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Danièle Fournier-Prunaret, Laura Gardini, & Ludwig Reich, Editors

SOME REMARKS ON THE GENERAL THEOREM OF THE EXISTENCE OF ITERATIVE ROOTS OF HOMEOMORPHISMS WITH A RATIONAL ROTATION NUMBER

Paweł Solarz [1]

Abstract. We show that the theorem proved in [8] generalises the previous results concerning orientation-preserving iterative roots of homeomorphisms of the circle with a rational rotation number (see [2], [6], [10] and [7]).

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Résumé. Nous montrons que le théorème prouvé dans [8] généralise les résultats précédents concernant les racines itérées préservant l'orientation d'homéomorphismes du cercle avec un nombre de rotation rationnel (voir [2], [6], [10] et [7]).
Mots clefs. itération, racine itérée, point périodique, nombre de rotation, application préservant l'orientation.

A function $g\colon X \to X$, where $X \neq \emptyset$ is called an *iterative root* of a given function $f\colon X \to X$ if $g^m(x) = f(x)$ for $x \in X$, here g^m denotes the m-th iterate of g and $m \geq 2$, called the *order of an iterative root*, is an integer.

It seems that it was M. C. Zdun, who first started dealing with the problem of the existence of continuous and orientation-preserving iterative roots of an orientation-preserving homeomorphism of the circle, the homeomorphism with periodic or fixed points. Recall that $x \in X$ is a *periodic point* of order $n \in \mathbb{N}$, $n > 1$ of f, if

$$f^n(x) = x \text{ and } f^k(x) \neq x \text{ for } k \in \{1, \ldots, n-1\}.$$

If $f(x) = x$ then x is said to be a *fixed point of* f. The set of all periodic (fixed) points of f will be denoted by $\operatorname{Per} f$ ($\operatorname{Fix} f$). In [9] Zdun solved the problem of the embedding of some homeomorphisms of the circle $S^1 := \{z \in \mathbb{C} : \ |z| = 1\}$ in a continuous flow. Form these results one may conclude that if $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $S^1 \neq \operatorname{Fix} F \neq \emptyset$, resp. $\operatorname{Per} F = S^1$, then F has infinitely many iterative roots of any order with fixed, resp. periodic points.

Three years later J.H. Mai in [5] gave some conditions for the existence of continuous iterative roots of F with $S^1 \neq \operatorname{Fix} F \neq \emptyset$ or $S^1 \neq \operatorname{Per} F \neq \emptyset$.

In 2003 W. Jarczyk proved in [2] that if $\operatorname{Fix} F = S^1$, i.e., F is the identity function, then F has infinitely many, depending on an arbitrary function, continuous and orientation-preserving iterative roots with periodic points. In the same year in [6] a similar result was proved for F such that $\operatorname{Per} F = S^1$.

In 2008 M.C. Zdun in [10] proved so called factorization theorem and as an application of it he gave the necessary and sufficient conditions for the existence of iterative roots with periodic points for F such that $S^1 \neq \operatorname{Fix} F \neq \emptyset$.

[1] Pedagogical University, Institute of Mathematics, ul. Podchorążych 2, PL-30-084 Krakó w, Poland; e-mail: psolarz@up.krakow.pl.

The mentioned Zdun's theorem was also used in [7] in the proof of Theorem 2 and for proving the general theorem in [8].

In this paper we show that the mentioned theorem from [8] generalises the results from [7], [10], [6] and [2]. Before that we recall some useful facts and definitions. From now on we set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and $\mathbb{Z}_n^* = \{1, \ldots, n-1\}$ for a suitable natural n.

Let $u, w \in S^1$ and $u \neq w$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + 1$ and $u = e^{2\pi i t_1}$ and $w = e^{2\pi i t_2}$. Put

$$\overrightarrow{(u, w)} := \{e^{2\pi i t},\ t \in (t_1, t_2)\},\ \overrightarrow{[u, w]} := \overrightarrow{(u, w)} \cup \{u, w\},\ \overrightarrow{[u, w)} := \overrightarrow{(u, w)} \cup \{u\}.$$

These sets are called arcs.

