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PERIODICITY OF β-EXPANSIONS FOR CERTAIN PISOT UNITS *

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Abstract. Given $\beta > 1$, let T_β

$$\begin{array}{rcl} T_\beta : [0,1[& \rightarrow & [0,1[\\ x & \rightarrow & \beta x - \lfloor \beta x \rfloor. \end{array}$$

The iteration of this transformation gives rise to the greedy β-expansion. There has been extensive research on the properties of this expansion and its dependence on the parameter β.

In [17], K. Schmidt analyzed the set of periodic points of T_β, where β is a Pisot number. In an attempt to generalize some of his results, we study, for certain Pisot units, a different expansion that we call *linear expansion*

$$x = \sum_{i \geq 0} e_i \beta^{-i},$$

where each e_i can be superior to $\lfloor \beta \rfloor$, its properties and the relation with $\mathrm{Per}(\beta)$.

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Keywords. beta-expansions, beta-representations, Pisot numbers.

Résumé. Soit $\beta > 1$, considérons T_β

$$\begin{array}{rcl} T_\beta : [0,1[& \rightarrow & [0,1[\\ x & \rightarrow & \beta x - \lfloor \beta x \rfloor. \end{array}$$

L'itération de cette transformation donne lieu à un développement en série en β . Il y a eu un grand nombre de recherches sur les propriétés de ce développement et de sa dépendance par rapport au paramètre β.

Dans [17], K. Schmidt a analysé l'ensemble des points périodiques de T_β, où β est un nombre de Pisot. Afin de généraliser certains de ces résultats, nous étudions, pour certains nombres de Pisot, un développement différent que nous appelons *développement linéaire*

$$x = \sum_{i \geq 0} e_i \beta^{-i},$$

où chaque e_i peut être supérieur à $\lfloor \beta \rfloor$; nous étudions également ses propriétés et la relation avec $\mathrm{Per}(\beta)$.

Mots clefs. beta-expansions, beta-représentations, nombres de Pisot.

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Introduction

The study of the β-expansions is an important tool to understand the qualitative and quantitative behaviour of the β-transformation. For instance, the relations between the arithmetic and algebraic properties of the parameter and the combinatoric properties of the associated subshift are not completely understood. In particular, the study of the periodicity of the β-expansions is very interesting.

In the first section we introduce some known results involving Pisot units and the periodicity of the orbits of the real numbers in $[0,1[$. The second section is dedicated to the presentation of a different representation of the real numbers, that we call linear expansion and some properties of these expansions for the rational numbers. In the third section we restrict ourselves to the case when β is the biggest root of the multinacci polynomials. Finally, in the fourth section we propose an algorithm which transforms the linear expansion of some rational numbers into their corresponding greedy β-expansion maintaining the period and the difficulties with the other rational numbers.

1. Greedy β-expansions

β-expansions were introduced by A. Renyi, [16] and combinatorialy characterized by Parry, in [14].

For every $x \in \mathbb{R}$, denote $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\{x\} = x - \lfloor x \rfloor$ the integer and the fractional part of x, respectively.

Let $\beta > 1$ be a real number and $\Sigma = \prod_{i=1}^{\infty}\{0, 1, \cdots, \lfloor \beta \rfloor\}$. If $x \in [0,1[$, let

$$\begin{array}{rcl} T_\beta : [0,1[& \to & [0,1[\\ x & \to & \beta x - \lfloor \beta x \rfloor \end{array}$$

and put $\varepsilon_k(x) = \lfloor \beta . T_\beta^{(k-1)}(x) \rfloor, k \geq 1$.

The sequence $\varepsilon = (\varepsilon_k(x) : k \geq 1) \in \Sigma$, obtained in this way, is called the greedy β-expansion of $x \in [0,1[$ and it satisfies

$$x = \pi(\varepsilon) := \sum_{k \geq 1} \varepsilon_k(x) \beta^{-k}$$

and

$$\varepsilon_k(T_\beta(x)) = \varepsilon_{k+1}(x) \quad \forall k.$$

It is the unique such representation of x for which

$$\sum_{k > n} \varepsilon_k(x) \beta^{-k} < \beta^{-n}, \quad \forall n.$$

Let the sequence $(a_n)_{n=1}^{\infty}$ be defined as follows: let $1 = \sum_{k=1}^{\infty} a'_k \beta^{-k}$ be the greedy β-expansion of 1, that is, $a'_k = \lfloor \beta T_\beta^{k-1}(1) \rfloor, k \geq 1$. If the tail of the sequence (a'_n) differs from 0^∞ then we put $a_n \equiv a'_n$. Otherwise let $k = \max\{j : a'_j > 0\}$ and $(a_1 a_2 ...) := (a'_1 a'_2 ... a'_{k-1}(a'_k - 1))^\infty$.

The natural order of $\{0, 1, \cdots, \lfloor \beta \rfloor\}$ induces a lexicographic order in Σ wich will be denoted by $\prec$.

Parry, [14], showed that each greedy β-expansion ε is lexicographically less than $(a_1 a_2 ...)$ for every $n \in \mathbb{N}$, and, conversely, that every sequence with this property is the greedy β-expansion for some $x \in [0,1[$.

