which will be used and give formulas for the coordinate functions of the C^1-smooth skew product of maps of the unit interval with above properties. In Sec.2 we formulate and prove the main result of this paper (theorem 2.1) and the auxiliary results (lemmas 2.4 – 2.7; corollaries 2.8 – 2.9) concerning the properties of the map constructed in Sec.1.

1. Preliminaries. Formulas for the skew product with required properties.

Let $I = I_1 \times I_2$ be a closed rectangle in the plane (I_1, I_2 are closed intervals). We consider a *skew product of interval maps* $F : I \to I$, i. e. the dynamic system

$$F(x, y) = (f(x), g_x(y)), \text{ where } g_x(y) = g(x, y), \ (x; y) \in I. \tag{1}$$

Here $f : I_1 \to I_1$ is said to be the *quotient map* of the dynamic system (1), and the map $g_x : I_2 \to I_2$ for any $x \in I_1$ is said to be the *map acting in the fiber* over the point x.
The dynamic system (1) preserves vertical fibers in the following natural sense:

$$F(\{x\} \times I_2) \subseteq \{f(x)\} \times I_2 \quad x \in I_1.$$

If the skew product of interval maps F is differentiable, then dynamic system (1) preserves also the field of vertical directions, i. e. for the image of the unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the differential DF of the map F in any point $(x, y) \in I$ the equality

$$DF(x, y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\partial}{\partial y} g_x(y) \end{pmatrix}$$

is valid. There are no other invariant directions under DF. It means that smooth dynamic systems of the skew product class are not hyperbolic.
Since the map (1) preserves the vertical fibers in the above sense, then the equality

$$F^n(x, y) = (f^n(x), g_{x,n}(y)), \text{ where } g_{x,n} = g_{f^{n-1}(x)} \circ \ldots \circ g_x \tag{2}$$

is correct for any natural number n and for any point $(x; y) \in I$.
We will use the notation $\widetilde{g}_x$ for the map $g_{x,n}$ if x is a periodic point of f ($x \in Per(f)$), and $n = n(x)$ is its (least) period. Denote by $T^1(I)$ the space of all C^1-smooth skew products of interval maps with the natural C^1-norm.

1.1. Main Definitions

Begin from the concept of an attractor (see [20]).

Definition 1.1. *An attractor* of a skew product $F \in T^1(I)$ is a closed set $A^* \subset I$ such that it has an absorbing neighborhood; that is a neighbourhood $U(A^*)$ satisfying

$$F(\overline{U(A^*)}) \subset U(A^*), \quad \text{and} \quad A^* = \bigcap_{n=1}^{+\infty} F^n(\overline{U(A^*)}), \text{ where } \overline{(\cdot)} \text{ is the closure of a set.}$$

In the study of the properties of an arbitrary dynamic system with a compact phase space its set of nonwandering points (the nonwandering set) plays an important role (see e.g. [20]).

Definition 1.2. Let $\Phi : X \to X$ be a continuous map of the compact metric phase space. A point $x \in X$ is called a *nonwandering point* for the map Φ if for any neighborhood $U(x)$ of the point x there exists a natural number m such that $U(x) \bigcap \Phi^m(U(x)) \neq \emptyset$. The set of all nonwandering points of the map Φ is called the *nonwandering set*. We denote this set by $\Omega(\Phi)$. A point $x' \in X \setminus \Omega(\Phi)$ is called a *wandering point* of Φ.

During last years dynamic systems on one-dimensional ramified continua, in particular, on dendrites (i. e. on locally connected continua containing no simple closed curves), are intensively investigating (see e.g. papers [21] – [24]). Give the main definitions (see [25]).

Definition 1.3. A space homeomorphic to the closed unit interval [0, 1] (resp. to the unit circle $\mathbf{S^1}$ in the complex plane $\mathbf{C}$) is said to be an *arc* (resp. a *simple closed curve*).
A compact connected metric space is said to be a *continuum*.

Let X be a one-dimensional continuum, and a point x be such that the number of components of the set $X \setminus \{x\}$ is finite. The number of these components is called *the order of a point* x. Denote by $ord_X(x)$ the order of a point x.
A point $x \in X$ of an order satisfying the inequality $ord_X(x) \geq 3$ is called a *ramification point* of X. Denote by $R(X)$ the set of all ramification points of X.

1.2. **Example**

Let the surjective logistic map $f(x) = 4x(1-x)$ be the quotient map of the skew product $F^* \in T^1([0,\ 1]^2)$. Introduce two right triangles

$$I_* = \{(x;\ y) \in [0,\ 1]^2 : x \in [0,\ 1], 0 \leq y \leq 1-x\} \quad \text{and} \quad I^* = \{(x;\ y) \in [0,\ 1]^2 : x \in (0,\ 1], 1-x < y \leq 1\}.$$

Define the fiber maps of the skew product F^* satisfying

$$g_x(y) = \begin{cases} y, & (x;\ y) \in I_*; \\ 1-x+\sin(y-1+x), & (x;\ y) \in I^*. \end{cases} \tag{3}$$

Note that $g_x(y)$ is C^1-smooth in $[0,\ 1]^2$. Nonhyperbolic fiber maps of the type (3) have been used in the other examples of skew products of interval maps in the papers [17] – [18].

2. Properties of the map

The following theorem contains the main result of the paper.