For every homeomorphism $F\colon S^1 \to S^1$ there exists a unique (up to a translation by an integer) homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$F\left(e^{2\pi i x}\right) = e^{2\pi i f(x)}$$

and

$$f(x+1) = f(x) + k$$

for all $x \in \mathbb{R}$, where $k \in \{-1, 1\}$. We call F *orientation-preserving* if $k = 1$, which is equivalent to the fact that f is increasing.

Moreover, for every continuous function $G\colon I \to J$, where $I = \{e^{2\pi i t},\ t \in [a, b]\}$ and $J = \{e^{2\pi i t},\ t \in [c, d]\}$ there exists a unique continuous function $g\colon [a, b] \to [c, d]$ such that

$$G\left(e^{2\pi i x}\right) = e^{2\pi i g(x)}, \qquad x \in [a, b].$$

In this case we also call g the lift of G and we say that G preserves the orientation if g is strictly increasing.

For any orientation-preserving homeomorphism $F\colon S^1 \to S^1$, the limit

$$\alpha(F) := \lim_{n \to \infty} \frac{f^n(x)}{n} \pmod 1, \qquad x \in \mathbb{R}$$

always exists and does not depend on the choice of x and f. This number is called the *rotation number* of F (see [1]). It is known that $\alpha(F)$ is a rational and positive number if and only if F has a periodic point (see for example [1]). If $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where q, n are positive integers with $0 < q < n$ and $\gcd(q, n) = 1$, then $\operatorname{Per} F$ contains only periodic points of order n (see [7], [4]). Moreover, there exists a unique number $p \in \mathbb{Z}_n^*$ satisfying $pq = 1 \pmod n$. This number will be called the *characteristic number of* F and denoted $\operatorname{char} F := p$ (see [10]). If $\operatorname{Fix} F \neq \emptyset$, then $\alpha(F) = 0$ and we define $\operatorname{char} F := 1$.

Recall that if G is an orientation-preserving iterative root of F, where $\alpha(F) = \frac{q}{n}$, $0 \leq q < n$ and $\gcd(q, n) = 1$, then there are an integer $l \geq 1$ and $q' \in \mathbb{Z}_{nl}$ such that $\alpha(G) = \frac{q'}{ln}$ and $\gcd(q', nl) = 1$ (see for example Lemma 2 in [8]) and thus G has periodic of fixed points of order nl.

In view of Theorem 5 (see [10]) every orientation-preserving homeomorphism $F\colon S^1 \to S^1$ possessing periodic points of order n is of the form

$$F(z) = \begin{cases} T^q(F^n(z)), & z \in I_0 = \overrightarrow{\langle z_0, F^{\operatorname{char} F}(z_0) \rangle}, \\ T^q(z), & z \in S^1 \setminus I_0, \end{cases}$$

where $z_0 \in \operatorname{Per} F$, $q = n\alpha(F)$ and $T\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $T^n = \mathrm{id}_{S^1}$. The function $T = T_{z_0}(F)$ is unique up to a periodic point of F and it is called the *Babbage function of* F (see [10]). Of course if $\operatorname{Per} F = S^1$, we have $T_{z_0}(F) = F^{\operatorname{char} F}$. For an orientation-preserving homeomorphism such that $\operatorname{Fix} F \neq \emptyset$ we may assume that the Babbage homeomorphism of F is the identity function.

For the convenience we recall Theorem 2 from [8].

Theorem 1.

Let $m \geq 2$ and $l \geq 1$ be integers and let $F\colon S^1 \to S^1$ be an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where $0 \leq q < n$ and $\gcd(q,n) = 1$. F has continuous and orientation-preserving iterative root of order m with periodic points of order ln if and only if the following conditions are fulfilled:

(i) *$\frac{m}{l} =: m' \in \mathbb{Z}$ and there is $q' \in \mathbb{Z}^*_{nl}$ such that $\gcd(q', ln) = 1$ and $q'm' = q\,(\mathrm{mod}\ n)$;*

(ii) *for some $z_0 \in \mathrm{Per}\,F$ there is a partition of $I_0 := \overrightarrow{\langle z_0, F^{\mathrm{char}\,F}(z_0)\rangle}$ onto l consecutive disjoint arcs $J_0, \ldots, J_{l-1}$ such that $F^n[J_i] = J_i$, $i \in \mathbb{Z}_l$ and if $l > 1$, then there exist orientation-preserving homeomorphisms $V_i\colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$ satisfying*

$$F^n_{|J_{i+1}} = V_i \circ F^n_{|J_i} \circ V_i^{-1}, \quad i \in \mathbb{Z}_{l-1}. \tag{1}$$