Define the set of all β-expansions for $x \in [0,1[$ as

$$X_\beta = \{\varepsilon \in \Sigma : (\varepsilon_n \varepsilon_{n+1} ...) \prec (a_1 a_2 ...), n \in \mathbb{N}\}$$

that can be endowed with the lexicographical order, the product topology and the one sided shift, σ.

The map $\pi : X_\beta \to [0,1[$ defined by

$$\pi(\varepsilon) := \sum_{k\geq 1} \varepsilon_k(x)\beta^{-k}$$

is bijective and continuous almost everywhere. Also we have,

$$T_\beta \circ \pi(\varepsilon) = \pi \circ \sigma(\varepsilon)$$

so there's a topological conjugacy between $([0,1[, T_\beta)$ and the sub-*shift* (X_β, σ).

Let Per(β) be the set of (eventually) periodic expansions, Pur(β) the set of purely periodic expansions and Fin(β) the set of expansions that ends in infinitely many zeros.

Definition 1.1. *[14] Let $\beta > 1$. If $\{T_\beta^n(1)\}_{n=0}^\infty$ is finite then β is a β-number. If $T_\beta^n(1) = 0$ for some n, then β is a simple β-number.*

For every $\beta > 1$, Per(β) $\subset \mathbb{Q}(\beta) \cap [0,1[$. If $\beta > 1$ is an integer, it is clear that Per(β) $= \mathbb{Q}(\beta) \cap [0,1[$. For $\beta \in \mathbb{N}$, it is known for a long time that rational numbers p/q with purely periodic β-expansions are exactly those such that q and β are coprime, the length of the period being the order of β in $(\mathbb{Z}/q\mathbb{Z})^*$. However, if one replaces β by an algebraic integer, the situation can be totally different.

Definition 1.2. *A number α is an algebraic integer if it is the root of a monic integer polynomial. There is a unique monic integer polynomial $p_{min}(x)$, called the minimal polynomial, for which α is a root and the degree of $p_{min}(x)$ is minimal.*

Definition 1.3. *If α is an algebraic integer, and $p_{min}(x)$ is its minimal polynomial, then we say that all of the other roots of $p_{min}(x)$ are the conjugates of α.*

Definition 1.4. *A Pisot number, α, is a real algebraic integer greater than 1, all of those conjugates are of modulus strictly less than one.*

Definition 1.5. *A Salem number. α, is a real algebraic integer greater than 1, such that all conjugates are less than or equal to one in modulus, and at least one conjugate is equal to one in modulus.*

If α is rational and if $\{\alpha\} \in$ Per(β), then β must be an algebraic number. However, if $1/q \in$Per(β) for some $q \geq 2$, then β is an algebraic integer. In [17], K. Schmidt analyzed Per(β) and showed that every Pisot number is a β-number.

Theorem 1.6. *[17] Let $\beta > 1$ be a real number and assume that $\mathbb{Q} \cap [0,1[\subset Per(\beta)$. Then β is either a Pisot or a Salem number.*

A. Bertrand proved this same result independently in [6, 7].

Theorem 1.7. *[17] Let $\beta > 1$ be a Pisot number. Then $Per(\beta) = \mathbb{Q}(\beta) \cap [0,1[$.*

Theorem 1.8. *[14, 17] Let β be an algebraic integer, and assume that β is neither a Pisot nor Salem number. Then $\mathbb{Q} \cap Per(\beta)$ is nowhere dense in $[0,1[$.*

It is widely known that the orbit of 1 is finite if β is a Salem number of degree 4. The problem that has been left unsolved, in [17], is how the Salem numbers behave in general. David W. Boyd analyzed the Salem numbers [8–11].

A natural question is how to determine the set of real numbers with finite expansions. In 1992, C. Frougny and B. Solomyak, [12], studied in a systematical way the base numbers β which give rise to finite β-expansions for large classes of numbers. They stated that $\beta > 1$ has property (F) if Fin(β) $= \mathbb{Z}[1/\beta] \cap [0, +\infty[$ and that (F) can hold only for Pisot numbers β. A complete characterization of β satisfying the finiteness property (F) is known when β is a Pisot number of degree 2 or 3 (see [4]). On the other hand, not all Pisot numbers have property (F) and so a geometrical meaning of (F) became interesting.

For the β-transformation, Thurston, in 1989, introduced the concept of tiles in $\mathbb{R}^{d-1}$ when β is a Pisot unit of degree d, see [18]. In 1999, Akiyama [3] and Praggastis [15] independently showed that these tiles form a tilling of $\mathbb{R}^{d-1}$ when β satisfies the finiteness property (F).

It turns out that if β is a Pisot unit and satisfies (F) then there exists a neighborhood of zero in $\mathbb{Q}_+$ whose elements have purely periodic β- expansions (see [2]).

Set

$$\gamma(\beta) = \sup\{c \in [0,1[, \forall 0 \leq p/q \leq c, d_\beta(p/q) \text{ is purely periodic } \}$$

where $d_\beta(x)$ is the β-expansion of x. In other words, what Akiyama proved in [2] is the unexpected result that if β is a Pisot unit satisfying (F), then $\gamma(\beta) > 0$.

