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Danièle Fournier-Prunaret, Laura Gardini, & Ludwig Reich, Editors

ON THE FORMAL FIRST COCYCLE EQUATION FOR ITERATION GROUPS OF TYPE II

Harald Fripertinger[1] and Ludwig Reich[1]

Abstract. Let x be an indeterminate over $\mathbb{C}$. We investigate solutions

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n,$$

$\alpha_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq 0$, of the first cocycle equation

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t, F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

in $\mathbb{C}[\![x]\!]$, the ring of formal power series over $\mathbb{C}$, where $(F(s,x))_{s\in\mathbb{C}}$ is an iteration group of type II, i.e. it is a solution of the translation equation

$$F(s+t,x) = F(s,F(t,x)), \qquad s,t \in \mathbb{C}, \tag{T}$$

of the form $F(s,x) \equiv x + c_k(s)x^k \mod x^{k+1}$, where $k \geq 2$ and $c_k \neq 0$ is necessarily an additive function. It is easy to prove that the coefficient functions $\alpha_n(s)$ of

$$\alpha(s,x) = 1 + \sum_{n\geq 1} \alpha_n(s)x^n$$

are polynomials in $c_k(s)$.

It is possible to replace this additive function c_k by an indeterminate. Finally, we obtain a formal version of the first cocycle equation in the ring $(\mathbb{C}[y])[\![x]\!]$. We solve this equation in a completely algebraic way, by deriving formal differential equations or an Aczél–Jabotinsky type equation. This way it is possible to get the structure of the coefficients in great detail which are now polynomials. We prove the universal character of these polynomials depending on certain parameters, the coefficients of the generator K of a formal cocycle for iteration groups of type II. Rewriting the solutions $\Gamma(y,x)$ of the formal first cocycle equation in the form $\sum_{n\geq 1} \psi_n(x)y^n$ as elements of $(\mathbb{C}[\![x]\!])[\![y]\!]$, we obtain explicit formulas for ψ_n in terms of the derivatives $H^{(j)}(x)$ and $K^{(j)}(x)$ of the generators H and K and also a representation of $\Gamma(y,x)$ similar to a Lie–Gröbner series. There are interesting similarities between the solutions $G(y,x)$ of the formal translation equation for iteration groups of type II and the solutions $\Gamma(y,x)$ of the formal first cocycle equation for iteration groups of type II.

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[1] Institut für Mathematik, Karl-Franzens-Universität Graz, Heinrichstr. 36/4, A–8010 Graz, Austria;
emails: `harald.fripertinger@uni-graz.at` & `ludwig.reich@uni-graz.at`.

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Résumé. Soit x une indéterminée dans $\mathbb{C}$. Nous étudions les solutions

$$\alpha(s,x)=\sum_{n\geq 0}\alpha_n(s)x^n,$$

$\alpha_n\colon \mathbb{C}\to\mathbb{C}$, $n\geq 0$, de la première équation de cocycle

$$\alpha(s+t,x)=\alpha(s,x)\alpha\big(t,F(s,x)\big),\qquad s,t\in\mathbb{C}, \tag{Co1}$$

dans $\mathbb{C}[\![x]\!]$, l'anneau des séries entières formelles sur $\mathbb{C}$, où $(F(s,x))_{s\in\mathbb{C}}$ est un groupe d'itération de type II, c'est-à-dire une solution de l'équation de translation

$$F(s+t,x)=F(s,F(t,x)),\qquad s,t\in\mathbb{C}, \tag{T}$$

de la forme $F(s,x)\equiv x+c_k(s)x^k \bmod x^{k+1}$, où $k\geq 2$ et $c_k\neq 0$ est nécessairement une fonction additive. Il est facile de démontrer que les fonctions coefficients $\alpha_n(s)$ de

$$\alpha(s,x)=1+\sum_{n\geq 1}\alpha_n(s)x^n$$

sont des polynômes dans $c_k(s)$.

Il est possible de remplacer cette fonction addtitive c_k par une indéterminée. Finalement, nous obtenons une version formelle de la première équation de cocycle dans l'anneau $(\mathbb{C}[y])[\![x]\!]$. Nous résolvons cette équation d'une manière complètement algébrique, en dérivant formellement les équations différentielles ou une équation de type Aczél–Jabotinsky. De cette manière il est possible d'obtenir la structure détaillée des coefficients qui sont maintenant des polynômes. Nous montrons le caractère universel de ces polynômes en fonction de certains paramètres, les coefficients du générateur K d'un cocycle formel pour les groupes d'itération de type II. En réécrivant les solutions $\Gamma(y,x)$ de la première équation de cocycle sous la forme $\sum_{n\geq 1}\psi_n(x)y^n$ comme des éléments de $(\mathbb{C}[\![x]\!])[\![y]\!]$, nous obtenons des formules explicites pour ψ_n en terme des dérivées $H^{(j)}(x)$ et $K^{(j)}(x)$ des générateurs H et K et également une représentation de $\Gamma(y,x)$ similaire à une série de Lie–Gröbner. Il y a des similarités intéressantes entre les solutions $G(y,x)$ de l'équation de translation formelle pour les groupes d'itération de type II et les solutions $\Gamma(y,x)$ de la première équation formelle de cocycle pour les groupes d'itération de type II.

Mots clefs. Première équation de cocycle, équations fonctionnelles formelles, groupes d'itération de type II, anneau des séries formelles sur $\mathbb{C}$.

1. Introduction

In [2] we introduce the method of "formal functional equations" to solve the translation equation (and the associated system of cocycle equations) in rings of formal power series over $\mathbb{C}$ in the case of iteration groups of type I. Let $\mathbb{C}[\![x]\!]$ be the ring of formal power series $F(x)=\sum_{\nu\geq 0}c_\nu x^\nu$ over $\mathbb{C}$ in the indeterminate x, and denote by $(\Gamma,\circ)$ the group of formal series which are invertible with respect to substitution $\circ$. We consider the translation equation

$$F(s+t,x)=F\big(s,F(t,x)\big),\qquad s,t\in\mathbb{C}, \tag{T}$$

for $F_t(x)=F(t,x)=\sum_{\nu\geq 1}c_\nu(t)x^\nu\in\Gamma$, $t\in\mathbb{C}$. (Cf. the introduction of [2] for the motivation to study (T) and basic results on its solutions $(F_t)_{t\in\mathbb{C}}$.) A family $(F_t)_{t\in\mathbb{C}}$ which satisfies (T) is called iteration group, and neglecting the trivial iteration group, there are two types of such groups, namely iteration groups of type I where the coefficient c_1 is a generalized exponential function different from 1, and iteration groups of type II, where $c_1 = 1$.