For any $z_0 \in \mathrm{Per}\,F$, m, l, q', arcs $J_0, \ldots, J_{l-1}$ and homeomorphisms $V_i\colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$ satisfying (1) the iterative root $G\colon S^1 \to S^1$ of F is of the form:

$$G(z) := \begin{cases} V^{q'}(G_0(z)), & z \in J_0, \\ V^{q'}(z), & z \in S^1 \setminus J_0, \end{cases} \tag{2}$$

where $G_0\colon J_0 \to J_0$ is an orientation-preserving homeomorphism such that $\mathrm{Fix}\,G_0 \neq \emptyset$ and $G_0^{m'} = F^n_{|J_0}$ and $V = \Psi^{\mathrm{char}\,F}$ if $l = 1$ or

$$V(z) := \begin{cases} V_i(z), & z \in J_i,\ i \in \mathbb{Z}_{l-1}, \\ V_{l-1}(z) := \Psi^{\mathrm{char}\,F} \circ V_0^{-1} \circ \ldots \circ V_{l-2}^{-1}(z), & z \in J_{l-1}, \\ \Psi^{d\mathrm{char}\,F} \circ V_i \circ \Psi^{-d\mathrm{char}\,F}(z), & z \in F^{d\mathrm{char}\,F}[J_i], \\ & \quad i \in \mathbb{Z}_l,\ d \in \mathbb{Z}^*_n \end{cases} \tag{3}$$

if $l > 1$, where $\Psi\colon S^1 \to S^1$ is given by

$$\Psi(z) := T^q \circ T^d \circ G_i^{\beta_{i,d}} \circ T^{-d}(z), \quad z \in F^{d\mathrm{char}\,F}[J_i],\ d \in \mathbb{Z}_n,\ i \in \mathbb{Z}_l, \tag{4}$$

where

$$G_j := V_j \circ G_{j-1} \circ V_j^{-1}, \quad j \in \mathbb{Z}^*_l, \tag{5}$$

$T = T_{z_0}(F)$ denotes the Babbage homeomorphism of F and

$$\beta_{i,d} := \begin{cases} m' - \left[\frac{m}{nl}\right] - 1, & \text{if } d = 0,\ i'_i \leq m - \left[\frac{m}{nl}\right] nl - 1, \\ m' - \left[\frac{m}{nl}\right], & \text{if } d = 0,\ i'_i > m - \left[\frac{m}{nl}\right] nl - 1, \\ -\left[\frac{m}{nl}\right] - 1, & \text{if } d \in \mathbb{Z}^*_n,\ i'_{i+dl} \leq m - \left[\frac{m}{nl}\right] nl - 1, \\ -\left[\frac{m}{nl}\right], & \text{if } d \in \mathbb{Z}^*_n,\ i'_{i+dl} > m - \left[\frac{m}{nl}\right] nl - 1 \end{cases} \tag{6}$$

for $i \in \mathbb{Z}_l$ with $i'_k \in \mathbb{Z}_{nl}$ uniquely determined by $(k + i'_k q')\,(\mathrm{mod}\ nl) = 0$ for $k \in \mathbb{Z}_{nl}$.

Moreover, every orientation-preserving iterative root of order m of F with periodic points of order nl (if exists) may be expressed by (2) – (6).

In order to obtain the first corollary, suppose that $F\colon S^1 \to S^1$ is an orientation-preserving homeomorphism such that $S^1 \neq \mathrm{Per}\,F \neq \emptyset$, $\alpha(F) = \frac{q}{n}$, where $\gcd(q,n) = 1$ and that $l = 1$. Then condition (i) of Theorem 1 takes the following form: there exists a $q' \in \mathbb{Z}^*_n$ such that $\gcd(q', n) = 1$ and $q'm = q\,(\mathrm{mod}\ n)$ which is equivalent to $\gcd(m,n) = 1$ (see Remarks 2 and 3 in [7]). As $l = 1$ we have $m' = m$, $V = \Psi^{\mathrm{char}\,F}$ and $J_0 = I_0 = \overrightarrow{\langle z_0, F^{\mathrm{char}\,F}(z_0)\rangle}$ for some $z_0 \in \mathrm{Per}\,F$. Moreover, in (4) and (6) we have $i = 0$, $d \in \mathbb{Z}_n$, thus $\beta_{i,d} = \beta_{0,d} =: \beta_d$ for $d \in \mathbb{Z}_n$. In addition, $i'_0 = 0 \leq m - \left[\frac{m}{n}\right] n - 1$. Therefore we get