In [1], the authors were concerned with the real numbers β such that all sufficiently small rational numbers have a purely periodic β-expansion, that is, such that $\gamma(\beta) > 0$. They proved the following:

Theorem 1.9. *[1] Let β be a cubic Pisot unit satisfying (F) and such that the number field $\mathbb{Q}(\beta)$ is not totally real. Then, $\gamma(\beta)$ is irrational. In particular, $0 < \gamma(\beta) < 1$.*

The proof of this result relies on some topological properties of the Thurston tillings associated with Pisot units.

In [5], the main goal of the authors was also the study of the set of the rational numbers having a purely periodic β-expansion, with β a Pisot number but not necessarily a unit. They extend the concepts of the unit case and present upper and lower bounds for $\gamma(\beta)$. They also have some results involving the conditions that the rational and real numbers need to satisfy to have purely periodic β-expansions.

2. Linear expansions and periodicity

Opposed to what happens in the integer basis case, it is unknown in what measure it is possible to characterize the purely periodic points, namely the purely periodic rational numbers, depending on β. Another problem related with the previous one is in which measure does the orbits period depend on the arithmetic properties of its points. For instance, in which cases does the period of a rational orbit depend only on its denominator?

Suppose that β is an algebraic integer with minimal polynomial

$$P_d(x) = x^d - \sum_{j=0}^{d-1} k_{d-j} x^j$$

satisfying the condition $k_1 \geq k_2 \geq \cdots \geq k_d > 0$ then β is a Pisot number and the set $Fin(\beta)$ coincides with $\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$.

Consider β satisfying the previous conditions and, additionally, $k_d = 1$ then β is a Pisot unit. More, letting q be a natural number, the set

$$\{x \in \mathbb{Q}(\beta) \cap [0,1[\,|\, qx \in \mathbb{Z}[\beta]\}$$

is T_β invariant. In which cases is the orbit period of the elements of this set (in particular the rational numbers) determined by q?

In [17], K. Schmidt proved that if β satisfies $\beta^2 = n\beta + 1$, $n \geq 1$ then every $\alpha = \dfrac{p}{q} \in \mathbb{Q} \cap [0,1[$ has a purely periodic β-expansion and the period depends only on q. Moreover, if we denote the period by $p(q^{-1}, \beta)$ and if $q = q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$ is the prime factorization of q, we have

Property $\mathcal{P}$:

$$p(q^{-1}, \beta) = lcm\{p(q_i^{-k_i}, \beta) : 1 \leq i \leq m\}.$$

The proof uses an approximation and renormalization technique, that we call linear expansion, that differs from the known expansions because it admits coefficients greater than $\lfloor \beta \rfloor$, and by a procedure to obtain the greedy β-expansion from it.

2.1. **Properties of the linear expansions**

Let us consider

$$P_d(x) = x^d - \sum_{j=0}^{d-1} k_{d-j} x^j$$

satisfying $k_1 \geq k_2 \geq \cdots \geq k_d = 1$.

These polynomials are irreducible and their biggest real root β is a Pisot unit. Let $\omega = (\beta^{-1}, \cdots, \beta^{-d})^t$ and A the integer matrix

$$A = \begin{bmatrix} k_1 & k_2 & \cdots & k_d \\ 1 & 0 & \cdots & 0 \\ & \ddots & & 0 \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

that satisfies $A\omega = \beta\omega$, that is, ω is a eigenvector of A associated to the eigenvalue β. In particular, if $u = (k_1, \cdots, k_d)$, we have $u \cdot \omega = 1$. Given $\alpha \in \mathbb{Q}(\beta)$, there exists only one representation

$$\alpha = \frac{1}{q} \sum_{i=1}^{d} v_i^{(0)} \beta^{-i} = \frac{1}{q} v^{(0)} \cdot \omega, \qquad v_i^{(0)} \in \mathbb{Z},\ gcd\{v_1^{(0)}, \cdots, v_d^{(0)}, q\} = 1.$$

Since $\{\alpha\} = \alpha - \lfloor \alpha \rfloor = \alpha - \varepsilon_0(\alpha)$,

$$\begin{aligned} T_\beta(\alpha) &= \beta\alpha - \lfloor \beta\alpha \rfloor = \beta\alpha - \varepsilon_1(\alpha), \\ T_\beta^{(2)}(\alpha) &= T_\beta(T_\beta(\alpha)) = \beta^2(\alpha - \varepsilon_0(\alpha)\beta^{-0} - \varepsilon_1(\alpha)\beta^{-1} - \varepsilon_2(\alpha)\beta^{-2}) \end{aligned}$$

For n, we have

$$T_\beta^{(n)}(\alpha) = \beta^n \left(\alpha - \sum_{k=0}^{n} \varepsilon_k(\alpha) \beta^{-k} \right) \tag{1}$$

and also

$$T_\beta^{(n)}(\alpha) = \frac{1}{q} \sum_{k=1}^{d} v_k^{(n)} \beta^{-k} = \frac{1}{q} v^{(n)} \cdot \omega.$$

To better understand the relationship of $v_k^{(n)}$, obtained by T_β, with the symbolic orbit of α we note that the first part of (1) is given by

$$\beta^n \alpha = \beta^n . \frac{1}{q} . v^{(0)} . \omega = \frac{1}{q} . v^{(0)} . \beta^n \omega = \frac{1}{q} v^{(0)} . A^n \omega.$$

For the same reason, the second part of (1) is

$$\beta^n \sum_{k=0}^{n} \varepsilon_k(\alpha) \beta^{-k} = \sum_{k=0}^{n} \varepsilon_k(\alpha) \beta^{n-k} .1 = \sum_{k=0}^{n} \varepsilon_k(\alpha) \beta^{n-k} .u.\omega = \sum_{k=0}^{n} \varepsilon_k(\alpha) u A^{n-k} \omega.$$

So both parts are a combination of integer coefficients of $\beta^{-1}, \cdots, \beta^{-d}$.