As it was already done in [3] also in the present paper we will only treat iteration groups of type II, and then it is known that for each iteration group of this type there exists an integer $k \geq 2$ such that

$$F_t(x) = x + c_k(t)x^k + \dots, \qquad t \in \mathbb{C},$$

where $c_k\colon \mathbb{C} \to \mathbb{C}$ is an additive function different from 0. In [3] it is shown that the translation equation (T) can be replaced by the formal translation equation in $(\mathbb{C}[y,z])[\![x]\!]$ for iteration groups of type II

$$G(y+z,x) = G\big(y, G(z,x)\big) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

for $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$, with polynomials $P_n(y) \in \mathbb{C}[y]$, $n > k$, and by the boundary condition

$$G(0,x) = x. \tag{B}$$

We had proved the following theorem: *Let* $c_k \neq 0$ *be an additive function. Then* $F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ *is a solution of* (T) *if and only if* $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ *is a solution of* ($\mathrm{T}_{\mathrm{formal}}$) *and* (B)*.*

In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(s,x))_{s\in\mathbb{C}}$ (cf. [1]) we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t, F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t, F(s,x)\big) + \beta\big(t, F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n.$$

Similarly as in [2] and [3], it is our main purpose to get *detailed information on the coefficient functions* α_n *of a solution of* (Co1) *as polynomials in* c_k*.*

In section 2 we see that (Co1) may be replaced by a "formal cocycle equation", namely

$$\Gamma(y+z,x) = \Gamma(y,x) + \Gamma\big(z, G(y,x)\big) \tag{$\mathrm{Co1}_{\mathrm{formal}}$}$$

$$\Gamma(0,x) = 0 \tag{B3}$$

where $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a formal iteration group of type II, and where $\Gamma(y,x) \in (\mathbb{C}[y])[\![x]\!]$ is a formal series in x whose coefficients are polynomials in y. Hence, ($\mathrm{Co1}_{\mathrm{formal}}$) is an identity in $(\mathbb{C}[y,z])[\![x]\!]$.

($\mathrm{Co1}_{\mathrm{formal}}$) can be solved by using various differentiation processes which are, however, purely algebraic operations. Hence we may again speak of a "method of formal functional equations" to solve (Co1) in the case of iteration groups of type II. This approach yields three types of differential equations, namely ($\mathrm{Co1D}_{\mathrm{formal}}$), ($\mathrm{Co1PD}_{\mathrm{formal}}$) and ($\mathrm{Co1AJ}_{\mathrm{formal}}$).

In section 4 ($\mathrm{Co1D}_{\mathrm{formal}}$) is solved in such a way that it yields rather detailed information on the coefficient functions of a solution of the formal first cocycle equation for iteration groups of type II (Theorem 9). In particular, it gives universal representations of the coefficient functions as polynomials in c_k (replacing y) where the coefficients of the generator $K(x) = \sum_{j\geq 1} \kappa_j x^j$ of the (formal first) cocycle serve as parameters (Theorem 9).

In section 5 we use the formal differential equation ($\mathrm{Co1PD_{formal}}$). The result (Theorem 10) is essentially the same as Theorem 9.

Since $\Gamma(y,x)$, a solution of ($\mathrm{Co1_{formal}}$), belongs to $(\mathbb{C}[y])[\![x]\!]$, it is also an element of $\mathbb{C}[\![y,x]\!]$, the ring of formal power series in two indeterminates over $\mathbb{C}$, and hence it may be seen as an element $\Gamma(y,x) = \sum_{n\geq 1}\psi_n(x)y^n$ of $(\mathbb{C}[\![x]\!])[\![y]\!]$. This approach is worked out in section 6. It follows from ($\mathrm{Co1PD_{formal}}$) that the family $(\psi_n(x))_{n\geq 1}$, a summable family of formal power series, is the solution of the rather simple recursion (2_n), (3). We obtain explicit formulas for ψ_n in terms of the derivatives $H^{(j)}(x)$ and $K^{(j)}(x)$ of the generators H and K (Theorem 17) and also a representation of $\Gamma(y,x)$ similar to a Lie–Gröbner series (Theorem 19).

At the end of this Introduction we mention that there are interesting similarities between the solutions $G(y,x)$ of the formal translation equation for iteration groups of type II studied in [3] and the solutions $\Gamma(y,x)$ of the formal first cocycle equation for iteration groups of type II.

2. The formal first cocycle equation in $\mathbb{C}[\![x]\!]$

Let x be an indeterminate over $\mathbb{C}$. We investigate solutions $\alpha(s,x) = \sum_{n\geq 0}\alpha_n(s)x^n$, $\alpha_n\colon \mathbb{C}\to\mathbb{C}$, $n\geq 0$, of the first cocycle equation

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

in $\mathbb{C}[\![x]\!]$, the ring of formal power series over $\mathbb{C}$, where $(F(s,x))_{s\in\mathbb{C}}$ is an iteration group of type II, i.e. it is a solution of the translation equation

$$F(s+t,x) = F(s,F(t,x)), \qquad s,t\in\mathbb{C}, \tag{T}$$

of the form $F(s,x)\equiv x + c_k(s)x^k \bmod x^{k+1}$, where $k\geq 2$ and $c_k\neq 0$ is necessarily an additive function.

In [2] we have proved

Lemma 1. *Let $(F(s,x))_{s\in\mathbb{C}}$ be an iteration group.*

If $\alpha(s,x) = \sum_{n\geq 0}\alpha_n(s)x^n$ is a solution of (Co1) *and* (B1)*, then α_0 is an exponential function and*

$$\hat\alpha(s,x) := \frac{\alpha(s,x)}{\alpha_0(s)} = 1 + \frac{\alpha_1(s)}{\alpha_0(s)}x + \cdots$$

is also a solution of (Co1) *and* (B1)*.*

If, conversely, α_0 is an exponential function and $\hat\alpha(s,x)\equiv 1 \bmod x$ is a solution of (Co1) *and* (B1)*, then $\alpha(s,x) := \alpha_0(s)\hat\alpha(s,x)$ satisfies* (Co1) *and* (B1)*.*

Consider $\hat\alpha(s,x)\equiv 1 \bmod x$. By substitution into the logarithmic series we obtain

Lemma 2. *$\gamma(s,x) := \log\big(\hat\alpha(s,x)\big) = \sum_{n\geq 1}\gamma_n(s)x^n$ is a solution of*

$$\gamma(s+t,x) = \gamma(s,x) + \gamma\big(t,F(s,x)\big) \tag{$\mathrm{Co1_{log}}$}$$

and

$$\gamma(0,x) = 0 \tag{B2}$$

if and only if $\hat\alpha(s,x)$ satisfies (Co1) *and* (B1)*.*

By comparing coefficients it is easy to prove

Lemma 3. *Let $F(s,x) = x + \sum_{n\geq k}P_n(c_k(s))x^n$ be an iteration group of type II, then each coefficient function $\gamma_n(s)$ of a solution γ of* ($\mathrm{Co1_{log}}$) *is a polynomial $\tilde P_n(c_k(s))$, $s\in\mathbb{C}$. Moreover for all $s,t\in\mathbb{C}$ we have*

$$\sum_{n\geq 1}\tilde P_n(c_k(s)+c_k(t))x^n = \sum_{n\geq 1}\tilde P_n(c_k(s))x^n + \sum_{n\geq 1}\tilde P_n(c_k(t))\left[x + \sum_{r\geq k}P_r(c_k(s))x^r\right]^n.$$