Theorem 2.1. *The nonwandering set $\Omega(F^*)$ of the skew product F^* possesses the properties:*
(2.1.1.) *$\Omega(F^*)$ is the global attractor of F^* (i. e. the whole phase space of F^* is the absorbing neighborhood of the attractor $A^* = \Omega(F^*)$);*
(2.1.2.) *$A^* = \Omega(F^*)$ is the one-dimensional ramified continuum such that the cardinality of the set of ramification points $R(A^*)$ equals continuum, and the equality $\overline{R(A^*)} = [0,\ 1] \times \{0\}$ holds*[1]*; in addition, $ord_{A^*}((x;0)) = 3$ for any $(x;0) \in R(A^*)$, and A^* contains no a simple closed curve;*
(2.1.3.) *any point from the residual subset of the residual set (in $[0,\ 1] \times \{0\}$) of points with everywhere dense trajectories in $[0,\ 1] \times \{0\}$ has the order 2; in addition, any point from this subset is a point of a local connectedness of A^*; at the same time any point $(x;\ y)$, where y belongs to the slice $(\Omega(F^*))(x) = \{y \in [0,\ 1] : (x;\ y) \in \Omega(F^*)\}$, is not the point of local connectedness for an arbitrary f-uniformly recurrent nonperiodic point x;*
(2.1.4.) *the map F^* demonstrates the mixed dynamics on the attractor A^* in the following sense: any closed invariant set of vertical branches of A^* over points $\{(x;\ 0), \ldots (f^{n-1}(x);\ 0)\}$ for any point $x \in Per(f)$ (n is the (least) period of x under f, which is equal to the (least) period of $(x;\ 0)$ under F^*) consists of F^*-periodic orbits of the same (least) period n; F^* is the chaotic map on the horizontal invariant branch $[0,\ 1] \times \{0\}$ of A^*.*

[1]Among points of the set $R(A^*)$ one can find, for example, the points $(x;\ 0)$, where $x \in Per(f) \setminus \{0\}$, x is a homoclinic point to a periodic point from the interval $(0,\ 1)$, x is f-uniformly recurrent points.

Denote by A^*_x the closed invariant set of vertical branches of A^* over points $\{(x; 0), \dots (f^{n-1}(x); 0)\}$ for any $x \in Per(f)$. Then, as it follows from the formulas for the map F^*, $h(F^*{}_{|A^*_x}) = 0$; at the same time $h(F^*_{|[0,\,1]\times\{0\}}) > 0$, where $h(\cdot)$ is the topological entropy of a map.

To prove theorem 2.1 we need some properties of the skew product F^* giving in the part 2.1.

2.1. **Formulations**

We will use the following properties of the quotient map (see e.g. [26] – [28]).

Lemma 2.2. *The map* $f(x) = 4x(1-x)$, $x \in [0,\, 1]$, *possesses*
(2.2.1.) *the residual set of everywhere dense trajectories;*
(2.2.2.) *everywhere dense set* $Per(f)$ *of periodic points such that for any natural number* n *there exists a point* $x \in Per(f)$ *with the (least) period* $n = n(x)$.
In addition, f*-periodic orbits uniformly approximate the interval* $[0,\, 1]$ *(i. e. for any partition of the unit interval there exists a periodic orbit of* f *such that any interval of the partition contains at least one point of this periodic orbit).*

Point out that the property of uniform approximability of the unit interval by f-periodic orbits follows from the claims (2.2.1), (2.2.2) of lemma 2.2 and results of the papers [27], [28].

Analogously [17] – [19], we will use the multifunctions $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ setting

$$\eta_n(x) = \Omega(g_{x,\, n}), \quad n \geq 1, \tag{4}$$

where $(2^{[0,\, 1]})_m$ is the space of all closed subsets of the unit interval $[0,\, 1]$ with Hausdorff metric *dist* (see [25]).

Definition 2.3. A map $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ is called a *continuous* in a point $x' \in [0,\, 1]$ if for any $\varepsilon > 0$ there is a positive number $\delta = \delta(\varepsilon,\, x')$ such that for any $x \in [0,\, 1]$ satisfying $|x - x'| < \delta$ the inequality

$$dist(\eta_n(x),\ \eta_n(x')) < \varepsilon$$

holds. As usually, a map $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ is continuous if it's continuous in any point of the unit interval.

We will use the following property of multifunctions $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ of the skew product of interval maps $F^* \in T^1([0,\, 1]^2)$ constructed above.

Lemma 2.4. *Multifunctions* $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ *defined for the skew product* $F^* \in T^1([0,\, 1]^2)$ *are continuous for any* $n \geq 1$.

Establish the correlation between the graphs of multifunctions $\eta_n : [0,\, 1] \to (2^{[0,\, 1]})_m$ in the unit square and the nonwandering set $\Omega(F^*)$ of the map $F^* \in T^1([0,\, 1]^2)$. For this goal we use direct products $F^*_{n,\, 1}(x,\, y) = (f^n(x),\, id(y))$ for any $n \geq 1$, where $id(y)$ is the identity map of the unit interval.
Introduce multifunctions $\eta_{n,\, 1} : [0,\, 1] \to (2^{[0,\, 1]})_m$ $(n \geq 1)$ setting

$$\eta_{n,\, 1}(x) = (F^*_{n,\, 1}(\eta_n))(x) \tag{5}$$

for any $x \in [0,\, 1]$, where $(F^*_{n,\, 1}(\eta_n))(x) = \{y : (x,\, y) \in F^*_{n,\, 1}(\eta_n)\}$ is the slice of the set $F^*_{n,\, 1}(\eta_n)$ by the fiber over an arbitrary point x.