We conclude that

$$v^{(n)} \cdot \omega = \left(v^{(0)} A^n - q \sum_{k \geq 0} \varepsilon_k(\alpha) u A^{n-k} \right) \cdot \omega$$

which implies that $v^{(n)} \equiv v^{(0)} A^n \mod q$. So the action of T_β in the invariant set

$$\{x \in \mathbb{Q}(\beta) \cap [0,1[\,|\, qx \in \mathbb{Z}[\beta]\}$$

can be related to the induced action of A in the finite torus $(\mathbb{Z}/q\mathbb{Z})^d$.

Even though for β in the previous conditions every $\alpha \in \mathbb{Q}(\beta) \cap [0, +\infty[$ can be represented as a combination of powers of β^{-1} with non negative rational coefficients, [12], it is not true that in general for any $\beta > 1$ the coefficients of the representation $\alpha = \frac{1}{q} v^{(0)} \cdot \omega$ are non negative.

In fact, as a consequence of Perron-Frobenius theorem, [13], $\alpha = \frac{1}{q} v^{(0)} \cdot \omega$ is bigger than zero if and only if there exists $k > 0$ such that $v^{(0)} A^k \in \mathbb{Z}_+^d$. If fact, denoting this condition by $v^{(0)} \succeq 0$, we have

$$\alpha \in [0,1[\Leftrightarrow v^{(0)} \succeq 0 \wedge q.(k_1, k_2, ..., k_d) - v^{(0)} \succeq 0.$$

It is interesting to notice that the β-expansion of α, $(\varepsilon_k(\alpha), k \geq 1) \in \Sigma$ can be obtained when determining $v^{(n)}$ by induction:

$$v^{(n)} \succeq 0;\ q.(k_1, k_2, ..., k_d) - v^{(n)} \succeq 0; \qquad v^{(n+1)} = v^{(n)} A - q\varepsilon_{n+1}(\alpha).(k_1, k_2, ..., k_d).$$

This shows in particular that, for β in the previous conditions, the β-expansions of the elements in $\mathbb{Q}(\beta)$ can be obtained using only integer numbers as in continuous fractions, that is, not using approximations.

Suppose that $v_i^{(0)} \geq 0$ for $1 \leq i \leq d$. Let $r^{(0)}$ and $c^{(0)}$ be vectors given by

$$v^{(0)} = r^{(0)} + qc^{(0)} \qquad 0 \leq r_i^{(0)} < q \qquad \forall\, 0 \leq i \leq d$$

we define two sequences $r = (r^{(n)}, n \geq 0)$ and $c = (c^{(n)}, n \geq 0)$ of vectors with integer coefficients

$$\begin{cases} 0 \leq r_i^{(n)} < q, & \forall\, 0 \leq i \leq d; \\ \\ r^{(n-1)} A = r^{(n)} + qc^{(n)}, & \forall\, n > 0. \end{cases}$$

The sequence $r = (r^{(n)}, n \geq 0)$ can be identified with the positive orbit of $r^{(0)}$ by the induced action of A in $\left(\mathbb{Z}_{/q}\right)^d = (\mathbb{Z}/q\mathbb{Z})^d$, and is purely periodic because β is a unit. The sequence $c = (c^{(n)}, n \geq 0)$, on the other hand, may not be purely periodic but only due to the first term, $c^{(0)} = (0, ..., 0)$ (null vector): the subsequence $(c^{(1)}, c^{(2)}, \cdots)$ is purely periodic. Hence, we have

$$\begin{aligned} \alpha &= \frac{1}{q}(r^{(0)} + qc^{(0)}) \cdot \omega = c^{(0)} \cdot \omega + \frac{1}{q}\beta^{-1} r^{(0)} \cdot A\omega = c^{(0)} \cdot \omega + \beta^{-1} c^{(1)} \cdot \omega + \frac{1}{q}\beta^{-1} r^{(1)} \cdot \omega = \cdots = \\ &= \sum_{i=0}^{n} \beta^{-i} c^{(i)} \cdot \omega + \frac{1}{q}\beta^{-n} r^{(n)} \cdot \omega = \sum_{i \geq 0} \beta^{-i} c^{(i)} \cdot \omega = \sum_{i \geq 0} e_i \beta^{-i}. \end{aligned}$$

The coefficients of the linear expansion satisfy the following conditions

$$\begin{aligned} e_0 &= 0; \\ e_i &= \sum_{j=1}^{i} c_j^{(i-j)} \quad \text{if} \quad 1 \leq i < d; \\ e_i &= \sum_{j=1}^{d} c_j^{(i-j)} \quad \text{if} \quad i \geq d. \end{aligned} \tag{2}$$

If α is a rational number in the unit interval then each e_i has a upper bound.