In [3] we have proved

Lemma 4. *Let a be a nontrivial additive function, $a \neq 0$. Then the following assertions hold true:*

(1) *a takes infinitely many values.*
(2) *Consider $P(x_1, x_2) \in \mathbb{C}[x_1, x_2]$. If $P(a(s), a(t)) = 0$ for all $s, t \in \mathbb{C}$, then $P = 0$.*

This observation allows to study formal equations by replacing $a(s)$ and $a(t)$ by indeterminates y and z. In $\mathbb{C}[y]$ we have the formal derivation with respect to y. In $(\mathbb{C}[y])[\![x]\!]$ we have the formal derivation with respect to x. Moreover the mixed chain rule is valid for formal derivations. Therefore, differentiation is a purely algebraic process in $(\mathbb{C}[y])[\![x]\!]$!

This way we obtain the formal first cocycle equation in $(\mathbb{C}[y, z])[\![x]\!]$ for iteration groups of type II of the form

$$\Gamma(y + z, x) = \Gamma(y, x) + \Gamma\big(z, G(y, x)\big) \qquad (\mathrm{Co1_{formal}})$$

together with

$$\Gamma(0, x) = 0 \qquad (\mathrm{B3})$$

for $\Gamma(y, x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$, where $G(y, x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a formal iteration group of type II.

As a consequence of Lemma 4 we easily obtain

Theorem 5. *Let $c_k \neq 0$ be an additive function. Then $\gamma(s, x) = \sum_{n\geq 1} \tilde{P}_n(c_k(s))x^n$ is a solution of* $(\mathrm{Co1_{log}})$ *satisfying $\gamma(0, x) = 0$ if and only if $\Gamma(y, x) = \sum_{n\geq 1} \tilde{P}_n(y)x^n$ is a solution of* $(\mathrm{Co1_{formal}})$ *satisfying* (B3).

3. Three differential equation derived from $(\mathrm{Co1_{formal}})$

In the sequel we solve the system consisting of the formal translation equation $(\mathrm{Co1_{formal}})$ and the boundary condition (B3). In $\mathbb{C}[y]$ we have the formal derivation with respect to y. In $(\mathbb{C}[y])[\![x]\!]$ we have the formal derivation with respect to x. Moreover the mixed chain rule is valid for formal derivations.

In the present context

$$\frac{\partial}{\partial y} G(y, x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x)$$

is called the formal generator of G. (We set $h_k := 1$.) Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1. Moreover, we call $K(x) := \frac{\partial}{\partial y}\Gamma(y, x)|_{y=0}$ the formal generator of Γ.

Differentiation of $(\mathrm{Co1_{formal}})$ with respect to z yields

$$\frac{\partial}{\partial t}\Gamma(t, x)|_{t=y+z} = \frac{\partial}{\partial z}\Gamma\big(z, G(y, x)\big).$$

For $z = 0$ we get

$$\frac{\partial}{\partial y}\Gamma(y, x) = K(G(y, x)). \qquad (\mathrm{Co1D_{formal}})$$

Differentiation of $(\mathrm{Co1_{formal}})$ with respect to y and application of the mixed chain rule yields

$$\frac{\partial}{\partial t}\Gamma(t, x)|_{t=y+z} = \frac{\partial}{\partial y}\Gamma(y, x) + \frac{\partial}{\partial t}\Gamma(z, t)|_{t=G(y,x)}\frac{\partial}{\partial y}G(y, x).$$

For $y = 0$ we get

$$\frac{\partial}{\partial z}\Gamma(z, x) = K(x) + H(x)\frac{\partial}{\partial x}\Gamma(z, x). \qquad (\mathrm{Co1PD_{formal}})$$

Combining $(\mathrm{Co1D_{formal}})$ and $(\mathrm{Co1PD_{formal}})$ yields an Aczél–Jabotinsky differential equation of the form

$$K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y, x) = K(G(y, x)). \qquad (\mathrm{Co1AJ_{formal}})$$

4. The differential equation ($\mathrm{Co1D_{formal}}$)

Now we solve ($\mathrm{Co1D_{formal}}$) together with (B3) in order to solve ($\mathrm{Co1_{formal}}$). In the sequel we use the notation $\int_0^y G(\sigma,x)d\sigma$ for $\int G(\sigma,x)d\sigma|_{\sigma=y} - \int G(\sigma,x)d\sigma|_{\sigma=0}$. If $A(\sigma,x) = \sum_{n\geq 0} a_n(\sigma)x^n$, $\sigma\in\mathbb{C}$, then $\int A(\sigma,x)d\sigma := \sum_{n\geq 0}\left(\int a_n(\sigma)d\sigma\right)x^n$, where $\int a_n(\sigma)d\sigma$ is a primitive function of a_n.

In a completely algebraic way it is possible to prove

Theorem 6. *Let $K(x) = \sum_{n\geq 1}\kappa_n x^n$ be a formal series of order at least 1, and $H(x)$ be the formal generator of the formal iteration group $G(y,x)$ of type II. Then the solution of* ($\mathrm{Co1D_{formal}}$) *and* (B3) *is*

$$\Gamma(z,x) = \sum_{j=1}^{k-1}\int \kappa_j\left[G(\sigma,x)\right]^j d\sigma + \tilde{E}\big(G(z,x)\big) - \tilde{E}(x),$$

where $\tilde{E}(x)$ of order at least 1 is given by $\dfrac{\partial}{\partial x}\tilde{E}(x) = \dfrac{\sum_{j\geq k}\kappa_j x^j}{H(x)}$.

Proof. From ($\mathrm{Co1D_{formal}}$) it follows directly that its solution which satisfies (B3) is of the form

$$\begin{aligned}\Gamma(z,x) &= \int_0^z K(G(\sigma,x))d\sigma\\ &= \sum_{j=1}^{k-1}\kappa_j\int_0^z [G(\sigma,x)]^j d\sigma + \sum_{j\geq k}\kappa_j\int_0^z [G(\sigma,x)]^j d\sigma.\end{aligned}$$

There exists exactly one series $\tilde{E}(x)$ of order at least 1 so that

$$\frac{\partial}{\partial x}\tilde{E}(x) = \frac{\sum_{j\geq k}\kappa_j x^j}{H(x)}. \tag{$*$}$$

According to [3] G satisfies ($\mathrm{D_{formal}}$), i.e. $H(G(z,x)) = \frac{\partial}{\partial z}G(z,x)$. Hence, from ($*$) we have $\sum_{j\geq k}\kappa_j x^j = H(x)\frac{\partial}{\partial x}\tilde{E}(x) = \frac{\partial}{\partial z}G(z,x)|_{z=0}\frac{\partial}{\partial x}\tilde{E}(x)$ which is the same as

$$\frac{\partial}{\partial z}\tilde{E}(G(z,x))|_{z=0} = \sum_{j\geq k}\kappa_j[G(0,x)]^j.$$