Lemma 2.5. *Multifunctions* $\eta_{n,\, 1} : [0,\, 1] \to (2^{[0,\, 1]})_m$ $(n \geq 1)$ *defined for the map* $F^* \in T^1([0,\, 1]^2)$ *are continuous.*

Lemma 2.6. *The equality*

$$\bigcap_{n=1}^{+\infty} \eta_{n,\, 1} = \Omega(F^*)$$

holds for the map $F^* \in T^1([0,\, 1]^2)$, *where* $\eta_{n,\, 1}$ $(n \geq 1)$ *are the graphs of multifunctions* $\eta_{n,\, 1} : [0,\, 1] \to (2^{[0,\, 1]})_m$ *in the unit square. In addition,* $\Omega(F^*)$ *is the global attractor of* F^*.

Lemma 2.7. *The set* $A^* = \Omega(F^*)$ *of the skew product of maps of the unit interval* $F^* \in T^1([0, 1]^2)$ *is the one-dimensional ramified continuum, and the claims* (2.1.2) – (2.1.4) *of theorem* 2.1 *are fulfilled.*

The correctness of theorem 2.1 follows from lemmas 2.4 – 2.7.

2.2. Proofs

We begin this section from the proof of lemma 2.4.

Proof. Let n be an arbitrary natural number. By the equality (3) the fiber map g_x for any $x \in [0, 1]$ is the increasing diffeomorphism of the interval [0, 1] onto the interval $g_x([0, 1])$ with respect to the variable y. Then $g_{x,\,n}$ is the increasing diffeomorphism of the interval [0, 1] onto the interval $g_{x,\,n}([0, 1])$ (see formula (2)). Set

$$x^*_n(x) = \max\{x,\, f(x),\, \ldots,\, f^{n-1}(x)\} \text{ for any } x \in [0, 1].$$

As it follows from (3), the equality

$$\Omega(g_{x,\,n}) = Fix(g_{x,\,n}) = [0,\, 1 - x^*_n(x)] \tag{6}$$

is valid for any $x \in [0, 1]$, where $Fix(\cdot)$ is the set of the fixed points of a map.
Use the uniform continuity of the quotient map f on the compact $[0, 1] = \Omega(f)$. For any $\varepsilon > 0$ find $\delta = \delta(\varepsilon) > 0$ such that for any $x, x' \in [0, 1]$ satisfying $|x - x'| < \delta$ the inequality

$$|f^i(x) - f^i(x')| < \varepsilon \tag{7}$$

holds for any $1 \le i \le n - 1$. Note that inequality (7) implies the inequality

$$|x^*_n(x) - x^*_n(x')| < \varepsilon. \tag{8}$$

In fact, the correctness of (8) follows immediately from (7), if $x^*_n(x) = f^{i^*}(x)$, $x^*_n(x') = f^{i^*}(x')$ for some $0 \le i^* \le n - 1$. Let $x^*_n(x) = f^{i^*}(x)$, $x^*_n(x') = f^{j^*}(x')$ for some $0 \le i^*, j^* \le n - 1$, $j^* \ne i^*$.
Suppose that for some $\varepsilon > 0$ and for any $\delta > 0$ one can find points $x, x' \in [0, 1]$, $|x - x'| < \delta$, such that the inequality

$$|x^*_n(x) - x^*_n(x')| = |f^{i^*}(x) - f^{j^*}(x')| \ge \varepsilon \tag{9}$$

is fulfilled.
Fix $\delta > 0$ and $x, x' \in [0, 1]$, $|x - x'| < \delta$, such that inequality (9) holds.
Let $f^{i^*}(x) < f^{j^*}(x')$. Since $f^{j^*}(x) < f^{i^*}(x)$, then $f^{j^*}(x') - f^{j^*}(x) \ge f^{j^*}(x') - f^{i^*}(x) \ge \varepsilon$. It contradicts (7).
Let $f^{j^*}(x') < f^{i^*}(x)$. Since $f^{i^*}(x') < f^{j^*}(x')$, then $f^{i^*}(x) - f^{i^*}(x') \ge f^{i^*}(x) - f^{j^*}(x') \ge \varepsilon$. It contradicts (7) too. Hence, the inequality (8) is valid.
Since the equality $\eta_n(x) = \Omega(g_{x,\,n})$ is valid for any $x \in [0, 1]$, then using formulas (6) and (8), we obtain the inequality

$$dist(\eta_n(x),\, \eta_n(x')) < \varepsilon$$

for any $x, x' \in [0, 1]$ satisfying $|x - x'| < \delta$. Thus, η_n is continuous function (see definition 2.3).
Lemma 2.4 is proved. □

Note that continuous multifunctions $\eta_n : [0, 1] \to (2^{[0,\,1]})_m$ have the real dynamic sense. In fact, succeeding [19], introduce the auxiliary skew products

$$F^*_n(x, y) = (id(x),\, g_{x,\,n}(y)) : [0, 1]^2 \to [0, 1]^2, \tag{10}$$

where $id(x)$ is the identity map of the unit interval. Then, as it follows from [29], $\eta_n = \Omega(F^*_n)$ for any $n \geq 1$ (here η_n is the graph of the function η_n in $[0, 1]^2$)[2]

By formula (5) the following equality holds

$$\eta_{n,\,1}(x) = \bigcup_{\overline{x} \in \{f^{-n}(x)\}} \eta_n(\overline{x}) = [0,\, 1 - x_n^{min}(x)] \tag{11}$$

for any $x \in [0, 1]$, $n \geq 1$. Here $\{f^{-n}(x)\}$ is the complete preimage of x with respect to f^n,

$$x_n^{min}(x) = \min_{\overline{x} \in \{f^{-n}(x)\}} \{x^*_n(\overline{x})\}.$$

The continuity of multifunction $\eta_{n,\,1}$ $(n \geq 1)$ follows from the equality (11) and previous lemma 2.4 (see [25]). Lemma 2.5 is correct.