Lemma 2.1. *Given* $\alpha \in \mathbb{Q} \cap [0,1[$ *written as*

$$\alpha = \sum_{i=0}^{\infty} e_i \beta^{-i}$$

each $e_h \leq (d-1)\max\{k_i, 1 \leq i \leq d-1\}$, $h \geq 1$.

Proof. Given

$$\alpha = \frac{p}{q} = pq^{-1}.1 = q^{-1}.(k_1 p, ..., k_{d-1}p, p)(\beta^{-1}, ..., \beta^{-(d-1)}, \beta^{-d})$$

the initial vector to consider is:

$$v^{(0)} = \begin{bmatrix} k_1 p & \ldots & k_{d-1}p & p \end{bmatrix} = \begin{bmatrix} c_1^{(0)} q + r_1^{(0)} & \ldots & c_{d-1}^{(0)} q + r_{d-1}^{(0)} & p \end{bmatrix} \tag{3}$$

Without loss of generality, since $r_i^{(0)} \geq 0$ we have

$k_i\ p = c_i^{(0)}\ q + r_i^{(0)} \Rightarrow c_i^{(0)}\ q \leq k_i\ p < k_i\ q$, so $c_i^{(0)} \leq k_i$.
If N is the period of the orbit of α, we have for all $1 \leq i < N$,

$$\begin{aligned} \begin{bmatrix} r_1^{(i)} & \ldots & r_{d-1}^{(i)} & r_1^{(i-1)} \end{bmatrix} A &= \begin{bmatrix} k_1 r_1^{(i)} + r_2^{(i)} & \ldots & k_{d-1} r_1^{(i)} + r_1^{(i-1)} & r_1^{(i)} \end{bmatrix} = \\ &= \begin{bmatrix} c_1^{(i+1)} q + r_1^{(i+1)} & \ldots & c_{d-1}^{(i+1)} q + r_{d-1}^{(i+1)} & r_1^{(i)} \end{bmatrix}. \end{aligned}$$

Once more, $0 \leq r_1^{(i)} < q$, $\ 0 \leq r_j^{(i+1)} < q$, $\ j = 1, ..., d-1$. For $j = 1, ..., d-2$,

$$k_j r_1^{(i)} + r_{j+1}^{(i)} = c_j^{(i+1)} q + r_j^{(i+1)} \Leftrightarrow k_j (r_1^{(i)} - q) = (c_j^{(i+1)} - k_j) q + r_j^{(i+1)} - r_{j+1}^{(i)} \tag{4}$$

Since $(r_1^{(i)} - q) < 0$, $(c_j^{(i+1)} - k_j) q + r_j^{(i+1)} < r_{j+1}^{(i)} < q$ so $c_j^{(i+1)} \leq k_j$. A similar argument is necessary for $j = d-1$. Therefore

$$\alpha = \frac{p}{q} = \sum_{i=0}^{\infty} e_i \beta^{-i} = \sum_{i=0}^{\infty} (c_1^i, ..., c_{d-1}^i, 0)(\beta^{-i-1}, ..., \beta^{-i-(d-1)}, \beta^{-i-d}). \tag{5}$$

From (2), each e_h, of the linear expansion is a finite sum of c_j^i and since $k_1 \geq k_2 \geq ... \geq k_{d-1} \geq 1$ we have

$$e_h \leq (d-1)\max\{k_i, 1 \leq i \leq d-1\}, \quad h \geq 1.$$

□

The linear and β-expansion of α can eventually coincide. In this case, the infinite sequence $e = (e_0 e_1 e_2 ...)$ is said β-admissible. The sequence $e = (e_0 e_1 e_2 ... e_{d-1}(e_d ... e_N)^{\infty})$ is eventually periodic.

Notice that if $\alpha = \dfrac{p}{q}$ is a rational number, the last d vectors of the periodic block of the sequence $r = (r^{(n)}, n \geq 0)$ are

$$(0, 0, \cdots, p), (0, 0, \cdots, p, 0), \cdots, (p, 0, \cdots, 0) \tag{6}$$

and the correspondent vectors $c^{(n)}$ are null, which implies that, in this case, the sequence $e = (e_i, i \geq 0)$ is purely periodic.

Moreover, since $v^{(n)} \equiv r^{(n)} \mod q$, the period of the linear expansion, $\pi(\alpha,\beta)$, divides the period of the greedy β-expansion, $p(\alpha,\beta)$.