Therefore,

$$\tilde{E}(x) = \tilde{E}(G(z,x))|_{z=0} = \sum_{j\geq k}\kappa_j\int [G(\sigma,x)]^j d\sigma|_{\sigma=0} + c$$

and

$$\begin{aligned}\tilde{E}(G(z,x)) &= \sum_{j\geq k}\kappa_j\int [G(\sigma,G(z,x))]^j d\sigma|_{\sigma=0} + c\\ &= \sum_{j\geq k}\kappa_j\int [G(\sigma+z,x)]^j d\sigma|_{\sigma=0} + c\\ &= \sum_{j\geq k}\kappa_j\int [G(\sigma,x)]^j d\sigma|_{\sigma=z} + c.\end{aligned}$$

Consequently

$$\tilde{E}\big(G(z,x)\big) - \tilde{E}(x) = \sum_{j\geq k} \kappa_j \int_0^z [G(\sigma,x)]^j \, d\sigma.$$

Theorem 7. *For each generator $K(x)$ the solution Γ of* (Co1D$_{\text{formal}}$) *together with* (B3) *is a solution of* (Co1$_{\text{formal}}$).

Proof. Since G satisfies (T$_{\text{formal}}$) and (B) and since $G(y+z,x) = G(z+y,x)$ we have

$$\begin{aligned}\Gamma(y+z,x) &= \int_0^{y+z} K(G(\sigma,x))d\sigma \\ &= \int_0^z K(G(\sigma,G(y,x)))d\sigma + \int_0^y K(G(\sigma,x))d\sigma \\ &= \Gamma(y,x) + \Gamma(z,G(y,x)).\end{aligned}$$

By applying the exponential series we obtain the corresponding result to [1, third item of Theorem 2.6]

Corollary 8. *Let $G(y,x)$ be a formal iteration group of type II with formal generator H and let $F(s,x) = G(c_k(s),x)$ be an iteration group of type II where $c_k \neq 0$ is an additive function. For any generator $K(x) = \sum_{n\geq 1} \kappa_n x^n$ there exists a unique solution of* (Co1) *and* (B2) *which is of the form*

$$\alpha_0(s) \frac{E\big(G(c_k(s),x)\big)}{E(x)} \prod_{j=1}^{k-1} \exp\Big(\int \kappa_j [G(\sigma,x)]^j d\sigma|_{\sigma=c_k(s)} \Big),$$

where α_0 is an exponential function, and $E(x) = \exp\big(\tilde{E}(x)\big) = 1 + \ldots$ and $\tilde{E}(x)$ is a series of order at least 1 given by

$$\frac{\partial}{\partial x} \tilde{E}(x) = \frac{\sum_{n\geq k} \kappa_n x^n}{H(x)}.$$

We also get detailed information about the coefficients of $\Gamma(y,x)$ which are according to Theorem 5 polynomials in y.

Theorem 9. *Let $K(x) = \sum_{n\geq 1} \kappa_n x^n$ be a formal series of order at least 1. The polynomials $\tilde{P}_n(y)$ describing the coefficients of the solution of* (Co1D$_{\text{formal}}$) *and* (B3) *are universal polynomials of the form*

$$\tilde{P}_n(y) = \begin{cases} \kappa_n y & n < k \\ \kappa_k y + \frac{\kappa_1}{2} y^2 & n = k \\ \kappa_n y + \frac{(n-k+1)\kappa_{n-k+1}}{2} y^2 + \tilde{Q}_n(y,\kappa_1,\ldots,\kappa_{n-k}) & n > k, \end{cases}$$

where $\tilde{Q}_n$ are polynomials in y and in the coefficients $\kappa_1,\ldots,\kappa_{n-k}$. They satisfy $\tilde{Q}_n(0,\kappa_1,\ldots,\kappa_{n-k}) = 0$. A formal degree of $\tilde{P}_n$ is $1 + \lfloor \frac{n-1}{k-1} \rfloor$, $n \geq 1$.

Proof. In [3, Lemma 6] we proved that for $\nu \geq 1$

$$[G(y,x)]^\nu = x^\nu + \nu P_k(y) x^{\nu+k-1} + \sum_{n>\nu+k-1} \Big(\nu P_{n-\nu+1}(y) + Q_n^{(\nu)}(y) \Big) x^n$$

where

$$Q_n^{(\nu)}(y) = \sum_{\substack{(j_1,j_k,\ldots,j_{n-\nu})\in\mathbb{N}_0^{n-\nu-k+2} \\ \sum j_i = \nu \\ \sum i j_i = n}} \binom{\nu}{j_1\, j_k \ldots j_{n-\nu}} \prod_{i=k}^{n-\nu} P_i(y)^{j_i}, \quad n \geq \nu + k.$$

Therefore, similar as in [3, Lemma 7] we have

$$K\big(G(y,x)\big) = \sum_{n=1}^{k-1} \kappa_n x^n + \big(\kappa_k + \kappa_1 P_k(y)\big)x^k + \sum_{n>k}\Big(\kappa_n + \kappa_{n-k+1}(n-k+1)P_k(y) + \Phi_n\big(P_k(y),\dots,P_n(y),\kappa_1,\dots,\kappa_{n-k}\big)\Big)x^n$$

with polynomials Φ_n of the form

$$\Phi_n(P_k(y),\dots,P_n(y),\kappa_1,\dots,\kappa_{n-k}) = \kappa_1 P_n(y) + \sum_{r=2}^{n-k}\kappa_r\big(rP_{n-r+1}(y) + \Phi_n^{(r)}(y)\big).$$

By [3, Lemma 8] for $n \geq \nu + k$ a formal degree of $Q_n^{(\nu)}(y)$ is equal to

$$\left\lfloor \frac{n-\nu}{k-1} \right\rfloor.$$

Since $P_n(y)$ are polynomials with a formal degree $\lfloor \frac{n-1}{k-1} \rfloor$ a formal degree of $\Phi_n(P_k(y),\dots,P_n(y),\kappa_1,\dots,\kappa_{n-k})$ as a polynomial in y is $\lfloor \frac{n-1}{k-1} \rfloor$.

Writing $\Gamma(y,x)$ in the form $\sum_{n\geq 1}\tilde{P}_n(y)x^n$, we obtain by comparison of coefficients that

$$\frac{\partial}{\partial y}\tilde{P}_n(y) = \begin{cases} \kappa_n & n<k \\ \kappa_k + \kappa_1 y & n=k \\ \kappa_n + \kappa_{n-k+1}(n-k+1)y + \Phi_n(P_k(y),\dots,P_{n-k}(y),\kappa_1,\dots,\kappa_{n-k}) & n>k. \end{cases}$$

Integration under the boundary condition (B3) yields the assertion. Of course, integration increases a formal degree by 1.