Corollary 2.8. *The graphs of multifunctions $\eta_{n,\,1} : [0, 1] \to (2^{[0,\,1]})_m$ defined for the map $F^* \in T^1([0, 1]^2)$ are connected sets for any $n \geq 1$.*

The continuity property of multifunctions $\eta_{n,\,1} : [0, 1] \to (2^{[0,\,1]})_m$ makes it possible to prove lemma 2.6. Get over the proof of lemma 2.6.

Proof. 1. Formulas (6) and (11) imply the inclusion

$$[0,\, 1 - x_{n'}^{min}(x)] \subset [0,\, 1 - x_n^{min}(x)] \tag{12}$$

for any $x \in [0, 1]$ and for any $n' > n \geq 1$. Thus, $\eta_{n',\,1}(x) \subseteq \eta_{n,\,1}(x)$, and the set $\bigcap_{n=1}^{+\infty} \eta_{n,\,1}$ is not empty. Consider the sequence $\{\eta_{n!,\,1}\}_{n \geq 1}$. Then by lemmas 2.2 and 2.5 the equalities

$$\bigcap_{n=1}^{+\infty} \overline{\eta_{n!,\,1|Per(f)}} = \bigcap_{n=1}^{+\infty} \eta_{n!,\,1} = \bigcap_{n=1}^{+\infty} \eta_{n,\,1} \tag{13}$$

hold. By (13) the inclusion

$$\bigcap_{n=1}^{+\infty} \eta_{n,\,1} \subseteq \Omega(F^*). \tag{14}$$

is valid. Prove the equality

$$\bigcap_{n=1}^{+\infty} \eta_{n,\,1} = \Omega(F^*). \tag{15}$$

In fact, the equality (15) follows from the inclusion (14) if $\bigcap_{n=1}^{+\infty} \eta_{n,\,1} = [0, 1]^2$. Suppose that $\bigcap_{n=1}^{+\infty} \eta_{n,\,1} \neq [0, 1]^2$. Then $\bigcap_{n=1}^{+\infty} \eta_{n!,\,1} \neq [0, 1]^2$. Using formula (11) we obtain

$$[0, 1]^2 \setminus \bigcap_{n=1}^{+\infty} \eta_{n!,\,1} = \bigcup_{n=1}^{+\infty} \Big(\bigcup_{x \in [0,\,1]} \{x\} \times (1 - x_{n!}^{min}(x),\, 1] \Big), \tag{16}$$

[2]The equality $\Omega(F) = \overline{\bigcup_{x \in Per(f)} \{x\} \times \Omega(g_{x,\,n(x)})}$ is not correct for skew products of maps of an interval with a closed set of periodic points in the base, here $n(x)$ is the (least) period of a point $x \in Per(f)$ (see [30]). At the same time the equality $\Omega(F_{|\Omega(f) \times I_2}) = \overline{\bigcup_{x \in Per(f)} \{x\} \times \Omega(g_{x,\,n(x)})}$ is valid. In the example of the paper $\Omega(f) \times I_2 = [0, 1]^2$, and $\Omega(F^*_n) = \overline{\bigcup_{x \in Per(f)} \{x\} \times \Omega(g_{x,\,n(x)})} = \overline{\eta}_n = \eta_n$. The correctness of the equality $\overline{\eta}_n = \eta_n$ follows from lemma 2.4.

where $[0,\,1]^2 \setminus \bigcap\limits_{n=1}^{+\infty} \eta_{n!,\,1}$ is the nonempty set.

Let $(x^0;\, y^0) \notin \bigcap\limits_{n=1}^{+\infty} \eta_{n!,\,1}$, where $x^0 \in Per(f)$, $n(x^0)$ is the least period of x^0. Then by (14) and (16) there exists a natural number n_0 such that $y^0 \in (1 - x_{n_0!}^{min}(x^0),\, 1]$. Since the inequality $x_n^{min}(x^0) \geq x_{n_0!}^{min}(x^0)$ is correct for any $n \geq n_0!$, then $y^0 \in (1 - x_{n!}^{min}(x^0),\, 1]$ for all $n \geq n(x^0)$. Note that

$$F^*_{n!,\,1}(\{x^0\} \times \Omega(g_{x^0,\,n!})) = F^*_{n!}(\{x^0\} \times \Omega(g_{x^0,\,n!})) = \{x^0\} \times \Omega(g_{x^0,\,n!}).$$