In the definition of $e = (e_i, i \geq 0)$, using $P_d(x)$, we put in evidence the denominator q. But we can obtain the linear expansion with $s^{(n)} = \frac{1}{q} r^{(n)}$. The linear action of A is in $(\mathbb{Q}/\mathbb{Z})^d$ and implies that the periods of the rational numbers satisfy the arithmetic conditions presented by K. Schmidt for the quadratic case:

Proposition 2.2. *Let α be a rational number. The period of the linear expansion, $\pi(q^{-1},\beta)$, depends only on its denominator q. If the* $\gcd(q_1,q_2) = 1$ *then*

$$\pi((q_1 q_2)^{-1},\beta) = lcm(\pi(q_1^{-1},\beta), \pi(q_2^{-1},\beta)).$$

Let us consider, for example, $\beta^3 = \beta^2 + \beta + 1$ and $x = 1/2;\ 1/3;\ 1/6$

- $x = 1/2;\ 1/3;\ 1/6$ both expansions are purely periodic and have the same period (per. 4; 13; $52 = 4 \times 13$, respectively);

Computations show that the period of the orbits of several rational numbers present the same behaviour. Nevertheless, there are other rational numbers, e.g, $\frac{999}{1000}$ which have an eventually periodic β- expansion and purely periodic linear expansion. But the period of both expansions is equal to the period of $1/1000$ for which both expansions are purely periodic. This kind of behaviour seems to occur in, $P_d(x)$, defined previously.

On the other hand, if we consider $\beta^5 = \beta^4 + \beta^3 + 1$ and $x = 1/2;\ 1/3;\ 1/6$ we have

(1) $x = 1/2$ the linear expansion is purely periodic with period 14, but the β-expansion has period 28.
(2) $x = 1/3$ both expansions are purely periodic with the same period (per. 121).
(3) $x = 1/6$ has β-expansion with period $1694 = 14 \times 121$.

The property $\mathcal{P}$ is satisfied but the period of the expansions don't match. Computations show that when we consider $P_d(x)$ with at least one null coefficient we see the previous behaviour, frequently.

So, it is unusual to find Pisot units, for which the orbits of the rational numbers present the same period in their linear and β-expansion.

Recall that for each $n > 0$ we have $c_j^{(n)} = \frac{1}{q}\left(k_j r_1^{(n-1)} + r_{j+1}^{(n-1)} - r_j^{(n)}\right)$, $\forall i < d$ and $c_d^{(n)} = 0$ so if $t_n = r_1^{(n-1)}$, then for all n we can rewrite the linear expansion as

$$e_i = \frac{1}{q}\left(\sum_{j=1}^{d-1} k_j r_1^{(i-j-1)} + r_d^{(i-d)} - r_1^{(i-1)}\right) = \frac{1}{q}\left(\sum_{j=1}^{d} k_j t_{i-j} - t_i\right), \quad i \geq d.$$

The initial condition $e_1, \cdots, e_{d-1}$ is determined using the vectors $c^{(0)}$ and $r^{(0)}$. For $\frac{p}{q}$ a rational number the linear expansion is

$$t_{-1} = p, t_{-2} = t_{-3} = \cdots = t_{-d} = 0, \qquad t_i + q e_i = \sum_{j=1}^{d} k_j t_{i-j}, \quad 0 \leq t_i < q,\ i = 0, ..., N. \tag{7}$$

The combinatoric characterization of the linear expansions is apparently non trivial, so in the remainder of this subsection we will consider the case of β belonging to the *multinacci* family, that is, for all $i = 1, ..., d$, $k_i = 1$:

$$p_d(x) = x^d - \sum_{i=0}^{d-1} x^i.$$

For all $\alpha \in \mathbb{Q} \cap [0,1[$,

$$\alpha = \frac{p}{q}.1 = \frac{p}{q}(1,1...,1)(\beta^{-1},...,\beta^{-d}) = \frac{1}{q}(p,p,...,p)(\beta^{-1},...,\beta^{-d}) = \frac{1}{q}\upsilon^{(0)}.\omega.$$

Since $1 \leq p < q$, $\upsilon^{(0)} = r^{(0)} + qc^{(0)}$ and $c^{(0)} = \{0,0,...,0\}$, we have for all $n \geq 1$

$$r^{(n-1)}.A = r^{(n)} + qc^{(n)}. \tag{8}$$

The list $c = (c^{(n)}, n \geq 0)$ is finite and some of the elements can be explicitly determined, after some calculations, using (8). This, enable us to see that the maximal list of vectors $\{c^{(0)}, c^{(1)},...,c^{(N)}\}$ that can occur is:

Lemma 2.3. *The maximal sequence of vectors for $\alpha \in \mathbb{Q} \cap [0,1[$ is:*

$$\{c^{(0)}, c^{(1)}, ..., c^{(N)}\} = \{\{0,...,0\}, \underbrace{\{c_1^{(1)},...,c_{d-1}^{(1)},0\},..,\{c_1^{(N-(d+3))},...,c_{d-1}^{(N-(d+3))},0\}}_{N-d-3},$$
$$\underbrace{\{0,...,0\},\{c_1^{(N-(d+1))},0,...,0\},\{1,...,1,0\}}_{3}, \underbrace{\{0,...,0\},...,\{0,...,0\}}_{d-1}\}$$

$c_j^{(i)} \in \{0,1\}$, $i = 1,...,N-(d+3)$, $j = 1,...,d-1$ e $c_1^{(N-(d+1))} \in \{0,1\}$.

So, the conditions of the linear expansion, in the multinacci case, are simpler:

$$\forall n,\ 0 \leq c_j^{(n)} \leq 1,\ j = 1,...,d \qquad \text{and} \qquad 0 \leq e_i \leq d-1, i = 0,...,N.$$

Moreover

$$e_{i+1} - e_i = \frac{1}{q}\left(2t_i - t_{i+1} - t_{i-d}\right)$$

which implies $|e_{i+1} - e_i| \leq 1$, that is, there are no "jumps" in the sequence e_i. For instance, it is not possible to have blocks like ...0130..., ...020... or any other of the same type.