5. The differential equation ($\mathrm{Co1PD_{formal}}$)

Now we solve ($\mathrm{Co1PD_{formal}}$) together with (B3) in order to solve ($\mathrm{Co1_{formal}}$). Again we write $\Gamma(y,x)$ as $\sum_{n\geq 1}\tilde{P}_n(y)x^n$ with $\tilde{P}_n(y) \in \mathbb{C}[y]$.

Theorem 10. *Let $K(x) = \sum_{n\geq 1}\kappa_n x^n$ be a formal series of order at least 1. The polynomials $\tilde{P}_n(y)$ describing the coefficients of the solution of* ($\mathrm{Co1PD_{formal}}$) *and* (B3) *are universal polynomials of the form*

$$\tilde{P}_n(y) = \begin{cases} \kappa_n y & n<k \\ \kappa_k y + \frac{\kappa_1}{2}\, y^2 & n=k \\ \kappa_n y + \frac{(n-k+1)\kappa_{n-k+1}}{2}\, y^2 + \tilde{Q}_n(y,\kappa_1,\dots,\kappa_{n-k}) & n>k, \end{cases}$$

where $\tilde{Q}_n$ are polynomials in y and in the coefficients $\kappa_1,\dots,\kappa_{n-k}$. They satisfy $\tilde{Q}_n(0,\kappa_1,\dots,\kappa_{n-k}) = 0$. A formal degree of $\tilde{P}_n$ is $1 + \lfloor \frac{n-1}{k-1} \rfloor$, $n \geq 1$.

Proof. From ($\mathrm{Co1PD_{formal}}$) we deduce

$$\begin{aligned}\sum_{n\geq 1}\tilde{P}_n'(z)x^n &= \sum_{n\geq 1}\kappa_n x^n + \left(\sum_{n\geq 1} n\tilde{P}_n(z)x^{n-1}\right)\left(\sum_{n\geq k} h_n x^n\right)\\ &= \sum_{n=1}^{k-1}\kappa_n x^n + \sum_{n\geq k}\left(\kappa_n + \sum_{r=k}^{n+k-1} h_r(n+1-r)\tilde{P}_{n+1-r}(z)\right)x^n.\end{aligned}$$

Comparing coefficients we obtain expressions for $\tilde{P}_n'(y)$ which yield by formal integration under the boundary condition (B3) the polynomials $\tilde{P}_n$. The assertion on the formal degree of $\tilde{P}_n$ is shown by induction.

Theorem 11. *Let $H(x)$ be the formal generator of G. For all generators $K(x)$ the solution Γ of* ($\mathrm{Co1PD_{formal}}$) *together with* (B3) *is a solution of* ($\mathrm{Co1_{formal}}$)*.*

Proof. Let $\Gamma(y,x)$ be a solution of ($\mathrm{Co1PD_{formal}}$) and (B3). We prove that $U(y,z,x) := \Gamma(y+z,x)$ and $V(y,z,x) := \Gamma(y,x) + \Gamma(z,G(y,x))$ both satisfy the system

$$\begin{aligned}\frac{\partial}{\partial y} f(y,z,x) &= K(x) + H(x)\frac{\partial}{\partial x} f(y,z,x)\\ f(0,z,x) &= \Gamma(z,x)\end{aligned}$$

and that this system has a unique solution. The proof for U is trivial. According to [3] G satisfies ($\mathrm{PD_{formal}}$), i.e. $\frac{\partial}{\partial y} G(y,x) = H(x)\frac{\partial}{\partial x} G(y,x)$, thus

$$\begin{aligned}\frac{\partial}{\partial y} V(y,z,x) &= \frac{\partial}{\partial y}\Gamma(y,x) + \frac{\partial}{\partial t}\Gamma(z,t)|_{t=G(y,x)}\frac{\partial}{\partial y}G(y,x)\\ &= K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x) + \Big(\frac{\partial}{\partial t}\Gamma(z,t)|_{t=G(y,x)}\Big)H(x)\frac{\partial}{\partial x}G(y,x)\\ &= K(x) + H(x)\frac{\partial}{\partial x}\Big(\Gamma(y,x) + \Gamma(z,G(y,x))\Big)\\ &= K(x) + H(x)\frac{\partial}{\partial x}V(y,z,x).\end{aligned}$$

The boundary conditions for $U$and V are obvious. Finally, we prove that the coefficient functions of a solution $f(y,z,x) = \sum_{n\geq 1} f_n(y,z)x^n$ of this system are uniquely determined by $f_n(y,z) = \tilde{P}_n(y+z)$, $n \geq 1$, where $\tilde{P}_n$ are the polynomials occurring in the previous theorem.

6. Reordering of the summands

From the particular representation of $\Gamma(y,x)$ as

$$\Gamma(y,x) = \sum_{n\geq 1}\tilde{P}_n(y)x^n$$

with polynomials $\tilde{P}_n$ which have a formal degree $1 + \lfloor (n-1)/(k-1)\rfloor = \lfloor (n+k-2)/(k-1)\rfloor$ and which satisfy $\tilde{P}_n(0) = 0$, we also get a representation of $\Gamma(y,x)$ as

$$\Gamma(y,x) = \sum_{n\geq 1}\psi_n(x)y^n \tag{1}$$

as an element of $(\mathbb{C}[\![x]\!])[\![y]\!]$. Writing the polynomials $\tilde{P}_r(y)$ as $\tilde{P}_r(y) = \sum_{j=1}^{d_r} \tilde{P}_{r,j} y^j$, where $d_r = \lfloor (r+k-2)/(k-1) \rfloor$ is a formal degree of $\tilde{P}_r(y)$, $r \geq 1$, then for $n \geq 1$ we have

$$\psi_n(x) = \sum_{r \geq 1} \tilde{P}_{r,n} x^r.$$

Since the degrees d_r are monotonically increasing, the sum $\sum_{n\geq 1} \psi_n(x)$ belongs to $\mathbb{C}[\![x]\!]$ and $(\psi_n(x)y^n)_{n\geq 1}$ is a summable family in $(\mathbb{C}[\![x]\!])[\![y]\!]$. This allows us to rewrite (Co1PD$_{\text{formal}}$) and (B3) as

$$\sum_{n\geq 1} n\psi_n(x)y^{n-1} = K(x) + H(x) \sum_{n\geq 1} \psi_n'(x) y^n \tag{2}$$

$$\sum_{n\geq 1} \psi_n(0) y^n = 0 \tag{3}$$

where $(\psi_n'(x)y^n)_{n\geq 1}$ is also a summable family. We note that (2) is satisfied if and only if

$$\psi_1(x) = K(x)$$

and

$$\psi_n(x) = \frac{1}{n} H(x) \psi_{n-1}'(x) \tag{2_n}$$

holds true for all $n \geq 2$. Therefore

$$\begin{aligned}
\psi_1(x) &= K(x), \\
\psi_2(x) &= H(x)K'(x)/2, \\
\psi_3(x) &= \big(H(x)H'(x)K'(x) + H(x)^2 K''(x)\big)/6.
\end{aligned}$$

Now, given the formal generator $H(x) = \sum_{n\geq k} h_n x^n$, $k \geq 2$, $h_k \neq 0$, of G and a generator $K(x) = \sum_{j\geq 1} \kappa_j x^j$, we want to solve the system $((2),(3))$ and obtain some further properties of its solutions.