Hence, $y^0 \in [0,\,1] \setminus \Omega(g_{x^0,\,n!})$ for all $n \geq n(x^0)!$, and a point $(x^0;\, y^0)$ wanders in the fiber $x^0 \times [0,\,1]$ with respect to auxiliary map $F^*_{n!,\,1}$ for any $n \geq n(x^0)$. By lemma 2.5 multifunctions $\eta_{n,\,1}$ are continuous. Therefore, a point $(x^0;\, y^0)$ is a wandering point of the map $F^*_{n!,\,1}$ for any $n \geq n(x^0)$. Then by the inclusion (12) there exists a universal neighborhood $U_{n(x^0)!}((x^0;\, y^0)) = U_{1,\,n(x^0)!}(x^0) \times U_{2,\,n(x^0)!}(y^0)$ of a point $(x^0;\, y^0)$ such that the equality

$$U_{n(x^0)!}((x^0;\, y^0)) \bigcap (F^*_{n!,\,1})^j (U_{n(x^0)!}((x^0;\, y^0))) = \emptyset$$

holds for all $n \geq n(x^0)$ and for all $j \geq 1$. Using formula (3), we obtain that for any $n \geq n(x^0)$, $j \geq 1$ the equality

$$U_{2,\,n(x^0)!}(y^0) \bigcap g^j_{x,\,n!}(U_{2,\,n(x^0)!}(y^0)) = \emptyset \tag{17}$$

is valid for all $x \in U_{1,\,n(x^0)!}(x^0)$. Using(17) for $j = 1$, we obtain the equality

$$U_{2,\,n_0}(y^0) \bigcap g_{x,\,n!}(U_{2,\,n_0}(y^0)) = \emptyset,$$

where $x \in U_{1,\,n(x^0)!}(x^0)$. Since any fiber map is a homeomorphism, then by the above the equality

$$U_{n(x^0)!}((x^0;\, y^0)) \bigcap (F^*)^n (U_{n(x^0)!}(x^0;\, y^0)) = \emptyset \tag{18}$$

holds for any $n \geq n(x^0)$. It means that $(x^0;\, y^0) \notin \Omega(F^*)$ (see definition 1.2) for any $x^0 \in Per(f)$, and all points $(x;\, y) \in U_{n(x^0)!}((x^0;\, y^0))$ are wandering for F^*.

Let $(x^0;\, y^0) \notin \bigcap\limits_{n=1}^{+\infty} \eta_{n!,\,1}$, where $x^0 \notin Per(f)$. By formula (3) the inequality

$$g_{x,\,n}(y) > g_{x,\,n+1}(y) \tag{19}$$

holds for any $(x;\, y) \in I^*$ (in particular, for $(x;\, y) = (x_0;\, y_0)$) and for $n \geq 1$. Then, using above, we obtain that $(x^0;\, y^0) \notin \Omega(F^*)$ for $x^0 \notin Per(f)$, i. e. the following inclusion

$$\Omega(F^*) \subseteq \bigcap_{n=1}^{+\infty} \eta_{n,\,1} \tag{20}$$

holds. The inclusions (14) and (20) prove the equality (15).

2. Note that $A^* = \Omega(F^*)$ is the global attractor of F^*. In fact, use inequality (19) for any $x \in [0,\,1]$ and $y = 1$. Then

$$[0,\,1]^2 \supset F^*([0,\,1]^2) \supset \ldots \supset (F^*)^n([0,\,1]^2) \supset \ldots \supset A^*. \tag{21}$$

Since $\omega_{F^*}((x;\, y)) \subset \Omega(F^*)$ for any point $(x;\, y) \in [0,\,1]^2$, where $\omega_{F^*}(\cdot)$ is the ω-limit set of a trajectory with respect to F^*, then (21) implies the equality $A^* = \bigcap\limits_{n=1}^{+\infty} (F^*)^n([0,\,1]^2)$. Hence, $A^* = \Omega(F^*)$ is the global attractor of F^* (see definition 1.1). Lemma 2.6 is proved. □

Using lemma 2.6, corollary 2.8 and [25], we obtain

Corollary 2.9. *The nonwandering set* $A^* = \Omega(F^*)$ *of the skew product* $F^* \in T^1([0,\,1]^2)$ *is the continuum.*

We finish the proof of theorem 2.1 giving the proof of geometric lemma 2.7.

Proof. 1. By lemma 2.2 and formula (3) the inclusion $[0,\,1] \times \{0\} \subset \Omega(F^*)$ holds. This inclusion, corollary 2.9 and Urysohn's definition of the dimension of a compact set (see e.g. [31]) imply the equality $dim(\Omega(F^*)) = 1$ for the topological dimension of $\Omega(F^*) = A^*$. Thus, A^* is the one-dimensional continuum.
By corollary 2.9 the slice $(\Omega(F^*))(x)$ of the nonwandering set $\Omega(F^*)$ for any $x \in [0,\,1]$ is a closed interval (possibly, degenerate); in addition, $0 \in (\Omega(F^*))(x)$.
2. Let x be an arbitrary f-periodic point, and $n = n(x)$ be its (least) period. Then we have