Another consequence of the same equality, which is related to the next subsection, is that if $e_{i+1} > e_i$ then

$$2t_i \geq 2t_i - t_{i-d} = q + t_{i+1} \geq q$$

and therefore $t_i \geq \frac{q}{2}$. Hence, if for a certain i and $0 < m < d$ we have

$$e_{i+1} > e_i \text{ and } e_{i+m} > e_{i+m-1},$$

then for all $m < s < d$ we have

$$\sum_{j=1}^{d} t_{i+s-j} \geq t_i + t_{i+m-1} \geq q$$

and therefore $e_{i+s} > 0$.

This means that for the linear expansions in the multinacci case the block

$$e_i = 0, e_{i+1} = \cdots = e_{i+m-1} = 1, e_{i+m} = 2$$

is necessarily followed by at least $d-m$ positive coefficients, that is, it has at least d non-null coefficients.

Another property of the linear expansions, that is verified in the multinacci case, we now present for the tribonacci case: given $\alpha = p/q \in [0,1[$ in the reduced form, by lemma (2.3) there exist some restrictions on some coefficients. Nevertheless, given

$$r_1^{(i-1)} + e_i q = r_1^{(i-2)} + r_1^{(i-3)} + r_1^{(i-4)} \tag{9}$$

obtained from (7) for $d = 3$, we can rewrite it as

$$r_1^{(i-1)} = f(i) - \left(\sum_{j=0}^{i-2} f(j) e_{(i-j)} \right) q.$$

As for $f(i)$ it is known that:

- $f(0) = f(1) = 1$;
- $f(2) = 2$;
- $f(i) = f(i-1) + f(i-2) + f(i-3), \ \ i \geq 3$;

The general solution of the difference equation above is $f(i) = c_1 \lambda_1^i + c_2 \rho^i \cos(\theta)^i + c_3 \rho^i \sin(\theta)^i$ and the roots λ_i, $i = 1, 2, 3$, are associated with $\lambda^3 - \lambda^2 - \lambda - 1$.

Since each $0 \leq r_1^{(i-1)} < q$ we have

$$0 \leq f(i)p - \left(\sum_{j=0}^{i-2} f(j) e_{(i-j)} \right) q < q \Leftrightarrow \frac{\sum_{j=0}^{i-2} f(j) e_{(i-j)}}{f(i)} \leq \frac{p}{q} < \frac{\sum_{j=0}^{i-2} f(j) e_{(i-j)} + 1}{f(i)}.$$

That is, from the moment we know the first k elements of the linear expansion, $e_0 e_1 e_2 ... e_k$ it is possible to determine the interval of rational numbers that contain the block $e_0 e_1 ... e_k$. In particular, the interval endpoints depend of all of them $(e_0 e_1 ... e_k)$. This fact, shows the difficulty of determining whether or not a given block $e_{[j,j+p]}$ can occur in the middle of the linear expansion.

Applying the lemma 2.3 and (9) to (5) we can calculate the first and some of the last terms, explicitly, for every number $\alpha \in \mathbb{Q} \cap [0, 1[$.

2.2. The tribonacci polynomial

K.Schmidt, in [17], transformed the linear expansion of the rational numbers in their β-expansion without changing the period, if β satisfied $\beta^2 = n\beta + 1$.

In the multinacci case the forbidden blocks of the linear expansion, are lexicographically equal or superior to the block of ones $11 \cdots 11$ of length d.

In [12], C. Frougny and B. Solomyak presented an algorithm that allows the finite β-representation of a number to be transformed in the corresponding finite β-expansion. Even though this algorithm can be adapted to the linear expansions of the multinacci, for ease of reading we present it for the tribonacci case. We focus on the rational numbers, although it is not strictly necessary.

Let β be the Pisot number satisfying $1 = \beta^{-1} + \beta^{-2} + \beta^{-3}$ and

$$(e_1 e_2 ... e_N)^\infty$$

the purely periodic linear expansion of $\alpha = \dfrac{p}{q}$ a rational number. Recall that $0 \leq e_i \leq 2$ and $|e_{i+1} - e_i| \leq 1$.

We will now describe the steps of the algorithm, in an informal way:

Step **I**: The first step of the algorithm consists in substituting all the occurrences in the periodic block, of the forbidden blocks,

$$e_i e_{i+1} e_{i+2} e_{i+3} \qquad e_i = 0,\ e_{i+j} > 0, j \in \{1, 2, 3\}$$

by

$$(e_i + 1)(e_{i+1} - 1)(e_{i+2} - 1)(e_{i+3} - 1)$$

For simplicity, we denote the coefficients of the expansion by e_i.

Each of these substitutions originates new blocks just ahead, but given that each application of step **I** reduces the value of

$$\sum_{i=1}^{N} e_i$$

by 2, this step can only be used a finite number of times.