Theorem 12. *Let H be the formal generator of G. For any generator K the system $((2),(3))$ has a unique solution $\Gamma(y,x) = \sum_{n\geq 1} \psi_n(x) y^n$. The order of $\psi_n(x)$ is equal to $(n-1)(k-1) + \operatorname{ord} K(x)$ and $\psi_n(0) = 0$ for all $n \geq 1$.*

Corollary 13. *We assume that $\sum_{n\geq 1} \psi_n(x)y^n = \sum_{r\geq 1} \tilde{P}_r(y)x^r$ is the solution of $((2),(3))$ for a given generator $K(x)$. Writing*

$$\tilde{P}_r(y) = \sum_{j\geq 0} \tilde{P}_{r,j} y^j, \quad r \geq 1, \text{ and } \psi_n(x) = \sum_{r\geq 1} \tilde{P}_{r,n} x^r, \quad n \geq 0,$$

we deduce that $\tilde{P}_r(y)$ is a polynomial which has a formal degree $\lfloor (r+k-2)/(k-1) \rfloor$ and which satisfies $\tilde{P}_r(0) = 0$. Consequently

$$\sum_{n\geq 1} \psi_n(x) y^n = \sum_{r \geq 1} \tilde{P}_r(y) x^r \in (\mathbb{C}\,[y])[\![x]\!].$$

Since $\psi_1(x) = K(x)$ and $\tilde{P}_n(0) = 0$ for $n \geq 1$ we have

$$\tilde{P}_n(y) = \kappa_n y + \sum_{j=2}^{\lfloor (n+k-2)/(k-1) \rfloor} \tilde{P}_{n,j} y^j, \qquad n \geq 1.$$

Now we want to describe the coefficients $P_{n,j}$ for $j \geq 2$, $n \geq 1$.

Consider the situation for $j = 2$. Let $K(x) = \sum_{j\geq\rho} \kappa_j x^j$, $\rho \geq 1$, $\kappa_\rho \neq 0$, be a generator. Then

$$\begin{aligned}
\psi_2(x) &= \frac{1}{2} H(x)K'(x) \\
&= \frac{1}{2} \left(\sum_{\nu\geq k} h_\nu x^\nu \right) \left(\sum_{\mu\geq\rho} \mu\kappa_\mu x^{\mu-1} \right) \\
&= \frac{1}{2} \sum_{r\geq k+\rho-1} \left(\sum_{\substack{\nu+\mu=r+1 \\ \nu\geq k,\ \mu\geq\rho}} \mu h_\nu \kappa_\mu \right) x^r \\
&= \frac{1}{2} \sum_{r\geq k+\rho-1} \left(\sum_{\mu=\rho}^{r+1-k} \mu\kappa_\mu h_{r+1-\mu} \right) x^r.
\end{aligned}$$

Corollary 14. *The coefficients $\tilde{P}_{r,2}$ for $r \geq k + \rho - 1$ are of the form*

$$\begin{aligned}
\tilde{P}_{k+\rho-1,2} &= \frac{1}{2} h_k \rho \kappa_\rho \\
\tilde{P}_{k+\rho,2} &= \frac{1}{2} \left(h_{k+1}\rho\kappa_\rho + h_k(\rho+1)\kappa_{\rho+1} \right) \\
\tilde{P}_{r,2} &= \frac{1}{2} \sum_{\mu=\rho}^{r+1-k} \mu\kappa_\mu h_{r+1-\mu}, \qquad r > k + \rho.
\end{aligned}$$

This proves the second summand of $\tilde{P}_n(y)$, $n \geq k$, in Theorem 9.

Using (2_n), it is possible to prove the following

Theorem 15. *If $\psi_n(x) = \sum_{r\geq(n-1)(k-1)+\rho} \tilde{P}_{r,n} x^r$ and $H(x) = \sum_{r\geq k} h_r x^r$, then*

$$\psi_{n+1}(x) = \frac{1}{n+1} \sum_{r\geq n(k-1)+\rho} \left(\sum_{\nu=(n-1)(k-1)+\rho}^{r+1-k} \nu h_{r+1-\nu} \tilde{P}_{\nu,n} \right) x^r, \qquad n \geq 0.$$

By induction this formula allows the computation of the coefficients $\tilde{P}_{r,n+1}$ of $\psi_{n+1}(x)$ for $r = n(k-1) + \rho$. We obtain

$$\tilde{P}_{n(k-1)+\rho,n+1} = \frac{h_k^n \kappa_\rho}{(n+1)!} \prod_{j=1}^{n} \big(\rho + (j-1)(k-1)\big), \qquad n \geq 0.$$

Now we describe the solutions of (2) and (3) still in another way. We need some preparatory remarks and definitions:

For $n \geq 2$ let I_n be the set of all nonnegative integer sequences $(i_j)_{j\geq-1}$ with $i_{-1} \geq 1$, $\sum_{j\geq0} i_j = n-1$, and $\sum_{j\geq0} j i_j = n - 1 - i_{-1}$. Therefore, only finitely many components i_j can be different from zero. To be more precise, $i_j = 0$ for $j \geq n-1$, thus

$$I_n = \left\{ (i_j)_{j\geq-1} \;\middle|\; \begin{array}{c} i_j \in \mathbb{Z},\ i_j \geq 0,\ i_{-1} \geq 1,\ i_j = 0 \text{ for } j \geq n-1, \\ \sum_{j=0}^{n-2} i_j = n-1,\ \sum_{j=1}^{n-2} j i_j = n-1-i_{-1} \end{array} \right\}.$$

The sets I_n are all finite. For instance I_2 contains only one sequence, namely $\iota := (1, 1, 0, \ldots)$.

Consider two integer sequences $u = (u_j)_{j\geq -1}$ and $v = (v_j)_{j\geq -1}$. We define

$$u \prec v$$

if either

$$u_1 = v_1 - 1,\ u_j = v_j \text{ for } j \neq 1, \tag{$\prec_1$}$$

or

$$u_0 = v_0 - 1,\ \exists s > 1 : u_{s-1} = v_{s-1} + 1,\ u_s = v_s - 1,\ u_j = v_j \text{ for } j \notin \{0, s-1, s\}\,, \tag{$\prec_2$}$$

or

$$u_{-1} = v_{-1} + 1,\ u_0 = v_0 - 1,\ u_j = v_j \text{ for } j > 1. \tag{$\prec_3$}$$

If we put $r := s - 1$ then $(\prec_2)$ reads as

$$v_0 = u_0 + 1, \exists r \geq 1 : v_r = u_r - 1,\ v_{r+1} = u_{r+1} + 1,\ v_j = u_j \text{ for } j \notin \{0, r, r+1\}\,.$$

Assume that $v \in I_n$, $n > 2$, and that $u \prec v$. If u is also supposed to be a sequence of nonnegative integers with $u_{-1} \geq 1$, then v must satisfy certain properties: For applying $(\prec_1)$ it is necessary to have $v_1 \geq 1$, for $(\prec_2)$ $v_0 > 1$ and $v_s \geq 1$ for some $s > 1$, for $(\prec_3)$ $v_{-1} > 1$ and $v_0 > 0$.