Corollary **1.**

Let $F\colon S^1 \to S^1$ *be an orientation-preserving homeomorphism such that* $\operatorname{Per} F \neq S^1$ *and* $\alpha(F) = \frac{q}{n}$*, where* $\gcd(q,n) = 1$*.* F *has orientation-preserving, continuous iterative root of order* $m \geq 2$ *with periodic point of order* n *if and only if* $\gcd(m,n) = 1$*.*

For every such an m *and* $z_0 \in \operatorname{Per} F$ *the iterative root* $G\colon S^1 \to S^1$ *of* F *is of the form*

$$G(z) := \begin{cases} \left(\Psi^{\operatorname{char} F}\right)^{q'}(G_0(z)), & z \in I_0, \\ \left(\Psi^{\operatorname{char} F}\right)^{q'}(z), & z \in S^1 \setminus I_0, \end{cases} \tag{7}$$

where $q' \in \mathbb{Z}_n^*$ *is such that* $q'm = q \,(\operatorname{mod}\, n)$*,* $G_0\colon I_0 \to I_0$ *is an orientation-preserving homeomorphism such that* $\operatorname{Fix} G_0 \neq \emptyset$*,* $G_0^m = F^n_{|I_0}$ *and* $\Psi\colon S^1 \to S^1$ *is given by*

$$\Psi(z) := T^q \circ T^d \circ G_0^{\beta_d} \circ T^{-d}(z), \qquad z \in F^{d \operatorname{char} F}[I_0],\ d \in \mathbb{Z}_n, \tag{8}$$

where $T = T_{z_0}(F)$ *is the Babbage'a homeomorphism of* F *and*

$$\beta_d := \begin{cases} m - \left[\frac{m}{n}\right] - 1, & d = 0, \\ -\left[\frac{m}{n}\right] - 1, & d \in \mathbb{Z}_n^*,\ i'_d \leq m - \left[\frac{m}{n}\right] n - 1, \\ -\left[\frac{m}{n}\right], & d \in \mathbb{Z}_n^*,\ i'_d > m - \left[\frac{m}{n}\right] n - 1 \end{cases} \tag{9}$$

with $i'_d \in \mathbb{Z}_n$ *such that* $(d + i'_d q') \,(\operatorname{mod}\, n) = 0$ *for* $d \in \mathbb{Z}_n$*.*

Moreover, every orientation-preserving iterative root of order m *of* F *with periodic points of order* n *(if exists) may be expressed by* (7) *–* (9)*.*

This is a combination of Lemma 2 and Theorem 2 form [7].

Assume $l > 1$ and let F be such that $\emptyset \neq \operatorname{Fix} F \neq S^1$. For such a homeomorphism we have $\alpha(F) = 0$, thus $q = 0$ and $n = 1$. By the definition, $\operatorname{char} F = 1$. Notice that for every m such that $\frac{m}{l} \in \mathbb{Z}$ and every $q' \in \mathbb{Z}_l^*$ such that $\gcd(q', l) = 1$ we have $q'm' = 0 \pmod 1$, therefore the condition (i) of Theorem 1 is satisfied if $\frac{m}{l} \in \mathbb{Z}$. Moreover, in this case $\overrightarrow{\langle z_0, F^{\operatorname{char} F}(z_0)\rangle} = \overrightarrow{\langle z_0, F(z_0)\rangle} = S^1$ and $T_{z_0}(F) = \operatorname{Id}_{S^1}$ for any $z_0 \in \operatorname{Fix} F$. Furthermore, in (4) and (6) we have $d = 0$, $i \in \mathbb{Z}_l$ hence $i'_i \geq 0 > m - \left[\frac{m}{l}\right] l - 1 = -1$ for $i \in \mathbb{Z}_l$ and consequently, $\beta_{i,0} = 0$ for $i \in \mathbb{Z}_l$. Thus $\Psi = \operatorname{Id}_{S^1}$ and we get