$$\Omega(\widetilde{g}_{f^i(x)}) = [0,\,1 - x^*_{n(x)}(x)]$$

for any $0 \le i \le n-1$. Using lemma 2.6 and equality (11), we obtain

$$(\Omega(F^*))(f^i(x)) = [0,\,1 - x^*_{n(x)}(x)]. \tag{22}$$

Since $x = 1$ is the homoclinic point to the fixed point $x = 0$ of the quotient map f, then for any $x \in Per(f)$ with the (least) period $n(x)$ the inequality $x^*_{n(x)}(x) \neq 1$ is valid. By (22) $(\Omega(F^*))(f^i(x))$ is the same nondegenerate closed interval for $0 \le i \le n-1$.
3. Succeeding [32], consider Ω-function $\zeta^{F^*} : [0,\,1] \to (2^{[0,\,1]})_m$ of the map F^*, i. e. the multifunction satisfying

$$\zeta^{F^*}(x) = (\Omega(F^*))(x) \text{ for any } x \in [0,\,1].$$

Note that the graph of ζ^{F^*} in $[0,\,1]^2$ coincides with the nonwandering set $\Omega(F^*)$. Since $[0,\,1]$ is a compact Hausdorff space, $\Omega(F^*)$ is the closed set, then Ω-function is upper semicontinuous, and continuity points of this function form a nonempty residual set $C(\zeta^{F^*})$ in $[0,\,1]$ (see [25]). Distinguish the residual (in $[0,\,1]$) subset $C_t(\zeta^{F^*}) \subset C(\zeta^{F^*})$ consisting of f-transitive points, i.e. points with dense f-trajectories in $[0,\,1]$.
Show that $(\Omega(F^*))(x) = \{0\}$ for any point $x \in C_t(\zeta^{F^*})$. In fact, let $\{\varepsilon_m\}_{m\ge1}$ be a sequence of positive numbers satisfying $\lim\limits_{m\to+\infty} \varepsilon_m = 0$. By lemma 2.2 there exists the sequence of f-periodic points $\{x_m\}_{m\ge1}$ such that $\lim\limits_{m\to+\infty} x_m = x$, and f-periodic orbits of the points $\{x_m\}_{m\ge1}$ uniformly approximate the interval $[0,\,1]$ up to ε_m[3] and have the minimal (least) period among all f-periodic orbits approximating $[0,\,1]$ up to ε_m. Then the equality $\lim\limits_{n\to+\infty} x^*_{n(x_m)}(x_m) = 1$ holds. Therefore, definition 2.3 and formula (22) imply

$$(\Omega(F^*))(x) = \underset{m\to+\infty}{Lim}(\Omega(F^*))(x_m) = \underset{m\to+\infty}{Lim}[0,\,1 - x^*_{n(x_m)}(x_m)] = \{0\}, \tag{23}$$

where $\underset{m\to+\infty}{Lim}(\cdot)_m$ is the topological limit of a sequence of sets (see [25]).
4. Prove the absence of an arc γ in A^* such that $pr_1(\gamma)$ is the nondegenerate interval, where pr_1 is the natural projection of $[0,\,1]^2$ onto the unit interval of x-axis, and $\gamma \bigcap ([0,\,1] \times \{0\}) = \emptyset$.
Suppose not. Then by item 3 the interval $pr_1(\gamma)$ contains a point $x \in C_t(\zeta^{F_*})$. In this case $(A^*)(x)$ is the nondegenerate closed interval containing the points $y = 0$ and $y = (\gamma)(x)$, where $(\gamma)(x) \neq 0$. It contradicts the equality (23). Hence, there is no an arc γ with the above properties. It means also that there is no a simple closed curve in A^*. This property of A^* with formulas (22) and (23) implies the correctness of two equalities: $ord_{A^*}(x;\,0) = 2$ for any point $x \in C_t(\zeta^{F_*})$ and $ord_{A^*}(x;\,0) = 3$ for any point $x \in Per(f) \setminus \{0\}$. The last equality means that $(Per(f) \setminus \{0\}) \times \{0\} \subset R(A^*)$. The property proved here implies also the inclusion

[3] We say that f-periodic orbits *uniformly approximate the unit interval up to* ε_m if for an arbitrary partition of $[0,\,1]$ with the parameter ε_m there exists a periodic orbit such that any interval of the partition contains at least one point of this periodic orbit.

$R(A^*) \subset [0,\,1] \times \{0\}$ and the equality $ord_{A^*}(x;\,0) = 3$ for any point $(x;\,0) \in R(A^*)$. Using lemma 2.2 we obtain the equality $\overline{R(A^*)} = [0,\,1] \times \{0\}$. Note that $ord_{A^*}(0;\,0) = 2$.
Let $\{\varepsilon_m\}_{m\geq 1}$ be a sequence of positive numbers such as in item 3. Let $(x;\,0)$ be an arbitrary point of the horizontal interval $[0,\,1] \times \{0\}$, and $\{(x_m;\,0)\}_{m\geq 1} \subset R(A^*) \bigcap Per(F^*)$ be any sequence of ramification points of A^*, where the sequence $\{x_m\}_{m\geq 1}$ is chose such as in item 3 too. Denote by γ_m vertical branches of A^* containing $(x_m;\,0)$. Let $l(\gamma_m)$ be the length of γ_m, $m \geq 1$. The equality

$$\lim_{m\to+\infty} l(\gamma_m) = 0 \tag{24}$$

follows immediately from item 3. It means, in particular, that in any neighborhood of any nondegenerate vertical branch of A^* there exists a countable set of vertical branches satisfying (24).
5. The topological structure of A^* is more complicated then the topological structure of a dendrite. In fact, the map $f(x) = 4x(1-x)$ has the set of continuum cardinality consisting of infinite minimal sets (see e.g. [26]). Let $M(f)$ be an arbitrary infinite minimal set of the quotient map f of the skew product F^*. Then $M(F^*) = M(f) \times \{0\}$ is an infinite minimal set of F^*. Let $x \in M(f)$ be an arbitrary uniformly recurrent point of f. Using formulas (11) and (15), we obtain the inclusion