Lemma 16. (1) *Let $v = (v_j)_{j\geq -1}$ be a nonnegative integer sequence with $v_{-1} \geq 1$. If $u = (u_j)_{j\geq -1}$ belongs to I_n, $n > 2$, and $v \prec u$, then $v \in I_{n-1}$.*

(2) *If u belongs to I_n and $u \prec v$, then $v \in I_{n+1}$.*

(3) *For each $v \in I_{n+1}$ there exists exactly one $u \in I_n$ such that $u \prec v$.*

(4)
$$I_{n+1} = \bigcup_{u \in I_n} \{v \mid u \prec v\}\,.$$

(5) *Let u belong to I_n. The set of all $v \in I_{n+1}$ with $u \prec v$ is the union of*

$$\{(u_{-1}, u_0, u_1 + 1, u_2, u_3, \ldots), (u_{-1} + 1, u_0 + 1, u_1, u_2 \ldots)\}$$

and

$$\big\{(u_{-1}, u_0 + 1, u_1, \ldots, u_{r-1}, u_r - 1, u_{r+1} + 1, u_{r+2}, u_{r+3}, \ldots) \bigm| 1 \leq r \leq n-2,\ u_r > 0\big\}\,.$$

The proof is left to the reader.

For $n \geq 2$, $u = (u_j)_{j\geq -1} \in I_n$, $v = (v_j)_{j\geq -1}$, and $u \prec v$ we define

$$\tilde{L}(u, v) := \begin{cases} u_0 & \text{if } (\prec_1) \text{ is applied} \\ u_{s-1} & \text{if } (\prec_2) \text{ is applied} \\ 1 & \text{if } (\prec_3) \text{ is applied.} \end{cases}$$

We note that in the first case $u_0 = v_0$ and in the second case $u_{s-1} = v_{s-1} + 1 = u_r = v_r + 1$ for $r = s - 1$.

For $v \in I_n$, $n > 2$, we set

$$L(v) := \sum_{\substack{u \in I_{n-1} \\ u \prec v}} \tilde{L}(u, v) L(u) \tag{4}$$

and $L(\iota) := 1$ for $\iota := (1, 1, 0, \ldots) \in I_2$. This definition determines the value $L(v)$ recursively.

Lemma 16 and (4) allow to give a rather explicit representation of the series $\psi_n(x)$ in terms of the derivatives of the generators H and K, namely

Theorem 17. *Let H be the generator of the formal iteration group G of type II, and let $K(x)$ be a generator. Then the sequence $(\psi_n)_{n\geq 1}$ given by $\psi_1(x) = K(x)$ and*

$$\psi_n(x) = \frac{1}{n!} \sum_{i\in I_n} L(i) \prod_{j=0}^{n-2} \left[H^{(j)}(x)\right]^{i_j} K^{(i_{-1})}(x), \qquad n \geq 2,$$

satisfies the system $((2),(3))$ where $f^{(j)}(x) = \frac{d^j}{dx^j} f(x)$, $j \geq 0$, $f \in \mathbb{C}[\![x]\!]$.

Proof. By induction we proof that (2_n) is satisfied for all $n \geq 2$. For $n = 2$ we know $\psi_2(x) = H(x)K'(x)/2$ which coincides with the asserted formula. Now let $n \geq 2$. We have

$$\begin{aligned}
\psi_{n+1}(x) &= \frac{1}{n+1} H(x)\psi_n'(x) \\
&= \frac{1}{(n+1)!} H(x)\frac{\partial}{\partial x} \left(\sum_{i\in I_n} L(i) \prod_{j=0}^{n-2} \left[H^{(j)}(x)\right]^{i_j} K^{(i_{-1})}(x) \right) \\
&= \frac{1}{(n+1)!} \sum_{i\in I_n} \left({\sum_{r=0}^{n-2}}{}' H(x)L(i)i_r \prod_{j=0}^{r-1} \left[H^{(j)}(x)\right]^{i_j} \right. \\
&\qquad \left[H^{(r)}(x)\right]^{i_r-1} \left[H^{(r+1)}(x)\right]^{i_{r+1}+1} \prod_{j=r+2}^{n-2} \left[H^{(j)}(x)\right]^{i_j} K^{(i_{-1})}(x) \\
&\qquad \left. + H(x)L(i) \prod_{j=0}^{n-2} \left[H^{(j)}(x)\right]^{i_j} K^{(i_{-1}+1)}(x) \right).
\end{aligned} \tag{5}$$

The symbol ${\sum_{r=0}^{n-2}}{}'$ should indicate that we take the sum for $r = 0$, and for those $r > 0$ where $i_r > 0$. Consider the summand for $r = 0$. The order of derivative of K and the sequence of exponents of $H^{(j)}(x)$ for $j \geq 0$ is $v = (i_{-1}, i_0, i_1 + 1, i_2, i_3, \ldots) \in I_{n+1}$. From Lemma 16.5 we deduce that $i \prec v$ and $\tilde{L}(i, v) = i_0$. For $r > 0$ and $i_r > 0$, this sequence is $v = (i_{-1}, i_0 + 1, i_1, \ldots, i_{r-1}, i_r - 1, i_{r+1} + 1, i_{r+2} \ldots) \in I_{n+1}$. Again we deduce that $i \prec v$ and $\tilde{L}(i, v) = i_r$. In the last summand this sequence is $v = (i_{-1} + 1, i_0 + 1, i_1, \ldots) \in I_{n+1}$. Here we have $i \prec_3 v$ and $L(i, v) = 1$.