Corollary **2.**

Let $m, l \geq 2$ *be integers. An orientation-preserving homeomorphism* $F\colon S^1 \to S^1$ *with* $\emptyset \neq \operatorname{Fix} F \neq S^1$ *has a continuous and orientation-preserving iterative root of order* m *with periodic point of order* l *if and only if:*

(i) $\frac{m}{l} =: m' \in \mathbb{Z}$*;*

(ii) *there is a partition of* S^1 *onto* l *consecutive arcs* $J_0, \ldots, J_{l-1}$ *such that* $F[J_i] = J_i$*,* $i \in \mathbb{Z}_l$ *and there are orientation-preserving homeomorphisms* $V_i\colon J_i \to J_{i+1}$*,* $i \in \mathbb{Z}_{l-2}$ *such that*

$$F_{|J_{i+1}} = V_i \circ F_{|J_i} \circ V_i^{-1}, \quad i \in \mathbb{Z}_{l-1}.$$

For every such a partition $\{J_i\}_{i \in \mathbb{Z}_l}$*, numbers* m, l*, homeomorphisms* V_i*,* $i \in \mathbb{Z}_{l-1}$ *and each* $0 < q' < l$ *such that* $\gcd(q', l) = 1$ *the function*

$$G(z) := \begin{cases} V^{q'}(G_0(z)), & z \in J_0, \\ V^{q'}(z), & z \in S^1 \setminus J_0, \end{cases} \tag{10}$$

where $G_0\colon J_0 \to J_0$*,* $G_0^{m'}(z) = F(z)$ *for* $z \in J_0$ *and* $\operatorname{Fix} G_0 \neq \emptyset$ *and*

$$V(z) := \begin{cases} V_i(z), & z \in J_i,\ i \in \mathbb{Z}_{l-1}, \\ V_0^{-1} \circ V_1^{-1} \circ \ldots \circ V_{l-2}^{-1}(z), & z \in J_{l-1}, \end{cases} \tag{11}$$

is the iterative root of F *such that* $\operatorname{Per} F \neq \emptyset$.
Moreover, every continuous and orientation-preserving iterative root of F *of order* m *having periodic points of order* l *is given by* (10) *and* (11).

Taking $l = m$ in the above corollary we get Theorem 11 from [10].

Now turn to the case $\operatorname{Per} F = S^1$ and $l > 1$. As $F^n = \operatorname{Id}_{S^1}$ the condition (ii) of Theorem 2 is fulfilled for any partition of $I_0 = \overrightarrow{\langle z_0, F^{\operatorname{char} F}(z_0) \rangle}$, where $z_0 \in S^1$ and any orientation-preserving homeomorphisms $V_i \colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$. In addition, $G_0 \colon J_0 \to J_0$ is the identity function as $\operatorname{Fix} G_0 \neq \emptyset$ and $G_0^{m'} = F^n_{|J_0} = \operatorname{Id}_{J_0}$. This implies

$$\Psi = T^q = \left(F^{\operatorname{char} F}\right)^q = F,$$

as $T = T_{z_0}(F) = F^{\operatorname{char} F}$. Hence

Corollary **3.**

Let $m, l \geq 2$ *be integers,* $F \colon S^1 \to S^1$ *be an orientation-preserving homeomorphism such that* $\operatorname{Per} F = S^1$ *and* $\alpha(F) = \frac{q}{n}$*, where* $0 < q < n$ *and* $\gcd(q, n) = 1$. F *has a continuous and orientation-preserving iterative root of order* m *with periodic point of order* nl *if and only if:*

(i) $\frac{m}{l} =: m' \in \mathbb{Z}$ *and there is* $q' \in \mathbb{Z}^*_{ln}$ *such that* $\gcd(q', ln) = 1$ *and* $q'm' = q \,(\operatorname{mod}\, n)$.