$$(\Omega(F^*))(x) \supset [0,\, s(M(f))], \tag{25}$$

where $s(M(f))$ is supremum of the set $M(f)$, $0 < s(M(f)) < 1$. Hence, $M(F^*) \subset R(A^*)$, and the set $R(A^*)$ has continuum cardinality. At the same time any dendrite has at most countable set of ramification points. Since $x = \lim\limits_{q\to+\infty} f^{p_q}(x)$ for some number sequence $\{p_q\}_{q\geq 1}$, then as it follows from (25) A^* is not the locally connected set in any point $(x;\,y)$ for $x \in M(f)$, $y \in (A^*)(x)$. At the same time any dendrite is a locally connected continuum.
6. Any point $(x;\,0)$, where $x \in C_t(\zeta^{F^*})$, is a point of the local connectedness of A^*. In fact, let a sequence $\{(x_i;\,0)\}_{i\geq 1} \subset R(A^*)$ be such that $\lim\limits_{i\to+\infty} x_i = x$. Note that the skew product F^* is a local diffeomorphism in $[0,\,1]^2 \setminus (\{1/2\} \times [0,\,1])$ and $(1/2;\,0) \notin R(A^*)$. Therefore, $R(A^*)$ is an invariant set (i. e. the inclusion $F^*(R(A^*)) \subset R(A^*)$ is correct), and $\{(x^*_{n(x_i)}(x_i);\,0)\}_{i\geq 1} \subset R(A^*)$. Since $\overline{\{f^j(x)\}_{j\geq 0}} = [0,\,1]$, then there is $\lim\limits_{i\to+\infty} x^*_{n(x_i)}(x_i)$, and $\lim\limits_{i\to+\infty} x^*_{n(x_i)}(x_i) = 1$. Hence, analogously item 4, the equality $\lim\limits_{i\to+\infty} l(\gamma_i) = 0$ holds for the lengths of vertical branches over the points $(x_i;\,0)$. Thus, A^* is the locally connected set in any point $(x;\,0)$ if $x \in C_t(\zeta^{F^*})$. Claims (2.1.2), (2.1.3) are correct.
Claim (2.1.4) follows from equalities (6), (22) and formulas for the coordinate functions of F^*.
Lemma 2.7 is proved. □

It completes the proof of theorem 2.1.

Remark 2.10. Note that strange hyperbolic attractor in the plane (attractor by R.V.Plykin [9]) is not locally connected. In the example of this paper there is no even partial hyperbolicity with respect to y in the global attractor of the constructed map F^*.

Give the other point of view on the example constructed in this paper. This point of view is based on the concept of Ω-function of a skew product of maps of an interval. As it follows from the equality (22) and item 4 of the proof of lemma 2.7, any periodic point of the quotient map is a discontinuity point of Ω-function ζ^{F^*}. By the equality (23) and item 6 of the proof of lemma 2.7 any point of the set $C_t(\zeta^{F^*})$ is a continuity point of Ω-function ζ^{F^*}. It means that ζ^{F^*} has maximally admissible set of discontinuity points (in the sense of its structure) for upper semicontinuous multifunction. In this sense ζ^{F^*} is the analog of Riemann's function in the real analysis.

Finishing the paper, we formulate some problems.
1. It would be interesting to calculate Hausdorff dimension of the global attractor of the skew product F^* constructed in the paper.

2. Does there exist (at least, continuous) skew product in the plane possessing the global attractor with the properties of A^* (see theorem 2.1) such that some trajectory of this skew product has A^* as its ω-limit set?

References

[1] I.Z. Bezbakh, B.G. Zakharov, I.A. Prokhorov, *"Radiographical Characterization of Microsegregation in Crystals [in Russian]"*, In: Proceedings of the 6th International Conference "Growth of Monocrystals and Heat-Mass Transfer", vol.2, Obninsk, 2005, p.p.352-361.

[2] V. Visintin, *"Models of Phase Transitions"*, Birkhauser, Boston, 1996.

[3] A.A. Golovin, S.H. Davis, A.A. Nepomnyashchy, *"A Convective Cahn-Hilliard Model for the Formation of Facets and Carners in Crystal Growth"*, Physica D, vol.118, 1998, p.p.202-230.

[4] E.A. Lukashov, E.V. Radkevich, "Solidification and Structuresation of Instability Zones", Applied Math., No 1, p.p. 159-178, 2010.

[5] D.V. Anosov, *"Dynamical Systems in the 1960s: the Hyperbolic Revolution"*, In "Mathematical Events of the Twentieth Century [in Russian]", Fazis, Moscow 2003, p.p.1-18; English transl., Springer-Verlag, Berlin 2006, p.p.1-17.

[6] S. Smale, *"Differentiable Dynamical Systems"*, Bul. Amer. Math. Soc., vol.73, 1967, p.p.741-817.

[7] L.P. Shil'nikov, *"Homoclinic Trajectories: from Poincarẽ to Our Days"*, In "Mathematical Events of the Twentieth Century [in Russian]", Fazis, Moscow 2003, p.p.466-489; English transl., Springer-Verlag, Berlin 2006, p.p.347-370.