Step **II**: The resulting expansion may still have forbidden blocks when we link the copies of the periodic blocks, for instance, $D_1 = (1...11)^\infty$ or $D_2 = (11...1)^\infty$. In this case, we use a new periodic block, obtained from the previous one by a cyclic shift, that is, we use the expansion

$$e_1 e_2 \cdots e_m (e_{m+1} \cdots e_N e_1 \cdots e_m)^\infty$$

for some $m < N$, $D_1 = (1...11)^\infty = 1(...111)^\infty$ and $D_2 = (11...1)^\infty = 11(...111)^\infty$. Then we apply step **I**.

Notice that the shift does not modify the sum of the terms of the periodic block, therefore, once again the sequence of steps **I-II** can only be repeated a finite number of times.

If the resulting expansion is the β-expansion of α then the period divides N, but since the period of the linear expansion is a divisor of the β-expansion, we conclude that the periods are the same. In particular, for rational numbers near zero applying these two steps is clearly enough to get their β-expansion. However, for some rational numbers, we get an intermediate expansion, that contains some small forbidden blocks.

In the tribonacci case there is only a small forbidden block, 0210, that occurs if the block 210 is preceded by a block of positive coefficients whose length is multiple of 3. In this case we apply a new step:

Step **III**: search for the first occurrence of 0210 and eliminate it applying the "push forward" equality:

If

$$...0\beta^{-i} + (1+1)\beta^{-(i+1)} + \beta^{-(i+2)} + 0\beta^{-(i+3)} + a\beta^{-(i+4)}...$$

then

$$...0\beta^{-i} + \beta^{-(i+1)} + \beta^{-(i+1)}.\mathbf{1} + \beta^{-(i+2)} + 0\beta^{-(i+3)} + a\beta^{-(i+4)}...$$

since $1 = \beta^{-1} + \beta^{-2} + \beta^{-3}$

$$...0\beta^{-i} + \beta^{-(i+1)} + \beta^{-(i+1)}.(\beta^{-1} + \beta^{-2} + \beta^{-3}) + \beta^{-(i+2)} + 0\beta^{-(i+3)} + a\beta^{-(i+4)}...$$

hence

$$...0\beta^{-i} + \beta^{-(i+1)} + 2.\beta^{-(i+2)} + 1.\beta^{-(i+3)} + (a+1)\beta^{-(i+4)}...$$

In sequence terms

$$0210a \to 0121(a+1) \to 1010(a+1)$$

If this step creates new "big" (length greater than three) forbidden blocks we return to step **I**. Otherwise, we search for the next small forbidden block. In fact, depending of the coefficients that follow 0210, step **III** can eliminate it or create a new forbidden block that may be of a new type. Next, we present all the forbidden blocks that can be created after applying step **III**. Notice that we have to consider the cases where one of the new forbidden blocks encounters the block 0210.

0210.110	$\to$	101.0210
0210.100	$\to$	101.0200
0210.101	$\to$	101.0201
0210.21010	$\to$	110001.020
0210.210110	$\to$	110001.0210
020.010	$\to$	100.020
020.110	$\to$	10.0120
020.0110	$\to$	100.0210
0120.010	$\to$	100.0120
0120.100	$\to$	100.0210
0120.110	$\to$	100.0220
0120.0210	$\to$	1001.0200
0220.010	$\to$	1.020.110
0220.100	$\to$	110.0210
0220.110	$\to$	110.0220
0220.0210	$\to$	1101.020.0

So, every time we find ...02101... step **III** makes the sequence admissible, until that index, or generates one of the three new forbidden blocks: ...020..., ...0120..., ...0220....

After applying steps **I, II, III** either we get β-expansion of α or and eventually periodic expansion with period N with only one forbidden block,

$$0210,\ 020,\ 0120,\ 0220$$

at the end of the periodic block.

This can occur for some rational numbers that aren't in a neighborhood of zero. In this case, we apply step **II** increasing the length of the pre-periodic block, considering as the periodic block the one obtained by cyclic shift. Then we apply step **III**.

Notice that, after the application of this step, we have only one forbidden block, so several of the previous cases can not be considered.

2.3. Conclusion

From [1], we know that a small interval in the neighborhood of zero has purely periodic orbit. For the rational numbers near zero, due to the properties that were found regarding the coefficients of the linear expansion using step **I** and **II**, the algorithm stops and $p(\alpha,\beta)$, the length of the period in the β-expansion of α is equal to $\pi(\alpha,\beta)$, the length of the period in the linear expansion of α. Therefore, we have

Proposition 2.4. *Let $\beta > 1$ be the Pisot number satisfying $1 = \beta^{-1} + \beta^{-2} + \beta^{-3}$. To each rational number α in a neighborhood of zero, $p(\alpha,\beta) = \pi(\alpha,\beta)$ and the period depends only on q. If $q = q_1^{k_1}...q_m^{k_m}$ is the prime factorization of q then*

$$p(q^{-1},\beta) = lcm\{p(q_i^{-k_i},\beta) : 1 \leq i \leq m\}.$$

For the other rational numbers in $[0,1[$, the algorithm can be applied but we were not able to prove that the algorithm stops even though computations show that the previous result hold for all $\alpha \in \mathbb{Q} \cap [0,1[$. So the problem remains unsolved, but we will continue to investigate it, in our future work.

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