For every m, q', l *satisfying (i) every partition* $\{J_i\}_{i \in \mathbb{Z}_l}$ *of* I_0 *and every orientation-preserving homeomorphisms* $V_i \colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$ *the iterative root* $G \colon S^1 \to S^1$ *of* F *is of the form:*

$$G(z) := V^{q'}(z), \quad z \in S^1, \tag{12}$$

where

$$V(z) := \begin{cases} V_i(z), & z \in J_i,\ i \in \mathbb{Z}_{l-1}, \\ V_{l-1}(z) := F^{\operatorname{char} F} \circ V_0^{-1} \circ \ldots \circ V_{l-2}^{-1}(z), & z \in J_{l-1}, \\ F^{d \operatorname{char} F} \circ V_i \circ F^{-d \operatorname{char} F}(z), & z \in F^{d \operatorname{char} F}[J_i], \\ & \quad i \in \mathbb{Z}_l,\ d \in \mathbb{Z}^*_n. \end{cases} \tag{13}$$

Moreover, every continuous and orientation-preserving iterative root of F *of order* m *having periodic points of order* nl *is given by* (12) *and* (13).

This is slightly modified Theorem 2 from [6].

Assume that $\operatorname{Per} F = S^1$ and $l = 1$. Similarly as in Corollary 1 the condition (i) is equivalent to $\gcd(m, n) = 1$ and the condition (ii) is not the case. Moreover, $\Psi = F$ and $V = F^{\operatorname{char} F}$. Hence

Corollary **4.**

Let $m \geq 2$ *be integer,* $F \colon S^1 \to S^1$ *be an orientation-preserving homeomorphism such that* $\operatorname{Per} F = S^1$ *and* $\alpha(F) = \frac{q}{n}$*, where* $0 < q < n$ *and* $\gcd(q, n) = 1$. F *has a continuous and orientation-preserving iterative root of order* m *with periodic point of order* n *if and only if* $\gcd(m, n) = 1$. *For every* $m \geq 2$ *and every* $q' \in \mathbb{Z}^*_n$ *such that* $\gcd(q', n) = 1$ *and* $q'm = q \,(\operatorname{mod}\, n)$ *the iterative root* $G \colon S^1 \to S^1$ *of* F *is of the form:*

$$G(z) = \left(F^{\operatorname{char} F}\right)^{q'}(z), \quad z \in S^1.$$

This is Corollary 4 from [6]. In [6] this result was obtained independently on Theorem 2.

Finally suppose that $l > 1$ and $\operatorname{Fix} F = S^1$, i.e., $F = \operatorname{Id}_{S^1}$. Since $q = 0$, $n = 1$, we have $\operatorname{char} F = 1$, $I_0 = S^1$ and $T = T_{z_0}(F) = \operatorname{Id}_{S^1}$ for any $z_0 \in S^1$. Notice that if $\frac{m}{l} \in \mathbb{Z}$ then every $q' \in \mathbb{Z}^*_l$ such that $\gcd(q', l) = 1$ satisfies $m'q' = 0 \,(\operatorname{mod}\, 1)$. The condition (ii) is obviously fulfilled for any partition $\{J_i\}_{i \in \mathbb{Z}_l}$ of S^1 and any orientation-preserving homeomorphisms $V_i \colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$. Moreover as $G_0 = \operatorname{Id}_{J_0}$ we have $\Psi = \operatorname{Id}_{S^1}$. Therefore from Theorem 1 it follows the Jarczyk's theorem from [2].

Corollary **5.**

The function $F = \mathrm{Id}_{S^1}$ *has continuous and orientation-preserving iterative roots of order* $m \geq 2$ *with periodic points of order* $l > 1$ *if and only if* $\frac{m}{l} \in \mathbb{Z}$.

For every such numbers m, l, *every partition* $\{J_i\}_{i\in\mathbb{Z}_l}$ *of* S^1 *and every orientation-preserving homeomorphisms* $V_i \colon J_i \to J_{i+1}$, $i \in \mathbb{Z}_{l-1}$, *the function*

$$G(z) := V^{q'}(z), \quad z \in S^1, \tag{14}$$

where $q' \in \mathbb{Z}_l^*$ *is such that* $\gcd(q', l) = 1$ *and*

$$V(z) := \begin{cases} V_i(z), & z \in J_i, \ i \in \mathbb{Z}_{l-1}, \\ V_0^{-1} \circ V_1^{-1} \circ \ldots \circ V_{l-2}^{-1}(z), & z \in J_{l-1} \end{cases} \tag{15}$$

is the orientation-preserving continuous iterative root of Id_{S^1} *of order* l. *Moreover, every continuous and orientation-preserving iterative root of* Id_{S^1} *of order* m *having periodic points of order* l *is given by* (14) *and* (15).

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