Moreover, from Lemma 16.5 and (5) we derive that

$$\begin{aligned}
\psi_{n+1}(x) &= \frac{1}{(n+1)!} \sum_{i\in I_n} \sum_{\substack{v\in I_{n+1} \\ i\prec v}} \tilde{L}(i,v)L(i) \prod_{j=0}^{n-1} \left[H^{(j)}(x)\right]^{v_j} K^{(v_{-1})}(x) \\
&= \frac{1}{(n+1)!} \sum_{v\in I_{n+1}} \sum_{\substack{i\in I_n \\ i\prec v}} \tilde{L}(i,v)L(i) \prod_{j=0}^{n-1} \left[H^{(j)}(x)\right]^{v_j} K^{(v_{-1})}(x) \\
&= \frac{1}{(n+1)!} \sum_{v\in I_{n+1}} L(v) \prod_{j=0}^{n-1} \left[H^{(j)}(x)\right]^{v_j} K^{(v_{-1})}(x).
\end{aligned}$$

□

There exists also an approach with Lie–Gröbner-series (cf. [4] or [5, chapter 1]) to solve $((2),(3))$. Define an operator

$$D\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \qquad D(f(x)) := H(x)f'(x).$$

Lemma 18. *Let H be the formal generator of the formal iteration group G of type II and let K be a generator. If $(\psi_n)_{n\geq 1}$ satisfies $((2),(3))$, then*

$$\psi_n(x) = \frac{1}{n!}\, D^{n-1}(K(x)), \qquad n \geq 1. \tag{6}$$

Proof. For $n = 1$ we know that $\psi_1(x) = K(x)$. Assume that $n > 1$ and that the assertion is true for $n-1$, then from (2_{n-1}) we derive

$$\begin{aligned} \psi_n(x) &= \frac{1}{n}\, D\big(\psi_{n-1}(x)\big) \\ &= \frac{1}{n}\, H(x)\psi'_{n-1}(x) \\ &= \frac{1}{n}\, H(x)\frac{\partial}{\partial x}\left(\frac{1}{(n-1)!}\, D^{n-2}\big(K(x)\big)\right) \\ &= \frac{1}{n!}\, D^{n-1}\big(K(x)\big) \end{aligned}$$

and the proof is finished. □

In [3, Theorem 24] we have shown that a formal iteration group G with formal generator H is a Lie–Gröbner series of the form

$$G(y,x) = \sum_{r\geq 0} \frac{1}{r!}\, D^r(x)y^r.$$

According to [4, Satz 7 (Vertauschungssatz)] or [5, Theorem 6 (Commutation Theorem) p. 17] $G(y,x)$ has as a Lie–Gröbner series the property

$$K\big(G(y,x)\big) = K\left(\sum_{r\geq 0}\frac{1}{r!}\, D^r(x)y^r\right) = \sum_{r\geq 0}\frac{1}{r!}\, D^r\big(K(x)\big)y^r = G\big(y,K(x)\big). \tag{7}$$

In our last result we prove that for a solution $\Gamma(y,x)$ of $(\mathrm{Co1}_{\mathrm{formal}})$, written as $\Gamma(y,x) = \sum_{n\geq 1}\psi_n(x)y^n$, the form (6) is not only necessary but also sufficient.

Theorem 19. *Let H be the formal generator of the formal iteration group G of type II and let K be a generator. The series*

$$\Gamma(y,x) := \sum_{n\geq 1}\frac{1}{n!}\, D^{n-1}\big(K(x)\big)y^n,$$

satisfies $(\mathrm{Co1}_{\mathrm{formal}})$ *and* (B3).

Proof. Let x, y, z be distinct indeterminates. The order of the generator H of G is $k \geq 2$. The family $\big(D^{n-1}\big(K(x)\big)\big)_{n\geq 1}$ is summable since for $n \geq 1$ we have

$$\operatorname{ord} D^n\big(K(x)\big) = \operatorname{ord}\big(D^{n-1}\big(K(x)\big)\big) + k - 1 > \operatorname{ord} D^{n-1}\big(K(x)\big)$$

.

Furthermore,

$$\begin{aligned}
\Gamma(z,\Gamma(y,x)) &= \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\left(K\big(G(y,x)\big)\right)z^n\\
&\overset{(7)}{=} \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\big(G(y,K(x))\big)z^n\\
&= \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\left(\sum_{r\geq 0}\frac{1}{r!}\,D^r\big(K(x)\big)y^r\right)z^n\\
&= \sum_{n\geq 1}\frac{1}{n!}\sum_{r\geq 0}\frac{1}{r!}\,D^{n-1+r}\big(K(x)\big)y^r z^n\\
&= \sum_{N\geq 1}\left(\sum_{r=0}^{N-1}\frac{1}{r!}\,\frac{1}{(N-r)!}\,y^r z^{N-r}\right)D^{N-1}\big(K(x)\big).
\end{aligned}$$

Moreover

$$\begin{aligned}
\Gamma(y+z,x) &= \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\big(K(x)\big)(y+z)^n\\
&= \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\big(K(x)\big)\sum_{r=0}^{n}\frac{n!}{r!(n-r)!}\,y^r z^{n-r}\\
&= \sum_{n\geq 1}\left(\sum_{r=0}^{n}\frac{1}{r!(n-r)!}\,y^r z^{n-r}\right)D^{n-1}\big(K(x)\big)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\Gamma(y,x)+\Gamma\big(z,G(y,x)\big) &= \sum_{n\geq 1}\frac{1}{n!}\,D^{n-1}\big(K(x)\big)y^n+\sum_{N\geq 1}\left(\sum_{r=0}^{N-1}\frac{1}{r!}\,\frac{1}{(N-r)!}\,y^r z^{N-r}\right)D^{N-1}\big(K(x)\big)\\
&= \Gamma(y+z,x).
\end{aligned}$$

It is obvious that $\Gamma(0,x)=0$. □

We note that Lie–Gröbner-series in the context of iteration groups have already been used by St. Scheinberg [7] and also in [6].

Remark 20. There are interesting similarities between the solutions of ($\mathrm{Co1}_{\mathrm{formal}}$) and the formal iteration groups of type II presented in [3]. For instance compare Theorem 9 or Theorem 10 with Theorem 4 or Theorem 9 of [3]. Moreover, the results of section 6 and section 6 of [3] are also very similar. Especially, the representation of Theorem 17 is a generalization of a similar formula for formal iteration groups given in Theorem 22 of [3].

References

[1] H. Fripertinger and L. Reich. On covariant embeddings of a linear functional equation with respect to an analytic iteration group. *International Journal of Bifurcation and Chaos*, 13(7):1853–1875, 2003.

[2] H. Fripertinger and L. Reich. The formal translation equation and formal cocycle equations for iteration groups of type I. *Aequationes Math.*, 76:54–91, 2008.

[3] H. Fripertinger and L. Reich. The formal translation equation for iteration groups of type II. *Aequationes Math.*, 79:111–156, 2010.

[4] W. Gröbner. *Die Lie-Reihen und ihre Anwendungen. 2., überarb. und erweit. Aufl.* Berlin: VEB Deutscher Verlag der Wissenschaften., 1967.

[5] W. Gröbner and H. Knapp. *Contributions to the method of Lie series.* B.I.-Hochschulskripten. 802/802a. Mannheim etc.: Bibliographisches Institut., 1967.

[6] L. Reich and J. Schwaiger. Über die analytische Iterierbarkeit formaler Potenzreihenvektoren. *Sitzungsberichte der Österr. Akademie der Wissenschaften, Abt. II*, 184:599–617, 1975.

[7] St. Scheinberg. Power Series in One Variable. *Journal of Mathematical Analysis and Applications*, 31:321–333, 1970.