[8] M. Barge, *"Homoclinic Intersections and Indecomposability"*, Proc. Amer. Math. Soc., vol.101, 1987, 541-544.

[9] R.V. Plykin, *"Sources and Sinks of A-Diffeomorphisms of Surfices"*, Math. in the USSR-Sbornik, vol.23(2), 1974, p.p.233-253.

[10] R.V. Plykin, *"On the Geometry of Hyperbolic Attractors of Smooth Cascades"*, Rus. Math. Surv., vol.39(6), 1984, p.p.85-131.

[11] Y.N. Minsky, *"On Rigidity, Limit Sets and Ends of Hyperbolic 3-manifolds"*, Journ. of Amer. Math. Soc., vol.7, 1994, p.p.539-588.

[12] B.H. Bowditch, *"Hausdorff Dimension, and Dendritic Limit Sets"*, Math. Ann., vol.332, 2005, p.p.667-676.

[13] M. Ding, C. Grebogy, E. Ott, *"Evolution of Attractors in Quasiperiodically Forced Systems: From Quasiperiodic to Strange Nonchaotic to Chaotic"*, Physical Review A, vol.39(5), 1989, p.p. 2593-2598.

[14] A. Pikovsky, U. Feudal, *"Characterizing Strange Nonchaotic Attractors"*, CHAOS, vol.5(1), 1995, p.p.253-260.

[15] T.H. Jäger, *"On the Structure of Strange Nonchaotic Attractors in Pinched Skew Products"*, Ergod. Theory and Dynam. Sys., vol.27, 2007, p.p.493-510.

[16] L.S. Efremova, *"Differential Properties and Attracting Sets of a Simplest Skew Product of Interval Maps"*, Sbornik: Mathematics, vol.201(6), 2010, p.p.873-907.

[17] L.S. Efremova, *"New Set-valued Functions in the Theory of Skew Products of Interval Maps"*, Nonlinear Anal., vol.47(8), 2001, p.p. 5297-5308.

[18] L.S. Efremova, *"Set-valued Functions and Dynamics of Skew Products of Interval Maps"*, Progress in Nonlinear Science. Intern. Conf. Dedicated to the 100th Anniversary of A.A.Andronov. Nizhny Novgorod. Russia. July 2-6, 2001. Proceedings, vol.1, Math. Problems of Nonlinear Dynamics, 2002, p.p.219-224.

[19] L.S. Efremova, *"Space of C^1-Smooth Skew Products of Maps of an Interval"*, Theor. and Math. Physics, vol.164(3), 2010, p.p. 1208-1214.

[20] A. Katok, B. Hasselblatt, *"Introduction to the Modern Theory of Dynamical Systems"*, Encyclopedia Math. Appl., vol.54, Cambridge Univ. Press, Cambridge, 1995.

[21] S. Baldwin, *"Continuous Itinerary Functions and Dendrite Maps"*, Topol. Appl., vol. 154, 2007, p.p.2889-2038.

[22] F. Balibrea, R. Hric, L. Snoha, *"Minimal Sets on Graphs and Dendrites"*, Intern. Journ. Bifurcations and Chaos, vol.13, 2007, p.p.1721-1725.

[23] L.S. Efremova, E.N. Makhrova, *"The Dynamics of Monotone Maps of Dendrites"*, Sbornik: Mathematics, vol.192(6), 2001, p.p.807-821.

[24] J.H. Mai, E.H. Shi, *"$\overline{R} = \overline{P}$ for Maps of Dendrites X with $card(End(X)) < \mathbf{c}$"*, Int. Journ. Bifurcations and Chaos, vol.19, 2009, p.p. 1391-1396.

[25] K. Kuratowski, *"Topology"*, vol.1,2, Academic Press, New York-London, 1966, 1968.

[26] A.N. Sharkovskii, Y.L. Maistrenko, E.Y. Romanenko, *"Difference Equations and Their Applications [in Russian]"*, Naukova Dumka, Kiev, 1986; English transl., Kluwer, Academic Publishers, Dordrecht, 1993.

[27] A.N. Sharkovskii, *"On the Attracting and Attractive Sets"*, Soviet Math. Dokl., vol.160 (5), 1965, p.p.1036-1038.

[28] E. D'Aniello, T. Smith, *"Approximating ω-limit Sets with Periodic Orbits"*, Aequationes Math., vol.75, 2008, p.p.93-102.

[29] L.S. Efremova, *On the Nonwandering Set and the Center of Some Skew Products of Interval Maps*, Russ. Math. (Iz.VUZ), vol.50(10), 2006, p.p.17-25.

[30] J.L.G. Guirao, R.G.Rubio, *Nonwandering set of points of skew product maps with base having closed set of periodic points*, JMAA, vol.362 (2), 2010, p.p.350-354.

[31] P. Urysohn, *"Memoire sur les multiplicités cantoriennes"*, Fundam. Math. vol.7, 1925, p.p. 30-139; vol.8, 1926, p.p. 225-359.

[32] L.S. Efremova, *"On the Concept of Ω-function for a Skew Product of Interval Maps [in Russian]"*, VINITI, Moscow, Itogi nauki i tekhniki. Covremennaya matematika i ee prilozheniya, vol.67, 1999, p.p.129-160; English transl., Journ Math. Sci. (New York), vol.105, 2001, p.p. 1779-1798.